Value at Risk (VaR)
[Tsay (2005), chapter 7]

Var is a single estimate of the amount by which an institution’s position in a risk category could decline due to general market movements during a given holding period.

It can be used by financial institutions to assess their risks or by a regulatory committee to set margin requirements. It’s used to ensure that the financial institutions can still be in business after a catastrophic event.

- View of the financial institutions = maximal loss of a financial position during a given time period for a rare (or extraordinary) event under normal market conditions.

- View of the regulatory committee = minimal loss under extraordinary market circumstances.
Risk of a financial position for the nest $l$ periods:

$$p = \Pr[\Delta V(\ell) \leq \text{VaR}] = F_\ell(\text{VaR})$$

$\Delta V(\ell)$ is the change in value of the assets in the financial position from time $t$ to $t+l$, measured in $\$$ and is a random variable at the time index $t$. $F_\ell(x)$ is the cumulative distribution function (CDF).

Since the holder of a long financial position suffers a loss when $\Delta V(\ell) < 0$, the VaR typically assumes a negative value when $p$ is small, i.e., a loss.

So, VaR is concerned with tail behavior of the CDF $F_\ell(x)$.

Let $x_p = \inf\{x | F_\ell(x) \geq p\}$ the $p$th quantile of $F_\ell(x)$

$$\text{VaR} = x_p.$$
**Practical applications:**

1. The probability of interest $p$, such as $p = 0.01$ or $p = 0.05$.
2. The time horizon $l$. It might be set by a regulatory committee, such as 1 day or 10 days.
3. The frequency of the data, which might not be the same as the time horizon $l$. Daily observations are often used.
4. The CDF $F_\ell(x)$ or its quantiles.
5. The amount of the financial position or the mark-to-market value of the portfolio.

Here in econometrics, it's important to model the CDF $F_\ell(x)$. 
Notes:

- Use log returns in data analysis. The VaR calculated from the quantile of the distribution of $r_{t+1}$ given information available at time $t$ is therefore in percentage. So, the $\$ amount of VaR is the cash value of the financial position times the VaR of the log return series: VaR = Value * VaR(of log returns). Or the approximation VaR = Value * [exp(VaR of log returns) - 1].

- VaR is a prediction concerning possible loss of a portfolio in a given time horizon and should be computed using the *predictive distribution* of future returns of the financial position.
RiskMetrics [J.P. Morgan]

\[ r_t | F_{t-1} \sim N(\mu_t, \sigma_t^2) \]

\[ \mu_t = 0, \quad \sigma_t^2 = \alpha \sigma_{t-1}^2 + (1 - \alpha) r_{t-1}^2, \quad 1 > \alpha > 0 \]

\[ p_t = \ln(P_t) \]

\[ p_t - p_{t-1} = a_t, \quad a_t = \sigma_t \epsilon_t \] is an IGARCH(1,1) process without drift

The value of \( \alpha \) is often in the interval (0.9, 1) with a typical value of 0.94.
If you use a $k$-horizon return, $r_t[k] = r_{t+1} + \cdots + r_{t+k-1} + r_{t+k}$, and under the special IGARCH(1,1), the conditional distribution $r_t[k] | F_t$ is normal with mean zero and variance $\sigma_t^2[k]$. Using independence assumption of $\epsilon_t$,

$$\sigma_t^2[k] = \text{Var}(r_t[k] | F_t) = \sum_{i=1}^{k} \text{Var}(a_{t+i} | F_t)$$

$$\text{Var}(a_{t+i} | F_t) = E(\sigma_{t+i}^2 | F_t)$$
can be obtained recursively.

Using $r_{t-1} = a_{t-1} = \sigma_{t-1}\epsilon_{t-1}$ we have

$$\sigma_t^2 = \sigma_{t-1}^2 + (1 - \alpha)\sigma_{t-1}^2(\epsilon_{t-1}^2 - 1)$$
for all $t$. 
In particular, we have

$$\sigma_{t+i}^2 = \sigma_{t+i-1}^2 + (1 - \alpha)\sigma_{t+i-1}^2(\epsilon_{t+i-1}^2 - 1) \quad \text{for} \quad i = 2, \ldots, k.$$ 

Since $E(\epsilon_{t+i-1}^2 - 1|F_t) = 0$ for $i \geq 2$, the prior equation shows that

$$E(\sigma_{t+i}^2|F_t) = E(\sigma_{t+i-1}^2|F_t) \quad \text{for} \quad i = 2, \ldots, k.$$ 

For the 1-step ahead volatility forecast: $\sigma_{t+1}^2 = \alpha \sigma_t^2 + (1 - \alpha)r_t^2$.

- $\text{Var}(r_{t+i}|F_t) = \sigma_{t+i}^2$ for $i \geq 1$.
- $\sigma_t^2[k] = k \sigma_{t+1}^2$
- $r_t[k]|F_t \sim N(0, k \sigma_{t+1}^2)$

So, under the special IGARCH(1,1) defined by the riskmetrics the conditional variance of $r_t[k]$ is proportional to the time horizon $k$. The conditional standard deviation of a $k$-period horizon log return is the $\sqrt{k} \sigma_{t+1}$.
Suppose that the financial position is a long position so that loss occurs when there is a big price drop (i.e., a large negative return). If the probability is set to 5%, then RiskMetrics uses $1.65\sigma_{t+1}$ to measure the risk of the portfolio; that is, it uses the one-sided 5% quantile of a normal distribution with mean zero and standard deviation $\sigma_{t+1}$. The actual 5% quantile is $-1.65\sigma_{t+1}$, but the negative sign is ignored with the understanding that it signifies a loss. Consequently, if the standard deviation is measured in percentage, then the daily VaR of the portfolio under RiskMetrics is

$$\text{VaR} = \text{Amount of position} \times 1.65\sigma_{t+1},$$

and that of a $k$-day horizon is $\text{VaR}(k) = \text{Amount of position} \times 1.65\sqrt{k}\sigma_{t+1}$, where the argument $(k)$ of VaR is used to denote the time horizon. Consequently, under RiskMetrics, we have

$$\text{VaR}(k) = \sqrt{k} \times \text{VaR}.$$

This is referred to as the \textit{square root of time rule} in VaR calculation under RiskMetrics.
See examples 7.1 and 7.2, chapter 7, Tsay (2005).

Advantages of RiskMetrics:

- Simplicity
- Easy to understand and apply
- Makes risk more transparent in the financial markets
- The square root of time rules

Disadvantages of RiskMetrics:

- As security returns tend to have heavy tails (or fat tails), the normality assumption used often results in underestimation of VaR.
- Is either the zero mean assumption or the special IGARCH(1,1) model assumption of the log returns fails, then the square root of time rule is invalid.
\[ r_t = \mu + \alpha_t, \quad \alpha_t = \sigma_t \epsilon_t, \quad \mu \neq 0, \]
\[ \sigma_t^2 = \alpha \sigma_{t-1}^2 + (1 - \alpha) \alpha_t^2, \]

where \( \{\epsilon_t\} \) is a standard Gaussian white noise series. The assumption that \( \mu \neq 0 \) holds for returns of many heavily traded stocks on the NYSE; see Chapter 1. For this simple model, the distribution of \( r_{t+1} \) given \( F_t \) is \( N(\mu, \sigma_{t+1}^2) \). The 5% quantile used to calculate the 1-period horizon VaR becomes \( \mu - 1.65 \sigma_{t+1}. \) For a \( k \)-period horizon, the distribution of \( r_t[k] \) given \( F_t \) is \( N(k \mu, k \sigma_{t+1}^2) \), where as before \( r_t[k] = r_{t+1} + \cdots + r_{t+k}. \) The 5% quantile used in the \( k \)-period horizon VaR calculation is \( k \mu - 1.65 \sqrt{k} \sigma_{t+1} = \sqrt{k} \left( \sqrt{k} \mu - 1.65 \sigma_{t+1} \right). \) Consequently, VaR(k) \( \neq \sqrt{k} \times \) VaR when the mean return is not zero. It is also easy to show that the rule fails when the volatility model of the return is not an \( \text{IGARCH}(1,1) \) model without drift.
**Multiple positions**: When the investor holds multiple positions and needs to compute the overall VaR of the positions.

Consider the case of two positions. Let \( \text{VaR}_1 \) and \( \text{VaR}_2 \) be the VaR for the two positions and \( \rho_{12} \) be the cross-correlation coefficient between the two returns,

\[
\rho_{12} = \frac{\text{Cov}(r_{1t}, r_{2t})}{\sqrt{\text{Var}(r_{1t})} \sqrt{\text{Var}(r_{2t})}}
\]

\[
\text{VaR} = \sqrt{\text{VaR}_1^2 + \text{VaR}_2^2 + 2\rho_{12} \text{VaR}_1 \text{VaR}_2}.
\]

Generalizing,

\[
\text{VaR} = \sqrt{\sum_{i=1}^{m} \text{VaR}_i^2 + 2 \sum_{i<j} \rho_{ij} \text{VaR}_i \text{VaR}_j},
\]

where \( \rho_{ij} \) is the cross-correlation coefficient between returns of the \( i \)th and \( j \)th instruments and \( \text{VaR}_i \) is the VaR of the \( i \)th instrument.
Econometric approach to Var Calculation

An ARMA(p,q)-GARCH(u,v) specification:

Consider the log return $r_t$ of an asset. A general time series model for $r_t$ can be written as

$$r_t = \phi_0 + \sum_{i=1}^{p} \phi_i r_{t-i} + a_t - \sum_{j=1}^{q} \theta_j a_{t-j}, \quad (7.5)$$

$$a_t = \sigma_t \epsilon_t,$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{u} \alpha_i a_{t-i}^2 + \sum_{j=1}^{v} \beta_j \sigma_{t-j}^2. \quad (7.6)$$
The 1-step ahead forecasts of the conditional mean and conditional variance of $r_t$:

$$\hat{r}_t(1) = \phi_0 + \sum_{i=1}^{p} \phi_i r_{t+1-i} - \sum_{j=1}^{q} \theta_j a_{t+1-j},$$

$$\hat{\sigma}_t^2(1) = \alpha_0 + \sum_{i=1}^{u} \alpha_i a_{t+1-i}^2 + \sum_{j=1}^{v} \beta_j \sigma_{t+1-j}^2.$$ 

If one further assumes that $\epsilon_t$ is Gaussian, then the conditional distribution of $r_{t+1}$ given the information available at time $t$ is $N[\hat{r}_t(1), \hat{\sigma}_t^2(1)]$. Quantiles of this conditional distribution can easily be obtained for VaR calculation. For example, the 5% quantile is $\hat{r}_t(1) - 1.65\hat{\sigma}_t(1)$. If one assumes that $\epsilon_t$ is a standardized Student-$t$ distribution with $v$ degrees of freedom, then the quantile is $\hat{r}_t(1) - t^*_v(p)\hat{\sigma}_t(1)$, where $t^*_v(p)$ is the $p$th quantile of a standardized Student-$t$ distribution with $v$ degrees of freedom.
The relationship between quantiles of a Student-\( t \) distribution with \( v \) degrees of freedom, denoted by \( t_v \), and those of its standardized distribution, denoted by \( t^*_v \), is

\[
p = \Pr (t_v \leq q) = \Pr \left( \frac{t_v}{\sqrt{v/(v-2)}} \leq \frac{q}{\sqrt{v/(v-2)}} \right) = \Pr \left( t^*_v \leq \frac{q}{\sqrt{v/(v-2)}} \right),
\]

where \( v > 2 \). That is, if \( q \) is the \( p \)th quantile of a Student-\( t \) distribution with \( v \) degrees of freedom, then \( q/\sqrt{v/(v-2)} \) is the \( p \)th quantile of a standardized Student-\( t \) distribution with \( v \) degrees of freedom. Therefore, if \( \varepsilon_t \) of the GARCH model in Eq. (7.6) is a standardized Student-\( t \) distribution with \( v \) degrees of freedom and the probability is \( p \), then the quantile used to calculate the 1-period horizon VaR at time index \( t \) is

\[
\hat{r}_t(1) + \frac{t_v(p)\hat{\sigma}_t(1)}{\sqrt{v/(v-2)}},
\]

where \( t_v(p) \) is the \( p \)th quantile of a Student-\( t \) distribution with \( v \) degrees of freedom and assumes a negative value for a small \( p \).

(See example 7.3)
Now, we want to compute the $k$-horizon VaR of an asset at time $h$.

Let $r_h[k] = r_{h+1} + \cdots + r_{h+k}$ the $k$-period log return at the forecast origin $h$.

Suppose an ARMA(p,q)-GARCH(u,v) model as defined in (7.5) and (7.6).

Then, the expected return is $\hat{r}_h[k] = r_h(1) + \cdots + r_h(k)$, where $r_h(l)$ is the $l$-step ahead forecast of the return at the forecast origin $h$.

Using the MA($\infty$) representation of a ARMA(p,q), we can write the $l$-step ahead forecast error at the forecast origin $h$ as

$$r_t = \mu + a_t + \psi_1a_{t-1} + \psi_2a_{t-2} + \cdots$$

$$e_h(l) = r_{h+l} - r_h(l) = a_{h+l} + \psi_1a_{h+l-1} + \cdots + \psi_{l-1}a_{h+1}$$
Then, the forecast error of the expected $k$-period return $\hat{r}_h[k]$ is the sum of 1-step to $k$-step forecast error of $r_t$ at the forecast origin $h$

$$e_h[k] = e_h(1) + e_h(2) + \cdots + e_h(k)$$

$$= a_{h+1} + (a_{h+2} + \psi_1 a_{h+1}) + \cdots + \sum_{i=0}^{k-1} \psi_i a_{h+k-i}$$

$$= a_{h+k} + (1 + \psi_1)a_{h+k-1} + \cdots + \left(\sum_{i=0}^{k-1} \psi_i \right) a_{h+1},$$

where $\psi_0 = 1$. 
Therefore, the volatility forecast of the \( k \)-period return at the forecast origin \( h \) is the conditional variance of \( e_h[k] \). Using the independent assumption of \( \epsilon_{t+i} \), we have

\[
V_h(e_h[k]) = V_h(a_{h+k}) + (1 + \psi_1)^2 V_h(a_{h+k-1}) + \cdots + \left( \sum_{i=0}^{k-1} \psi_i \right)^2 V_h(a_{h+1})
\]

\[
= \sigma_h^2(k) + (1 + \psi_1)^2 \sigma_h^2(k - 1) + \cdots + \left( \sum_{i=0}^{k-1} \psi_i \right)^2 \sigma_h^2(1), \quad (7.8)
\]

where \( V_h(z) \) denotes the conditional variance of \( z \) given \( F_h \) and \( \sigma_h^2(\ell) \) is the \( \ell \)-step ahead volatility forecast at the forecast origin \( h \). If the volatility model is the GARCH model in Eq. (7.6), then these volatility forecasts can be obtained recursively by the methods discussed in Chapter 3.
Example:

consider the special time series model

\[ r_t = \mu + a_t, \quad a_t = \sigma_t \epsilon_t, \]

\[ \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2. \]

Then we have \( \psi_i = 0 \) for all \( i > 0 \). The point forecast of the \( k \)-period return at the forecast origin \( h \) is \( \hat{r}_h[k] = k\mu \) and the associated forecast error is

\[ e_h[k] = a_{h+k} + a_{h+k-1} + \cdots + a_{h+1}. \]

Consequently, the volatility forecast for the \( k \)-period return at the forecast origin \( h \) is

\[ \text{Var}(e_h[k] | F_h) = \sum_{\ell=1}^{k} \sigma_h^2(\ell). \]
Using the forecasting method of GARCH(1,1) models

\[
\sigma^2_h(1) = \alpha_0 + \alpha_1 a^2_h + \beta_1 \sigma^2_h,
\]
\[
\sigma^2_h(\ell) = \alpha_0 + (\alpha_1 + \beta_1) \sigma^2_h(\ell - 1), \quad \ell = 2, \ldots, k.
\]

we obtain that for the case of \(\psi_i = 0\) for \(i > 0\),

\[
\text{Var}(e_h[k]|F_h) = \frac{\alpha_0}{1 - \phi} \left[ k - \frac{1 - \phi^k}{1 - \phi} \right] + \frac{1 - \phi^k}{1 - \phi} \sigma^2_h(1), \quad (7.10)
\]

where \(\phi = \alpha_1 + \beta_1 < 1\). If \(\psi_i \neq 0\) for some \(i > 0\), then one should use the general formula of \(\text{Var}(e_h[k]|F_h)\) in Eq. (7.8). If \(\epsilon_t\) is Gaussian, then the conditional distribution of \(r_h[k]\) given \(F_h\) is normal with mean \(k \mu\) and variance \(\text{Var}(e_h[k]|F_h)\). The quantiles needed in VaR calculation are readily available. If the conditional distribution of \(a_t\) is not Gaussian (e.g., a Student-\(t\) or generalized error distribution), simulation can be used to obtain the multiperiod VaR.

Conclusion: the square root of time rule used by RiskMetrics holds only for the special white noise IGARCH(1,1).

(See Example 7.3 (continued), pg 298)
Quantile Estimation

It provides a nonparametric approach to VaR calculation. It makes no specific distributional assumption on the return of a portfolio except that the distribution continues to hold within the prediction period.

2 methods:

- empirical quantile directly
- quantile regression
1) **Empirical quantile**

Assume that the distribution of return in the prediction period is the same as that in the sample period. So, one can use the empirical quantile of the return $r_t$ to calculate VaR.

First, ordering the sample returns,

$$ r_{(1)} \leq r_{(2)} \leq \cdots \leq r_{(n)} $$

to denote the arrangement and refer to $r_{(i)}$ as the $i$th order statistic of the sample. In particular, $r_{(1)}$ is the sample minimum and $r_{(n)}$ the sample maximum.

Assume that the returns are iid with continuous distributions e $r_{(l)}$ the order static, where $l = np$ with $0 < p < 1$. 
Cox and Hinkley (1974):

Result. Let \( x_p \) be the \( p \)th quantile of \( F(x) \), that is, \( x_p = F^{-1}(p) \). Assume that the pdf \( f(x) \) is not zero at \( x_p \) (i.e., \( f(x_p) \neq 0 \)). Then the order statistic \( r(\ell) \) is asymptotically normal with mean \( x_p \) and variance \( p(1 - p)/[n f^2(x_p)] \). That is,

\[
r(\ell) \sim N \left[ x_p, \frac{p(1 - p)}{nf(x_p)^2} \right], \quad \ell = np.
\] (7.11)

One can use \( r(\ell) \) to estimate the quantile \( x_p \), where \( l = np \).

Problem: \( np \) can’t be a positive integer.
Solution:
1. use a simple interpolation to obtain quantile estimates.
2. For noninteger \( np \), let \( l_1 \) and \( l_2 \) be the two neighboring positive integers such that \( l_1 < np < l_2 \).
3. Define \( p_i = l_i/n \).
4. \( r_{(li)} \) is a consistent estimate of the quantile \( x_{pi} \).
Therefore, the quantile \( x_p \) can be estimated by

\[
\hat{x}_p = \frac{p_2 - p}{p_2 - p_1} r(\ell_1) + \frac{p - p_1}{p_2 - p_1} r(\ell_2).
\]

(see examples 7.4 and 7.5, pg 299)