Models with idiosyncratic and aggregate shocks and incomplete markets

Numerical Methods

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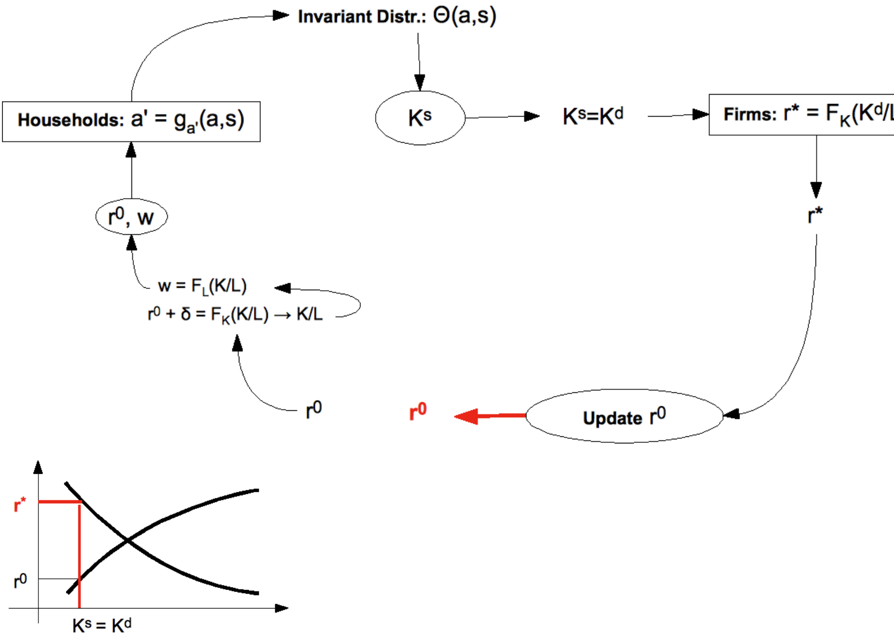
Recap

- Aiyagari 1994, production economy, incomplete markets, idiosyncratic income shocks, no aggregate risk, borrowing limit
- How to compute the equilibrium?
- Computational approach: Do a fixed-point iteration on r
- Equilibrium r: Consistency between households' supply of, and firms' demand for capital, $K^s = K^d$
- Equilibrium w: Can be backed out from $w = F_L(K, L)$ due to fixed labor supply



Invariant Distr.: Θ(a,s)
Households: a' = g_a'(a,s)
K^s
K^s=K^d
Firms: r* = F_K(K^d/L
r*
r^0, w
w = F_L(K/L)
r^0 + δ = F_K(K/L) → K/L
r^0
r^0
Update r^0
r*
r^0
K^s = K^d

An example

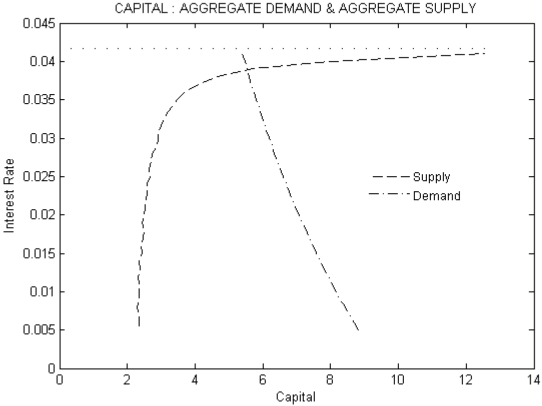
- Continuum of agents
- $u(c_t) = c_t^{1-\gamma}/(1-\gamma)$ with $\gamma = 3$
- $s_t = (1, 0)$ - a model of unemployment
- Probability transition matrix:

$$P = \begin{bmatrix} 0.95 & 0.05 \\ 0.95 & 0.05 \end{bmatrix}$$

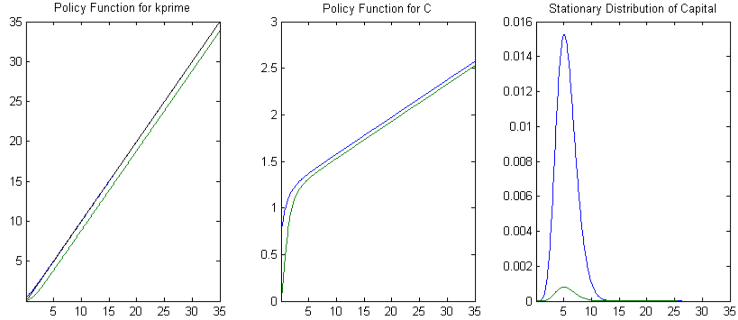
ie. the probability of being unemployed is 5 percent no matter whether you're employed today or not

- $\beta = 0.96$, $\alpha = 0.4$, $\delta = 0.1$
- $a_t \in (a_{min}, .., a_{max})$
- Here we need to impose that $a_{min} > 0$ - else marginal utility goes to infinity since the natural borrowing limit is zero
- I impose that $a_{min} = 0.001$ and that $a_{max} = 40$ and use 1000 equally spaced grid points
- Converges in around 3.046 seconds (on an old 2.9GHz Mac)
- Equilibrium interest rate is 3.897 percent per year as compared to 4.167 percent in the non-stochastic economy

Capital demand and supply



Policy fcts. and the stat. distribution



Models with Idiosyncratic and Aggregate Uncertainty

- The incomplete markets models we considered so far did not contain aggregate risk
- $\Rightarrow$ They are, generally, easy to solve because (a) in the stationary equilibrium the distribution over asset holdings and employment states is constant $\Rightarrow$ (b) The aggregate state (K, L) is constant $\Rightarrow$ (c) Prices are constant over time
- At the individual level, there is a large degree of uncertainty and ex-post heterogeneity **BUT** ...
- ... the individual uncertainty disappears at the aggregate level because of the law of large numbers.
- What if we consider an incomplete markets economy with both, idiosyncratic **and** aggregate risk?
- Seems a small extension, but it's not!

- Why is it interesting to study those models?
- Models with aggregate **and** idiosyncratic uncertainty are a natural synthesis of the prototypical RBC model used to study aggregate fluctuations (à la Kydland and Prescott (1982)) and Bewley-Huggett-Aiyagari models used to address distributional issues.
- Models of this sort can be used to study a variety of issues:
- The dynamics of the income distribution over the business cycle, Castaneda et al. (1998)
- Welfare costs of business cycles, Storesletten et al. (2001)
- The effect of productivity shocks on labor market variables, Nakajima (2008)
- Social security issues, Harenberg and Ludwig (2011)
- ....

The Computational Challenge ...

- Aggregate shock affects all individuals alike $\Rightarrow$ shock does not wash out but it affects the aggregate state variables
- Aggregate state determines factor prices, and hence every household's wealth will depend on the history of aggregate shocks and the sequence of individual shocks
- Implication: Cross-sectional distribution of (a, s) will vary over time and so will wages and interest rates
- Individuals need to forecast future prices since they affect current decisions. Prices depend on the aggregate state $\Rightarrow$
- Must forecast aggregate capital stock. To do that they need to forecast cross-sectional distribution of capital
- This is VERY difficult: Mapping distributions into distributions and looking for a fixed point to find the equilibrium ......

- Implication: the cross-sectional distribution $\Theta(a, s)$ becomes an aggregate state variable that changes over time

- This is potentially an enormous object. How to deal with it?

- Krusell and Smith (1998): Approximate the equilibrium by assuming "bounded rationality"

- Strategy: Approximate the cross-sectional distribution using some moments of the distribution

- Key result: A reasonably high precision can be obtained by using the mean of the distribution only

The Setup I

- Large number of infinitely lived agents that value consumption
- One type of assets a which yields a state non-contingents return r_t in period t
- Two sources of uncertainty: (1) aggregate productivity shock z_t, (2) idiosyncratic employment shock s_t
- An agent can be either employed $s_t = 1$ or unemployed $s_t = 0$
- s_t follows a first order Markov process with the transition probabilities $p^s_{s,s'|z,z'}$
- The transition probabilities are conditional on z and z'. Thanks to this assumption we do not have to keep track of the economy's aggregate level of employment

The Setup II

- There is a representative firm which has access to the following CRS technology:

$$Y_t = z_t F(K_t, L_t)$$

- $z_t \in Z = \{z_1, z_2, ...\}$ follows a first order Markov process with the transition probabilities $p^z_{z,z'}$

- Aggregate labor supply (i.e. the proportion of the employed) is a function of z only, i.e. $L(z_t)$

The Recursive Formulation

$$V(z,x,s,a) = \max_{c,a'} \left\{ u(c) + \beta \sum_{z'} \sum_{s'} p^{z}_{z,z'} p^{s}_{s,s'|z,z'} V(z',x',s',a') \right\}$$

$$\text{subject to} \quad c + a' = a(1 + r(z,x)) + w(z,x)s$$

$$x' = \phi_x(z,x)$$

- x is the cross-sectional distribution of agents over (a,s)
- $\phi_x(z,x)$ denotes the law of motion of the distribution
- The solution to this problem consists of optimal decision rules for consumption $c = g_c(z,x,s,a)$ and asset holdings $a' = g_a(z,x,s,a)$
- Notice: r and w are functions of the aggregate state (z,x)

- The optimal decision of the representative firm imply:

$$r = z_t F_K(K, L), \qquad w = z_t F_L(K, L)$$

- The aggregate capital stock and labor supply are given by

$$K = \int_X a \, dx, \qquad L = \int_X e \, dx$$

- As before the aggregate state (z, x) are sufficient for computing the factor prices w and r. The difference is that, now, those will not be constant in equilibrium.

Recursive Competitive Equilibrium

A recursive competitive equilibrium consists of pricing functions $r(z,x)$, $w(z,x)$, a transition function for x, $\phi_x(z,x)$, value function $V(z,x,s,a)$, decision rules $g_c(z,x,s,a)$ and $g_{a'}(z,x,s,a)$, such that:

- $V(z,x,s,a)$ and $g_c(z,x,s,a)$ and $g_{a'}(z,x,s,a)$ solve the household's problem
- $r(z,x)$ and $w(z,x)$ are consistent with firm's optimization

$$r(z,x) = zF_K(K,L) \qquad w(z,x) = zF_L(K,L)$$

- The transition function $\phi_x(z,x)$ is consistent with the actual law of motion implied by $g_{a'}(z,x,s,a)$ and the processes for z and s

Krusell and Smith (1998)

- How shall we deal with such a large dimensional object like x?
- Solution: Use a finite set of moments of the asset distribution
- Krusell and Smith (1998) show that, for the current model, it is enough to use only the first moment (mean asset holding) to represent the asset distribution
- Adding higher moments does not change the result substantially.
- Intuition: The optimal decision rule for asset holdings is close to linear (for $s = 0$ and $s = 1$)
- Let's represent x by the mean asset holding, i.e. $K = \int_X a \, dx$
- The law of motion for x becomes $K' = \phi_K(z, K)$

Solution Strategy

- The big unknown here is the transition function $\phi_K(z, K)$
- The strategy is to first, parametrize $\phi_K(\cdot)$ and then iterate on the parameters that characterize the function
- A functional form suggested by Krusell and Smith (1998) and used by many other applications is the log-linear form:

$$\log K' = \phi_{0,z} + \phi_{1,z} \log K$$

- Suppose z can take on two values, $\{z_{low}, z_{high}\}$ then there are four parameters that characterize the transition function

$$\phi_{0,z_{low}} \qquad \phi_{1,z_{low}} \qquad \phi_{0,z_{high}} \qquad \phi_{1,z_{high}}$$

Algorithm, Step I: The Agent's Problem

1. Choose the set of moments that represent the type distribution. Here we follow KS (1998) and take the mean asset holding K
2. Parameterize the forecasting function:

$$\log K' = \phi_{0,z} + \phi_{1,z} \log K$$

3. Set grid points for the space of K. KS (1998) use 25 points only
4. Start with a guess on $\phi^0_{0,z}$ and $\phi^0_{1,z}$
5. A reasonable guess is: $\phi^0_{0,z} = \log \bar{K}$ and $\phi^0_{1,z} = 0$, where $\bar{K}$ is the steady state of aggregate asset holdings
6. Using those guesses solve the household's optimization problem for $g_c(z, x, s, a)$ and $g_{a'}(z, x, s, a)$

Algorithm, Step II: The Simulation Step

1. Set the number of periods T and the number of periods which will be cut T_0. KS (1998) use $T = 11000$ and $T_0 = 1000$
2. Draw a sequence of z_t for $t = 0, 1, ..., T$, using a random number generator. *[Use the same sequence in every simulation]*
3. Set the initial distribution x_0. A good choice is $x_0 = \bar{x}$, where $\bar{x}$ is the distribution associated with the non-stochastic steady state
4. Using x_0, compute K_0, which is the aggregate (mean) capital stock
5. Use $g_{a'}(z, x, s, a)$, $p^s_{s,s'|z,z'}$ and x_0 to obtain x_1.
6. Compute K_1 from x_1
7. Repeat until $t = T$. The result is a set of sequences for x_t, z_t and K_t
8. Drop the first T_0 periods from the sample

Algorithm, Step III: Updating

1. Take $\{z_t, K_t\}$ and run an OLS regression to obtain the new coefficients: $\phi^1_{0,z}$ and $\phi^1_{1,z}$

2. Compare the old with the new set of coefficients. If the distance is small enough, stop. Otherwise update $\phi_{0,z}$ and $\phi_{1,z}$ and go to step 6 two slides back

3. The updating should be done conservatively: (with λ being small)

$$\phi_{0,z} = \lambda \phi^1_{0,z} + (1-\lambda)\phi^0_{0,z}$$

$$\phi_{1,z} = \lambda \phi^1_{1,z} + (1-\lambda)\phi^0_{1,z}$$

4. Once $\phi_{0,z}$ and $\phi_{1,z}$ are found we can check the goodness of fit of the approximation. If the fit is not satisfactory we can try richer functional form for the transition function, or increase the set of moments

Aggregate shocks and machine learning

- Fernández-Villaverde, Hurtado & Nuño (2020):
  - Krusel-Smith-type world; more complicated
  - Continuous-time setup (we'll talk about this later)
  - Updating using machine learning (neural networks)
  - Model estimation