Models with idiosyncratic shocks and incomplete markets

Numerical Methods

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Acknowledgements

- This set of slides is heavily based on notes by Georg Dürnecker.

Competitive Equilibrium

- So far: social planner's problem - use of the fundamental welfare theorems
- Convenient, especially from computational perspective
- However, if the competitive equilibrium is not Pareto optimal, we cannot use this approach
- We have to compute the (competitive) equilibrium directly

Competitive Equilibrium

- So far: social planner's problem - use of the fundamental welfare theorems
- Convenient, especially from computational perspective
- However, if the competitive equilibrium is not Pareto optimal, we cannot use this approach
- We have to compute the (competitive) equilibrium directly
- Many ways of doing this depending on how we define the market/asset structure
- Let's start with complete markets and aggregate risk only
- Later: idiosyncratic (and aggregate) risk and incomplete markets

Environment

- Large number of identical households
- Large number of identical firms
- Firms rent capital and labor from the households at the prices r (for capital) and w (for labor)
- The aggregate capital stock is given by K
- Aggregate uncertainty through productivity shock θ
- Measure of households is normalized to 1, in equilibrium individual and aggregate variables need to be the same

- Aggregate state is given by (K, θ)
- K and θ determine the factor prices: $r(K, \theta)$ and $w(K, \theta)$
- Firm j's problem is static and it consists of

$$\max_{k_j, l_j} f(k_j, l_j, \theta) - r(K, \theta) k_j - w(K, \theta) l_j$$

- Firms take all factor prices as given. Output price is normalized to 1
- No market power
- The first-order conditions are given by

$$f_k(k_j, l_j, \theta) = r(K, \theta)$$

$$f_l(k_j, l_j, \theta) = w(K, \theta)$$

- Household i's problem consist of

$$\max_{\left\{c_{i,s},x_{i,s}\right\}_{s=t}^{\infty}} E_t \sum_{s=t}^{\infty} \beta^{s-t} u(c_{i,t})$$

$$s.t.: \quad c_{i,s} + x_{i,s} = w_s + r_s k_{i,s}$$

$$k_{i,s+1} = (1-\delta) k_{i,s} + x_{i,s}$$

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- or in recursive formulation

$$v(k_i, K, \theta) \quad = \quad \max_{c_i, x_i} \{u(c_i) + \beta E [v(k_i', K', \theta')]\}$$

$$s.t. \quad : \quad c_i + x_i = w(K, \theta) + r(K, \theta) k_i$$

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$$s.t. \quad : \quad c_i + x_i = w(K,\theta) + r(K,\theta)k_i$$

$$k_i' = (1-\delta)k_i + x_i$$

- The household knows the law of motion for the aggregate productivity shock and for the aggregate capital stock

$$K' = (1-\delta)K + X$$

Definition of the Equilibrium

- A *Recursive Competitive Equilibrium* consists of **price functions** $r(K,\theta)$ and $w(K,\theta)$, **value functions** $v(k_i,K,\theta)$, **optimal decision rules** $c(k_i,K,\theta)$, $x(k_i,K,\theta)$ for each household, and the corresponding **aggregate decision rules** $C(K,\theta)$, $X(K,\theta)$ which are such that:

- They solve the household's optimization problem

- They are consistent with firms maximizing profits, i.e.

  $f_k(K,1,\theta) = r = r(K,\theta)$ and $f_l(K,1,\theta) = w = w(K,\theta)$

- Consistency between individual and aggregate decisions, i.e.

  $k_i = k = K \quad l_j = l = 1 \quad c(K,K,\theta) = C(K,\theta) \quad x(K,K,\theta) = X(K,\theta)$

- The aggregate resource constraint holds

$$C(K,\theta) + X(K,\theta) = Y(K,\theta) \qquad \text{for all } (K,\theta)$$

- From the household's problem it follows that in the optimum

$$u_c(c_i(k_i, K, \theta)) = \beta E u_c(c_i(k_i', K', \theta'))(r(K', \theta') + (1 - \delta))$$

- Next, we know that in the RCE

$$c(K, K, \theta) = C(K, \theta)$$

$$r(K, \theta) = f_k(K, 1, \theta)$$

- Using these:

$$u_c(C(K, \theta)) = \beta E u_c(C(K', \theta'))(f_k(K', 1, \theta') + 1 - \delta)$$

which is the same as in the planner's problem

Now:

- Complete markets + idiosyncratic risk
- Incomplete markets + idiosyncratic risk
- Heterogeneity

Complete markets and idiosyncratic risk

- Under idiosyncratic risk we require the existence of complete markets if we want that "competitive = social planner allocation" holds

- In multiple agent models, the planner's problem involves maximizing:

$$U_{planner} = \sum_{i=1}^{N} \phi_i u_i(c_i)$$

  where $\phi_i \geq 0$ is the welfare weight that the planner associates to agent of type i and N denotes the number of different types of agents

- The planner maximizes only subject to the resource constraint. Take an endowment example:

$$\sum_i c_i \leq \sum_i w_i$$

- Hence, the planner's problem will include the first-order condition

$$\phi_i u_i'(c_i) = \lambda$$

where λ is the multiplier on the resource constraint

- Therefore

$$\phi_i u_i'(c_i) = \phi_j u_j'(c_j)$$

- Whatever the state, the weighted marginal utilities are equalized across agents

- In other words: All idiosyncratic risk is shared

- In a competitive world **this requires complete markets**

- We can say nothing about inequality (it is assumed through welfare weights) and inequality does not matter for the aggregate allocation (and vice versa)
- Idiosyncratic risk has no consequence for the allocation neither at the aggregate level nor at the individual level
- Complete markets: Useful starting point
- Tells us how the economy would behave in the special case where markets can provide full insurance against idiosyncratic risk

Incomplete Markets

- Individuals: no access to insurance markets, can purchase single, risk-free asset with non-state-contingent payoff
- Asset holdings as "self-insurance" (precautionary savings)
- Individuals cannot **fully** insure against idiosyncratic risk
- Many different reasons and variations of such models
- Main implications:
  - Marginal rates of substitution may differ across agents - imperfect risk sharing
  - Agents differ in wealth and consumption - model can say something about inequality
  - Aggregate equilibrium may depend on distribution
- First models by Bewley in the 1970s and 80s

A savings problem

- Suppose there is a continuum of households
- Each household faces idiosyncratic labor income risk - for example employed or unemployed, labor income ws_t - but there is no aggregate uncertainty
- Particular example: interpreting s_t as the individual employment level, or alternatively as the efficiency level of an individual agent, the cross-sectional average of s_t can be interpreted as the aggregate employment level (in efficiency units) which is constant
- Labor income risk cannot fully be insured - savings only through a single asset a_t with state non-contingent payoffs - thus there are incomplete markets

- We assume that s_t is discrete and evolves according to an n-state homogeneous Markov chain with transition probability matrix P

- Also discretize assets: $a_t \in A = [a_1 < .. < a_n]$

- This is distinctively different from discretizing aggregate assets: A_t

- This discretization implies the existence of borrowing constraints through the assumption on the minimum level of assets

The households problem:

$$\begin{aligned} & \max E_0 \textstyle\sum_{t=0}^{\infty} \beta^t u(c_t) \\ st \quad & : \quad c_t + a_{t+1} = (1+r)a_t + ws_t \\ a_{t+1} \quad & \in \quad A \end{aligned}$$

where we make standard assumptions about preferences

Bellman's equation for the household's problem:

$$v(a,s) = \max_{a' \in A} \left\{ u\left(ws + (1+r)a - a'\right) + \beta E\left(v(a',s')\right) \right\}$$

or using the Markov structure:

$$v(a_l, s_j) = \max_{a' \in A} \left\{ u\left(ws_j + (1+r)a_l - a'\right) + \beta \sum_{h=1}^{n} p_{j,h} v(a', s_h) \right\}$$

- This problem can be solved using standard procedures
- Borrowing limits: built into it through the grid imposed on a
- Borrowing limits are a necessary condition for the existence of an equilibrium - were there no borrowing limits, agents could keep on accumulating debt (Ponzi schemes)
- Which debt limit to impose?
- It must be a debt limit that we know that the agent can actually observe: The formulation above assumes that the agent always pays her debt
- But, still - not clear how to impose the debt limit (there would be many different ones that would reassure that the agent can always observe the debt)

- Aiyagari - natural debt limit: consumption must be positive

$$c_t \geq 0$$

- From the budget constraint we have that

$$\begin{aligned} c_t + a_{t+1} &= ws_t + (1+r)a_t \Rightarrow \\ a_t &= \tfrac{1}{1+r} \textstyle\sum_{i=0}^{\infty} (1+r)^{-i} (c_{t+i} - ws_{t+i}) \end{aligned}$$

- so $c_t \geq 0$ implies that:

$$a_t \geq -\frac{1}{1+r} \sum_{i=0}^{\infty} (1+r)^{-i} ws_{t+i}$$

- However, the right hand side here is a stochastic variable since it depends on future idiosyncratic earnings shocks
- And: We cannot impose simply that it holds in expected value - in that case there will be states where it does not hold!

- But - we can impose it by using the worst state forever

$$s_1 = \min s$$

- the natural debt limit is

$$\phi = -\frac{1}{1+r}\sum_{i=0}^{\infty}(1+r)^{-i}\, w s_1 = -\frac{s_1 w}{r}$$

- Of course: We may impose stronger debt limits - these would provide even stronger limits on the maximum debt and thus can be implemented leaving the agent with non-negative consumption

Distributions

- We can find the stationary distribution of agents over assets and over employment states (a, s), $\pi_\infty(a, s)$, from iterating on:

$$\pi_{t+1}(a', s') = \sum_s \sum_a \pi_t(a, s) P(s', s) \mathscr{I}(g_a(a, s) = a')$$

- π_∞ can also be interpreted as the fraction of agents that at each point in time are characterized by (a_I, s_j)

- From $\pi_\infty(a, s)$ we can derive the wealth distributing from:

$$H_\infty(a) = \pi_\infty(a, s) G(s)$$

  where $G(s)$ is the ergodic probability of state s

- The model can say something about wealth inequality given assumptions about preferences and asset markets

- There will in general be incomplete risk sharing: households will not be able fully to insure their idiosyncratic risk

Models with no aggregate fluctuations

- **Example 1: Huggett 1993, JEDC**
  - Pure exchange economy with many agents
  - Only asset is a private state non-contingent one-period debt contract
  - Agents always pay their debt
  - Exogenous debt limit: Debt cannot exceed $\phi > 0$, $a \in A = [a_1 < .. < a_n]$, $a_1 = -\phi$
- **Example 2: Aiyagari 1994, QJE**
  - Production economy with capital

Aiyagari 1994, QJE

- Large number of households that face idiosyncratic risk in the form of an employment shock, s_t
- Example: $s = 1$ ($s = 0$) $\Rightarrow$ employed (unemployed)
- No aggregate uncertainty
- Households can save in an asset (capital) with state non-contingent payoff $\Rightarrow$ precautionary savings motive
- Representative firm rents capital K and labor L from households
- Factor prices: interest rate $r(K, L)$ and wages $w(K, L)$ are functions of the aggregate state (K, L)
- K and L are constant in equilibrium

Households ...

- Recursive formulation of the household's problem

$$V(s,a) = \max_{c,a'} \left\{ u(c) + \beta \sum_{s'} p_{s|s'} V(s',a') \right\}$$

$$\text{subject to} \quad c + a' = (1+r)a + ws \qquad c \geq 0 \qquad a' \geq \bar{a}$$

- r and w are taken as given by the household
- s follows a discrete-state Markov process with transition probabilities $p_{s|s'}$
- $\bar{a}$ is a borrowing limit for the household
- The solution to this problem consists of a set of optimal decision rules for consumption and assets: $g_c(s,a)$, $g_{a'}(s,a)$
- There is ex-post heterogeneity in asset holdings a and the employment state s described by the distribution $\Theta(s,a)$

Firms ...

- Wages w and the aggregate capital stock K are determined endogenously
- Production takes place within a representative firm
$$Y = AF(K,L) = AK^{\alpha}N^{1-\alpha} \qquad A \geq 1 \quad \alpha \in [0,1]$$
- Aggregate productivity A is constant
- Individual labor supply is determined by the employment shock s
- Aggregate employment L is fixed and determined by the ergodic probabilities of the Markov chain
- Perfect competition implies
$$\begin{aligned} (1-\alpha)AK^{\alpha}N^{-\alpha} &= w \\ \alpha AK^{\alpha-1}N^{1-\alpha} &= r+\delta \end{aligned}$$

Definition of the Equilibrium

A stationary equilibrium consists of **prices** $r(K,L)$, $w(K,L)$, **value functions** $V(s,a)$, **optimal decision rules** $g_c(s,a)$, $g_{a'}(s,a)$ and the **invariant distribution** $\Theta(s,a)$, such that:

- $V(s,a)$ and $g_c(s,a)$, $g_{a'}(s,a)$ solve the household's optimization problem
- $r(K,L)$ and $w(K,L)$ solve the firm's (static) optimization problem: $r+\delta = F_K(K,L)$, $w = F_L(K,L)$
- $\Theta(s,a)$ is a stationary distribution consistent with the optimal decision rule $g_{a'}(s,a)$ and the Markov process for s
- The aggregate capital stock K is consistent with the stationary distribution $\Theta(s,a)$:

$$K = \sum_S \int_A g_{a'}(s,a) d\Theta(s,a)$$

- How to compute the equilibrium?
- In equilibrium all markets have to be cleared. 3 markets:
  - Y: Goods market / Price p
  - L: Labor market (fixed supply) / Price w
  - K: Capital market / Price r
- Thanks to Walras' Law it suffices to focus on (the last) 2 markets
- Computational approach: Do a fixed-point iteration on r
- Equilibrium r: Consistency between households' supply of, and firms' demand for capital, $K^s = K^d$
- Equilibrium w: Can be backed out from $w = F_L(K, L)$ due to fixed labor supply

The Computational Approach in Words:

1. Start with a guess on the interest rate r^0 and compute how much capital K^s households are willing to supply given r^0
   [To achieve that we need to know (1) how much each (a, s)-type of households supplies: $g_{a'}(a, s)$, and (2) how many households of each (a, s)-type exist: $\Theta(s, a)$]

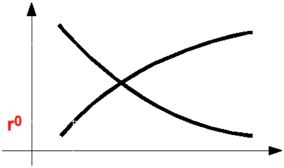
2. Determine for which interest rate r^* firms are willing to purchase the capital supplied by households K^s
   [This can be inferred from the firm's optimality condition $r^* + \delta = F_K(K^s, L)$]

3. Use the implied interest rate r^* to update the initial guess r^0
   [The updating should be done in a conservative way to ensure convergence]

4. The equilibrium is found if $r^* = r^0$

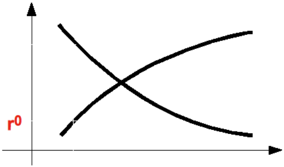
A graphical illustration

r^0



$$r^0 + \delta = F_K(K/L) \rightarrow K/L$$

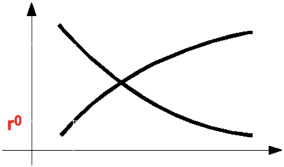
r^0



$$w = F_L(K/L)$$

$$r^0 + \delta = F_K(K/L) \rightarrow K/L$$

r^0

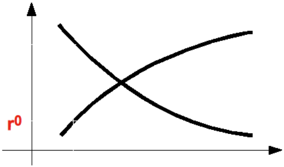


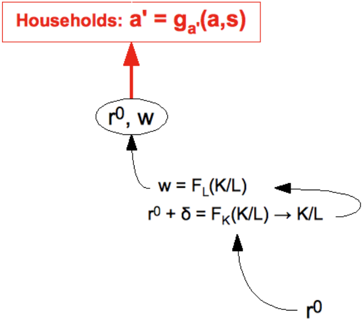
r^0, w

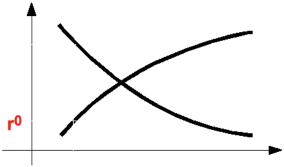
$w = F_L(K/L)$

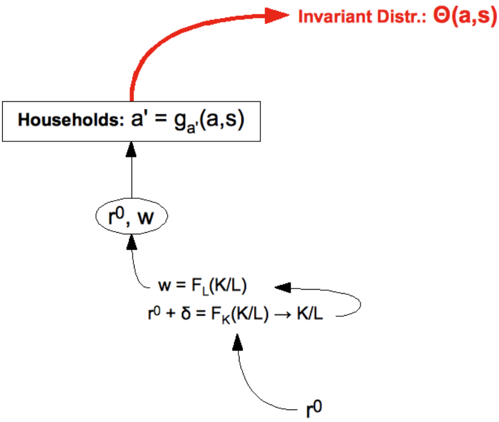
$r^0 + \delta = F_K(K/L) \rightarrow K/L$

r^0




Households: $a' = g_{a'}(a,s)$
r^0, w
$w = F_L(K/L)$
$r^0 + \delta = F_K(K/L) \rightarrow K/L$
r^0


r^0


Invariant Distr.: $\Theta(a,s)$
Households: $a' = g_{a'}(a,s)$
r^0, w
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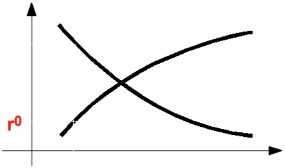
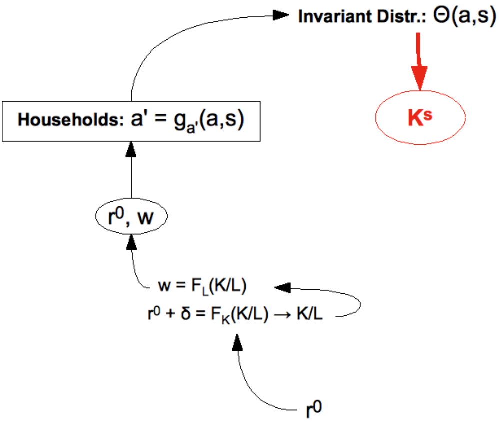
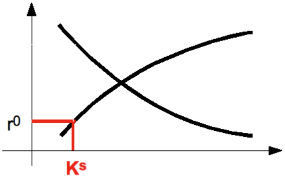
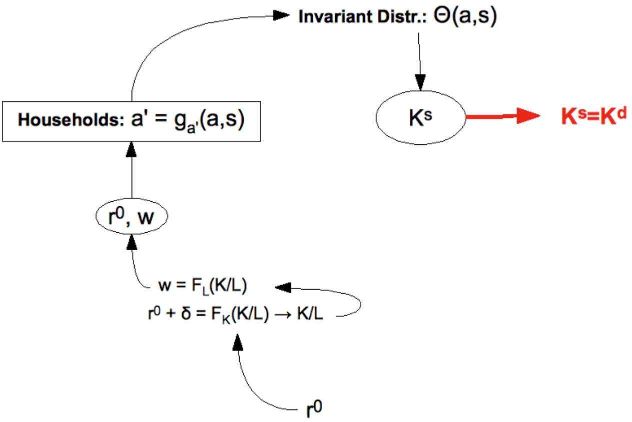
$r^0 + \delta = F_K(K/L) \rightarrow K/L$
r^0


r^0


Invariant Distr.: Θ(a,s)
K^s
Households: a' = g_a'(a,s)
r^0, w
w = F_L(K/L)
r^0 + δ = F_K(K/L) → K/L
r^0


r^0
K^s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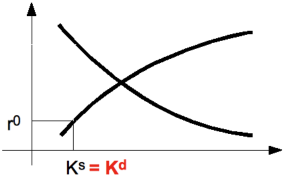
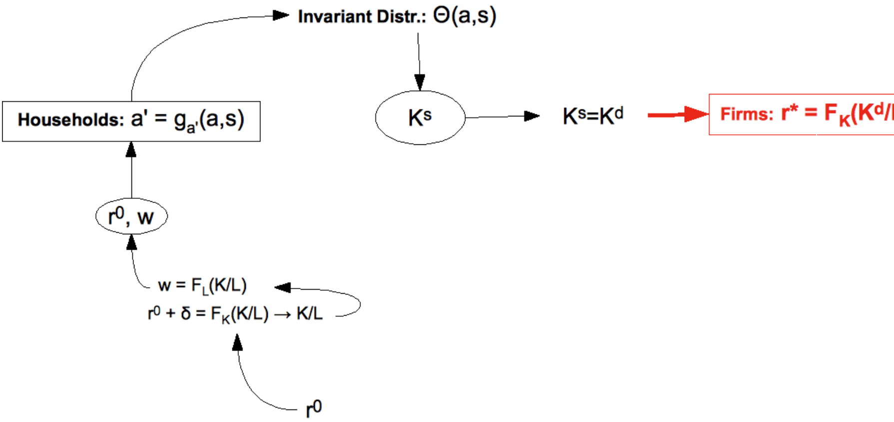
Households: a' = g_a'(a,s)
Ks
Ks=Kd
r0, w
w = FL(K/L)
r0 + δ = FK(K/L) → K/L
r0


r0
Ks = Kd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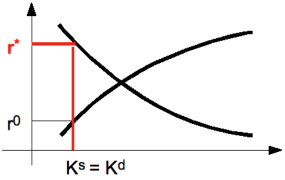
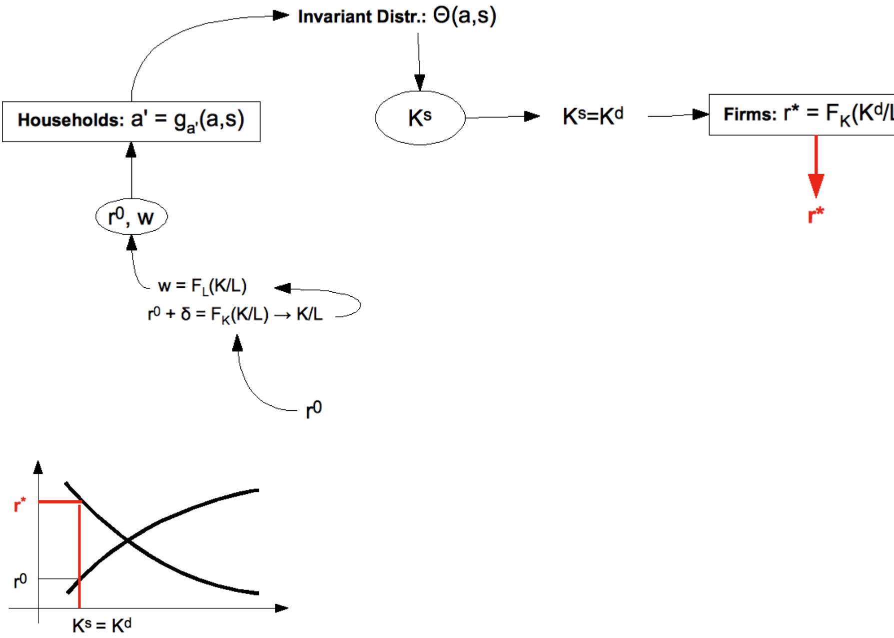
r^0


r^*
r^0
$K^s = K^d$



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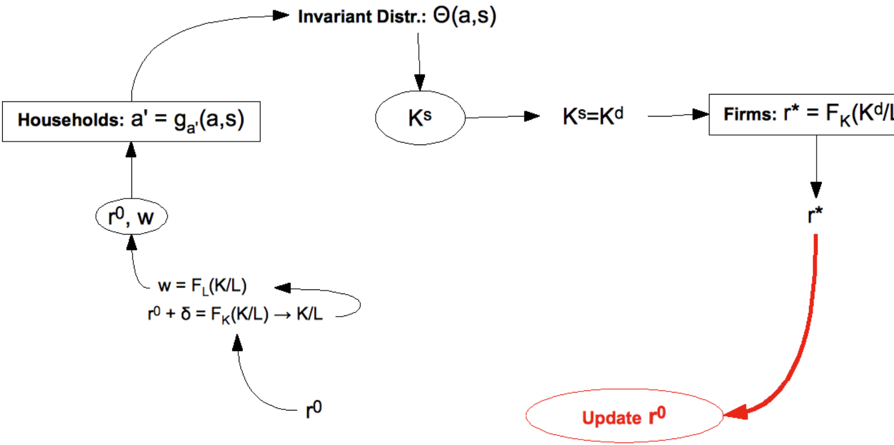
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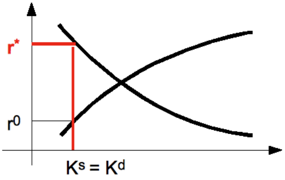
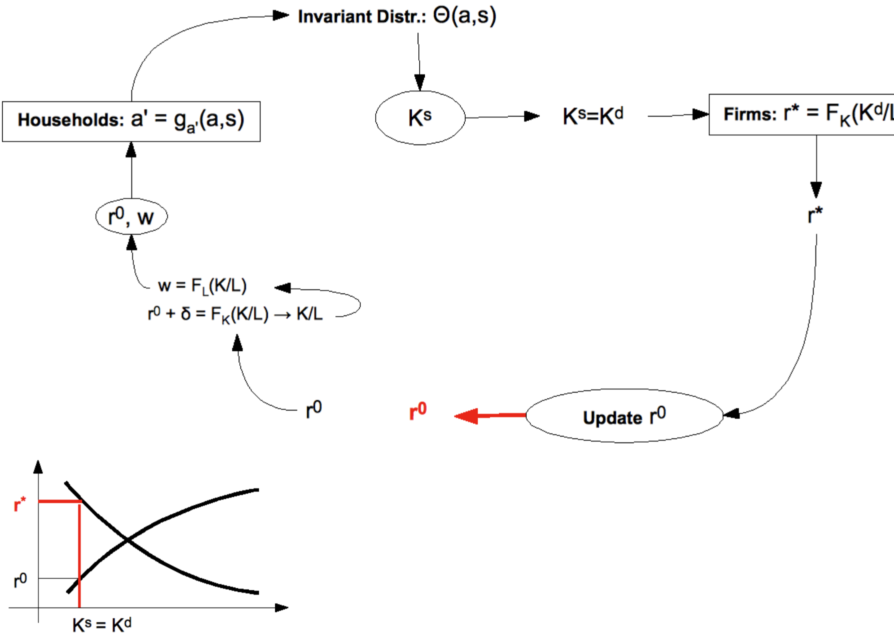
w = F_L(K/L)
r^0 + δ = F_K(K/L) → K/L
r*
r^0
Update r^0


r*
r^0
K^s = K^d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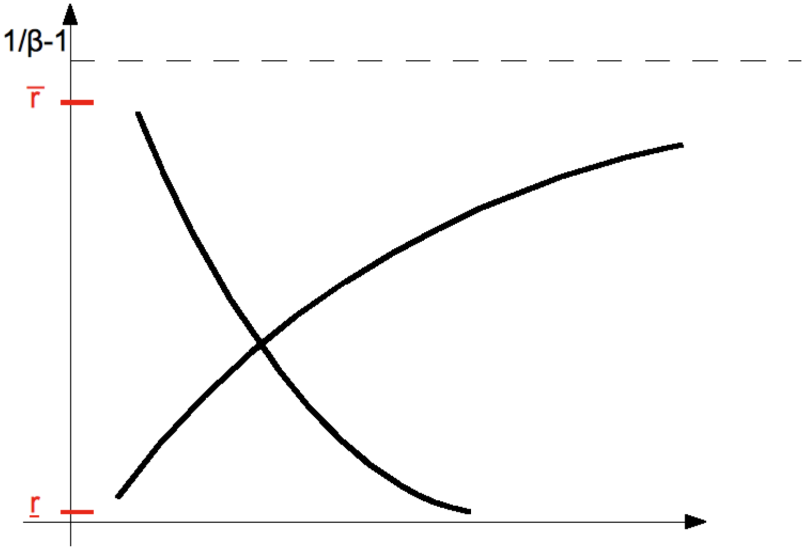
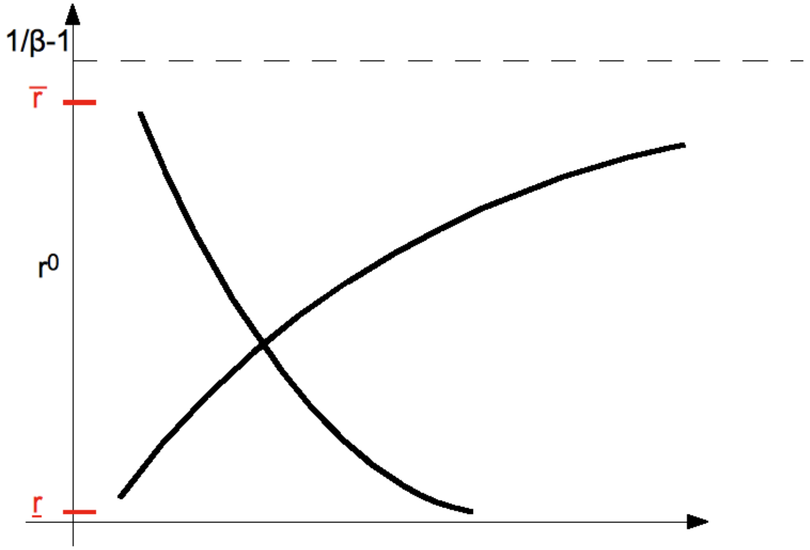
Ks=Kd
Firms: r* = FK(Kd/L
r0, w
r*
w = FL(K/L)
r0 + δ = FK(K/L) → K/L
r0
r0
Update r0
r*
r0
Ks = Kd

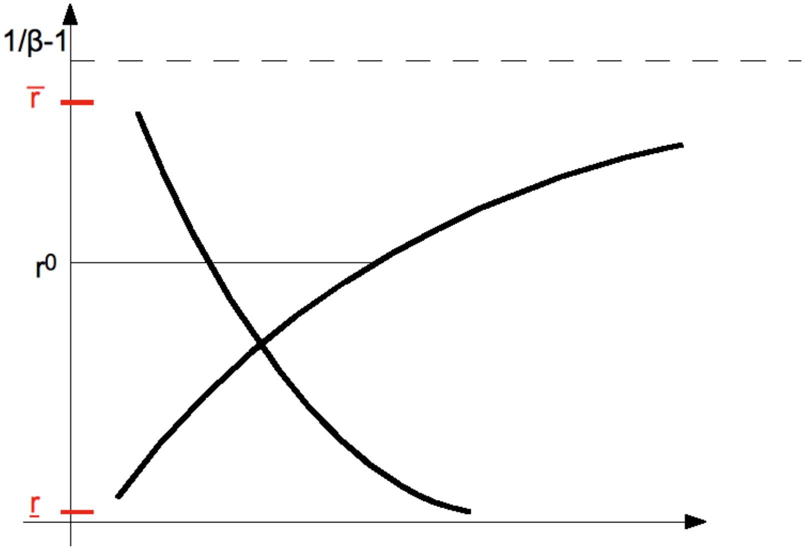
A Note on the updating ...

- Using $r^0 = r^*$ as new guess is inadvisable as it easily leads to divergence
- A good way to do the updating is bisection (aka bracketing)
- Idea:
  1. Determine an upper and a lower bound on the interest rate, denoted by $\underline{r}$, $\bar{r}$
     [For the case at hand we know that $r \in (0, 1/\beta - 1)$. Hence, we choose $\underline{r}$ and $\bar{r}$ accordingly]
  2. Start with $r^0 = (\underline{r} + \bar{r})/2$
  3. If (in the next round) the implied interest rate r^* is above r_0 then replace $\underline{r}$ with r_0
  4. If the implied interest rate r^* is below r_0 replace $\bar{r}$ with r_0
  5. Repeat Steps (2)-(4) until $|r^0 - r^*| < \varepsilon$

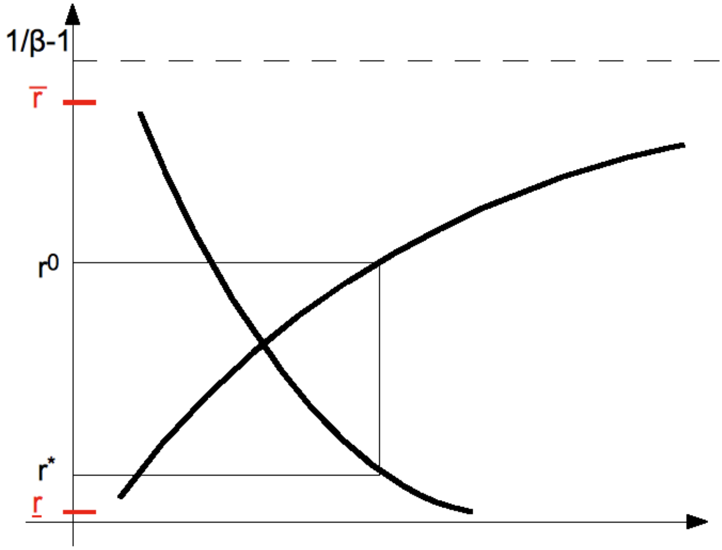

1/β-1
r̄
r̲



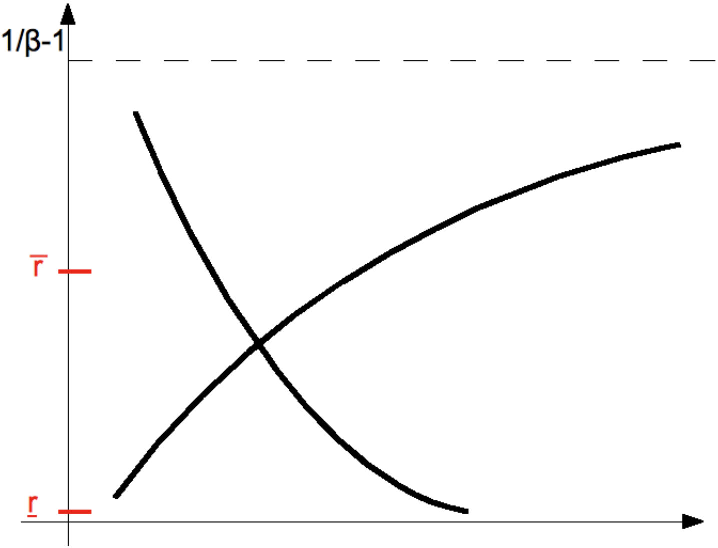
1/β-1
r̄
r⁰
r̲



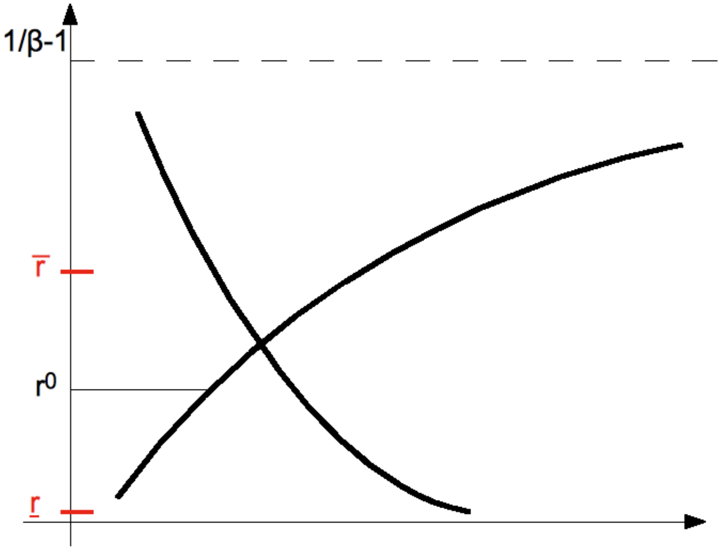
1/β-1
r̄
r⁰
r̲



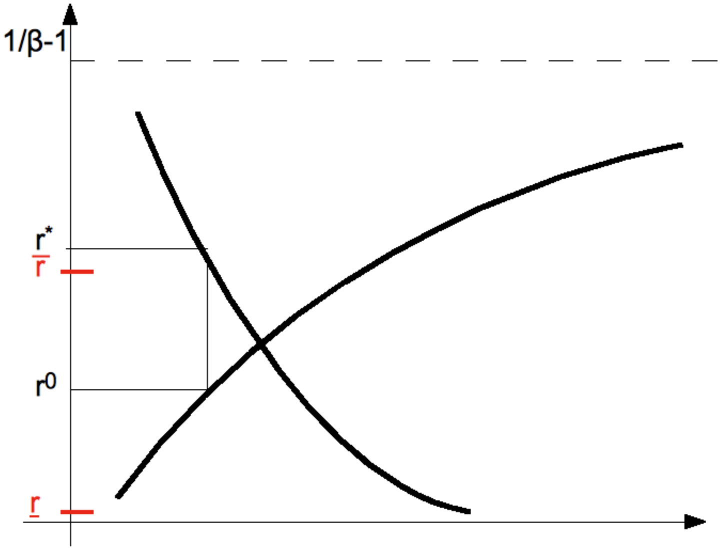
1/β-1
r̄
r0
r*
r



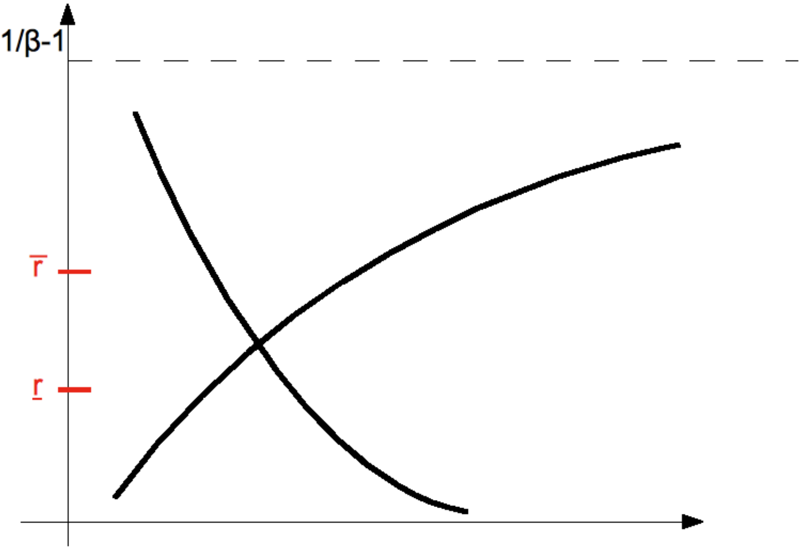
1/β-1
$\bar{r}$
$\underline{r}$

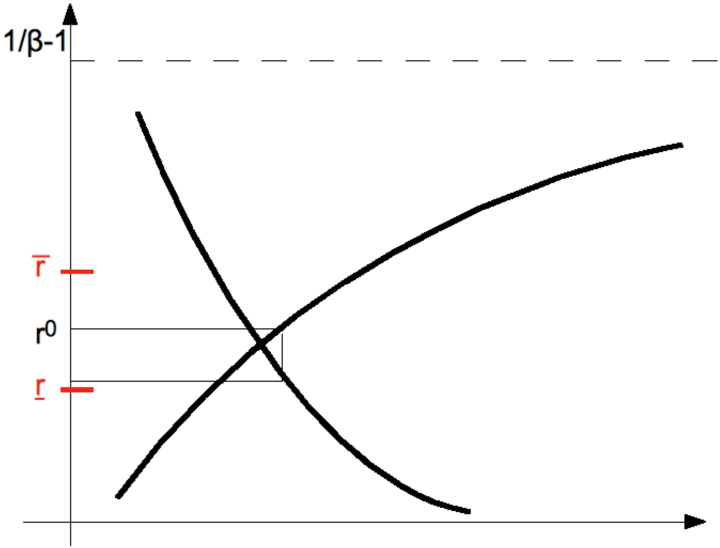


1/β-1
r̄
r⁰
r̲

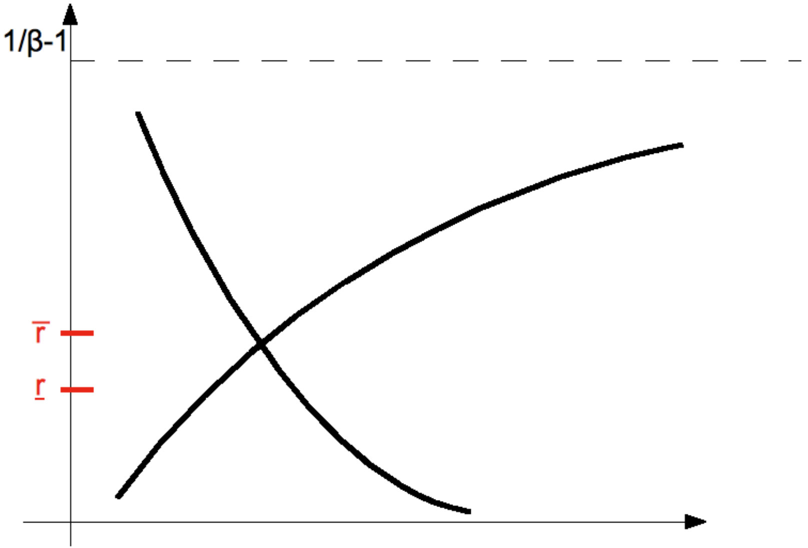


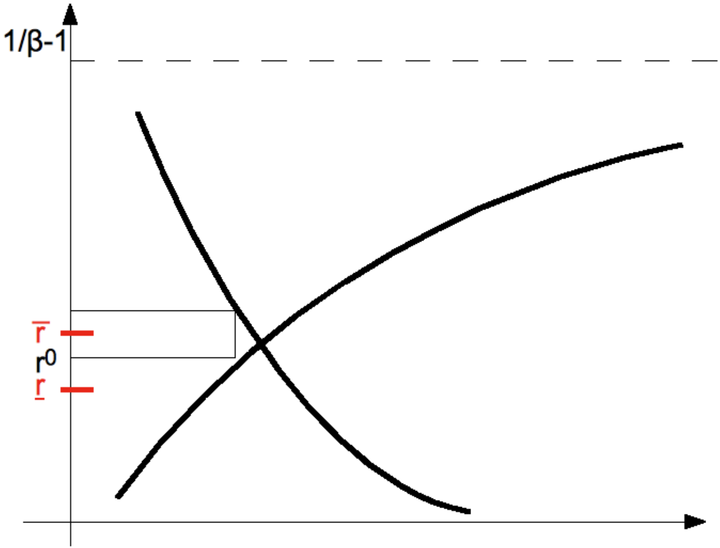
1/β-1
r*
r̄
r0
r̲


1/β-1
r̄
r̲

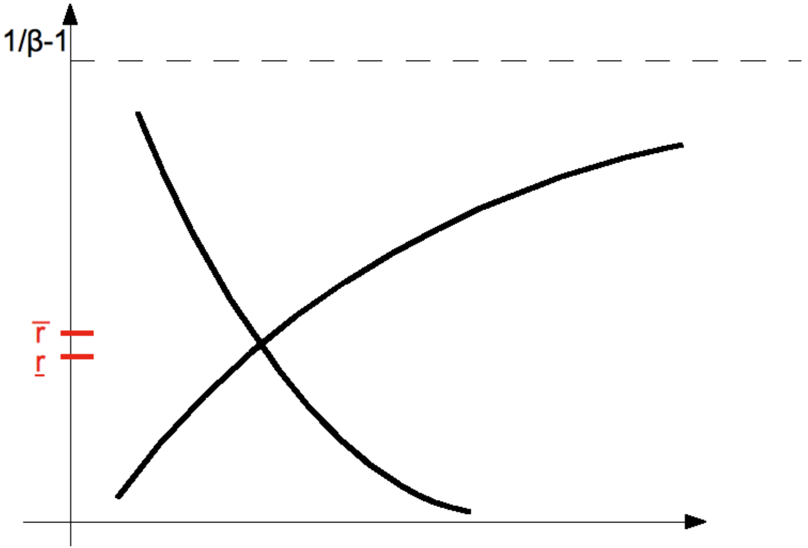


1/β-1
r̄
r0
r̲


1/β-1
$\overline{r}$
$\underline{r}$



1/β-1
r̄
r^0
r̲


1/β-1
r̄
r̲

Structure of the Algorithm

1. Start with r^0
2. Use $r^0 + \delta = F_K(K/L) = \alpha A \left(\frac{K}{L}\right)^{\alpha-1}$ to compute K/L associated with r^0
3. Use K/L and $w = F_L(K/L) = (1-\alpha)A\left(\frac{K}{L}\right)^{1-\alpha}$ to compute the implied wage $w(K/L)$
4. Take (r^0, w) and solve the households problem for $g_{a'}(a,s)$
5. Use $g_{a'}(a,s)$ and the law of motion for s to compute the stationary distribution $\Theta(s,a)$ and the aggregate capital supply K^s
6. With K^s at hand derive the interest rate implied by $r^* + \delta = F_K(K^s/L)$
7. Update r^0 using the information provided by r^*

Next

- Consider incomplete markets economy with both idiosyncratic and aggregate risk?
- Seems a small extension, but it's not!
- Approach by Krusell and Smith