Chapter 1

1.1 Let \( U(.) \) be a well-behaved utility function that represents the preferences of an agent. Let \( f(U) \) be a monotone transformation of the original utility function \( U \). Why is an increasing function \( g = f(U( )) \) also a utility function representing the same preferences as \( U( ) \)?

1.2 Assume a well-behaved utility function. The maximizing choice for a consumer is preserved under increasing monotone transformations. Show this using the (first-order) optimality condition (MRS = price ratio) for a typical consumer and give an economic interpretation.

1.3 The MRS (marginal rate of substitution) is not constant in general. Give an economic interpretation. When will the MRS be constant? Give an example (a utility function over two goods) and compute the MRS. Is this of interest for us? Why? What undesirable properties does this particular utility function (and/or the underlying preferences) exhibit? In a two goods/two agents setting, what about the Pareto set when indifference curves are linear for both agents?

1.4 Consider a two goods/two agents pure exchange economy where agents’ utility functions are of the form: \( U(c^j_1, c^j_2) = (c^j_1)^{\alpha} \cdot (c^j_2)^{1-\alpha} \), \( j = 1, 2 \) with \( \alpha = 0.5 \). Initial endowments are \( e^1_1 = 6, e^1_2 = 4, e^2_1 = 14, e^2_2 = 16 \) (superscripts represent agents).

a. Compute the original utility level for both agents. Compute the original MRS and give the (first-order) optimality conditions. What is your conclusion?

b. Describe the Pareto set (the set of Pareto optima).

c. Assume that there exists a competitive market for each good. What is the equilibrium allocation? What are equilibrium prices? Comment. What are the utility levels and MRS after trading? What do you conclude?

d. Assume that the utility functions are now given by \( U(c^j_1, c^j_2) = \ln((c^j_1)^{\alpha} \cdot (c^j_2)^{1-\alpha}), \alpha \in [0,1], j = 1, 2 \). What is the optimality relation for the typical consumer? How does it compare with that obtained in part a? Compute the original utility levels and MRS. What can you say about them as compared to those obtained in part a?

e. Same setting as in part d: What is the equilibrium allocation if agents can trade the goods on competitive markets? What are the equilibrium prices? What are the after-trade utility levels and MRS? How do they compare with those obtained under a? What do you conclude?
1.5 Figure 1.1 shows an initial endowment point \( W \), the budget line, and the optimal choices for two agents. In what direction will the budget line move? Why?

**Figure 1.1 The Edgeworth-Bowley Box: Initial Endowment \( W \)**

Chapter 2

2.1 Utility function: Under certainty, any increasing monotone transformation of a utility function is also a utility function representing the same preferences. Under uncertainty, we must restrict this statement to linear transformations if we are to keep the same preference representation. Give a mathematical as well as an economic interpretation for this.

Check it with this example. Assume an initial utility function attributes the following values to 3 perspectives:

\[
\begin{align*}
B & \quad u(B) = 100 \\
M & \quad u(M) = 10 \\
P & \quad u(P) = 50
\end{align*}
\]

a. Check that with this initial utility function, the lottery \( L = (B, M, 0.50) \succ P \).

b. The proposed transformations are \( f(x) = a + bx, a \geq 0, b > 0 \) and \( g(x) = \ln(x) \). Check that under \( f \), \( L \succ P \), but that under \( g \), \( P \succ L \).
2.2  Lotteries: Discuss the equivalence between \((x, z, \pi)\) and \((x, y, \pi + (1-\pi)\tau)\) when \(z = (x, y, \tau)\). Can you think of circumstances under which they would not be viewed as equal?

2.3  Inter-temporal consumption: Consider a two-date (one-period) economy and an agent with utility function over consumption:

\[
U(c) = \frac{c^{1-\gamma}}{1-\gamma}
\]

at each period. Define the inter-temporal utility function as \(V(c_1, c_2) = U(c_1) + U(c_2)\). Show (try it mathematically) that the agent will always prefer a smooth consumption stream to a more variable one with the same mean, that is,

\[
U(\bar{c}) + U(\bar{c}) > U(c_1) + U(c_2)
\]

if \(\bar{c} = \frac{c_1 + c_2}{2}\).

Chapter 3

3.1  Risk aversion: Consider the following utility functions (defined over wealth \(Y\)):

\[
\begin{align*}
(1) & \quad U(Y) = -\frac{1}{Y} \\
(2) & \quad U(Y) = \ln Y \\
(3) & \quad U(Y) = -Y^{-\gamma} \\
(4) & \quad U(Y) = -\exp(-\gamma Y) \\
(5) & \quad U(Y) = \frac{Y^{\gamma}}{\gamma} \\
(6) & \quad U(Y) = \alpha Y - \beta Y^2
\end{align*}
\]

a. Check that they are well behaved (\(U' > 0, U'' < 0\)) or state restrictions on the parameters so that they are (utility functions (1)—(5)). For utility function (6), take positive \(\alpha\) and \(\beta\), and give the range of wealth over which the utility function is well behaved.

b. Compute the absolute and relative risk aversion coefficients.

c. What is the effect of the parameter \(\gamma\) (when relevant)?

d. Classify the functions as increasing/decreasing risk aversion utility functions (both absolute and relative).
3.2 Certainty equivalent:

\[
U(Y) = \begin{cases} 
\frac{1}{Y} & \text{for utility function (1)}, \\
\ln Y & \text{for utility function (2)}, \\
\frac{Y^\gamma}{\gamma} & \text{for utility function (3)}.
\end{cases}
\]

Consider the lottery \( L_1 = (50,000; 10,000; 0.50) \). Determine the lottery \( L_2 = (x; 0; 1) \) that makes an agent indifferent to lottery \( L_1 \) with utility functions (1), (2), and (3) as defined. For utility function (3), use \( \gamma = \{0.25, 0.75\} \). What is the effect of changing the value of \( \gamma \)? Comment on your results using the notions of risk aversion and certainty equivalent.

3.3 Risk premium: A businesswoman runs a firm worth CHF 100,000. She faces some risk of having a fire that would reduce her net worth according to the following three states, \( i = 1, 2, 3 \), each with probability \( \pi(i) \) (Scenario A).

<table>
<thead>
<tr>
<th>Scenario A</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>State</td>
<td>Worth</td>
<td>( \pi(i) )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.01</td>
</tr>
<tr>
<td>2</td>
<td>50,000</td>
<td>0.04</td>
</tr>
<tr>
<td>3</td>
<td>100,000</td>
<td>0.95</td>
</tr>
</tbody>
</table>

Of course, in state 3, nothing detrimental happens, and her business retains its value of CHF 100,000.

a. What is the maximum amount she will pay for insurance if she has a logarithmic utility function over final wealth? (Note: The insurance pays CHF 99,999 in the first case; CHF 50,000 in the second; and nothing in the third.)

b. Do the calculations with the following alternative probabilities:

<table>
<thead>
<tr>
<th>Scenario B</th>
<th>Scenario C</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi(1) )</td>
<td>0.01</td>
</tr>
<tr>
<td>( \pi(2) )</td>
<td>0.05</td>
</tr>
<tr>
<td>( \pi(3) )</td>
<td>0.94</td>
</tr>
</tbody>
</table>

Is the outcome (the comparative change in the premium) a surprise? Why?
3.4 Consider two investments A and B. Suppose that their returns, \( \tilde{r}_A \) and \( \tilde{r}_B \), are such that \( \tilde{r}_A = \tilde{r}_B + \vartheta \), where \( \vartheta \) is a non-negative random variable. Show that A FSD B.

3.5 A four-part question:

a. Explain intuitively the concept of first-order stochastic dominance.

b. Explain intuitively the concept of second-order stochastic dominance.

c. Explain intuitively the mean variance criterion.

d. You are offered the following two investment opportunities.

<table>
<thead>
<tr>
<th>Investment A</th>
<th>Investment B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payoff</td>
<td>Payoff</td>
</tr>
<tr>
<td>Probability</td>
<td>Probability</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>0.25</td>
<td>0.333</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>0.5</td>
<td>0.333</td>
</tr>
<tr>
<td>9</td>
<td>8</td>
</tr>
<tr>
<td>0.25</td>
<td>0.333</td>
</tr>
</tbody>
</table>

Apply concepts a through d. Illustrate the comparison with a graph.

3.6 An individual (operating in perfect capital markets) with a zero initial wealth, and the utility function \( U(Y) = Y^{1/2} \) is confronted with the gamble (16, 4; 1/2).

a. What is his certainty equivalent for this gamble?

b. If there were an insurance policy that, together with the original gamble, guaranteed him the expected payoff of the gamble, what is the maximum premium he would be willing to pay for it?

c. What is the minimum required increase (the probability premium) in the probability of the high payoff state so that he will not be willing to pay any premium for such an insurance policy? (Note that the insurance policy still pays the expected payoff of the unmodified gamble)

d. Now assume that he is confronted with the gamble (36, 16; 1/2). Calculate the certainty equivalent, the insurance premium, and the probability premium for this case as well. Explain what is going on, and why?

3.7 Refer to Table 3.2. Suppose the return data for investment 3 was as follows:
Consider two investments with the following characteristics:

<table>
<thead>
<tr>
<th>States</th>
<th>( \theta_1 )</th>
<th>( \theta_2 )</th>
<th>( \theta_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi )</td>
<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
</tr>
<tr>
<td>( z )</td>
<td>10</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>( y )</td>
<td>0</td>
<td>10</td>
<td>20</td>
</tr>
</tbody>
</table>

a. Is there state by state dominance between these two investments?
b. Is there FSD between these two investments?

- **Chapter 4**

4.1 Consider the portfolio choice problem of a risk-averse individual with a strictly increasing utility function. There is a single risky asset, and a risk-free asset. Formulate an investor's choice problem and comment on the first-order conditions. What is the minimum risk premium required to induce the individual to invest all his wealth in the risky asset? (Find your answer in terms of his initial wealth, absolute risk aversion coefficient, and other relevant parameters.)

Hint: Take a Taylor series expansion of the utility of next period's random wealth.

4.2 Portfolio choice (with expected utility): An agent has \( Y = 1 \) to invest. On the market two financial assets exist. The first one is riskless. Its price is one and its return is 2. Short selling on this asset is allowed. The second asset is risky. Its price is 1 and its return \( z \), where \( z \) is a random variable with probability distribution:

\[
\begin{align*}
z = 1 & \quad \text{with probability} \quad \pi_1 \\
z = 2 & \quad \text{with probability} \quad \pi_2 \\
z = 3 & \quad \text{with probability} \quad \pi_3
\end{align*}
\]

No short selling is allowed on this asset.
a. If the agent invests $a$ in the risky asset, what is the probability distribution of the agent’s portfolio return ($\tilde{R}$)?

b. The agent maximizes a Von Neumann-Morgenstern utility ($U$). Show that the optimal choice of $a$ is positive if and only if the expectation of $\tilde{z}$ is greater than 2.
Hint: Find the first derivative of $U$ and calculate its value when $a = 0$.

c. Give the first-order condition of the agent's problem.

d. Find $a$ when $U(Y) = 1 - \exp(-bY)$, $b > 0$ and when $U(Y) = \frac{1}{1 - \gamma} Y^{1-\gamma}$, $0 < \gamma < 1$. If $Y$ increases, how will the agent react?

e. Find the absolute risk aversion coefficient ($R_A$) in either case.

4.3 Risk aversion and portfolio choice: Consider an economy with two types of financial assets: one risk-free and one risky asset. The rate of return offered by the risk-free asset is $r_f$. The rate of return of the risky asset is $\tilde{r}$. Note that the expected rate of return $E(\tilde{r}) > r_f$.

Agents are risk-averse. Let $Y_0$ be the initial wealth. The purpose of this exercise is to determine the optimal amount $a$ to be invested in the risky asset as a function of the Absolute Risk Aversion Coefficient (Theorem 4.4).

The objective of the agents is to maximize the expected utility of terminal wealth:

$$\max_a E\left(U\left(Y\right)\right)$$

where: $E$ is the expectation operator, $U(.)$ is the utility function with $U'' > 0$ and $U''' < 0$, $Y$ is the wealth at the end of the period, $a$ is the amount being invested in the risky asset.

a. Determine the final wealth as a function of $a$, $r_f$, and $\tilde{r}$.

b. Compute the FOC. Is this a maximum or a minimum?

c. We are interested to determine the sign of $\frac{da^*}{dY_0}$. Calculate first the total differential of the FOC as a function of $a$ and $Y_0$. Write the expression for $\frac{da^*}{dY_0}$. Show that the sign of this expression depends on the sign of its numerator.
d. You know that $R_A$, the absolute risk aversion coefficient, is equal to $-\frac{U''(.)}{U'(.)}$. What does it mean if $R'_A = \frac{dR_A}{dY} < 0$?

e. Assuming $R'_A < 0$, compare $R_A(Y)$ and $R_A(Y_0(1 + r_f))$: is $R_A(Y) > R_A(Y_0(1 + r_f))$ or vice-versa? Don't forget there are two possible cases:
\[
\begin{cases}\bar{r} \geq r_f \\
\bar{r} < r_f
\end{cases}
\]

f. Show that
\[
U''(Y_0(1 + r_f) + a(\bar{r} - r_f))(\bar{r} - r_f) > -R_A(Y_0(1 + r_f))U'(Y_0(1 + r_f) + a(\bar{r} - r_f))(\bar{r} - r_f)
\]
for both cases in point e.

g. Finally, compute the expectation of $U''(Y)(\bar{r} - r_f)$. Using the FOC, determine its sign.

What can you conclude about the sign of $\frac{da^*}{dY}$? What was the key assumption for the demonstration?

4.4 Suppose that a risk-averse individual can only invest in two risky securities A and B, whose future returns are described by identical but independent probability distributions. How should he allocate his given initial wealth (normalized to 1 for simplicity) among these two assets so as to maximize the expected utility of next period's wealth?

4.5 An individual with a well-behaved utility function and an initial wealth of $1$ can invest in two assets. Each asset has a price of $1$. The first is a riskless asset that pays $1$. The second pays amounts $a$ and $b$ (where $a < b$) with probabilities of $\pi$ and $(1-\pi)$, respectively. Denote the units demanded of each asset by $x_1$ and $x_2$, respectively, with $x_1, x_2 \in [0,1]$.

a. Give a simple necessary condition (involving $a$ and $b$ only) for the demand for the riskless asset to be strictly positive. Give a simple necessary condition (involving $a$, $b$, and $\pi$ only) so that the demand for the risky asset is strictly positive.

b. Assume now that the conditions in item a are satisfied. Formulate the optimization problem and write down the FOC. Can you intuitively guess the sign of $dx_1/da$? Verify your guess by assuming that $x_1$ is a function of a written as $x_1(a)$, and taking the total differential of the FOC with respect to $a$. Can you also conjecture a sign for $dx_1/d\pi$? Provide an economic interpretation without verifying it as done previously.
Chapter 5

5.1 Comment fully on the following statement: If a portfolio has a high $\beta$, then further diversification is possible.

5.2 Consider an equally weighted portfolio of three stocks, each of which is independently distributed of the others (that is, $\text{cov}(r_i, r_j)=0$ for different securities $i$ and $j$). Assume also that each stock has the same total risk ($\sigma$). What fraction of each stock's risk is diversified away by including it in this portfolio?

5.3 What is the difference between the relationship implied by the Capital Market Line (CML) and the Security Market Line (SML)? Consider a particular portfolio $P$ with risk $\sigma_p$. Under what circumstances will the CML and the SML give the same $E\tilde{r}_p$?

5.4 Among your numerous assets, you are the owner of a finance company that extends one-year loans to people to buy appliances and other household goods. A young finance whiz that you just hired suggests that since the default risk of your loans is entirely diversifiable, you should charge your customers (those that are borrowing from you) the risk-free rate.

a. What do you think of the suggestion?

b. Assume the risk-free rate is 10%, and that the probability of default is 5% for the next year on a typical loan. In addition, assume that if a borrower defaults, all the principal but no interest is repaid. What rate should your finance company charge for loans over the next year?

c. Suppose the reclaimed appliances have lost 20% of their original value, and that in the event of default no interest is paid. What rate should you set?

5.5 At the moment, all of your assets are invested in asset A with the following return and risk characteristics:

\[
E\tilde{r}_A = 10% \\
\sigma_A = 10%
\]

Another asset (call it "B") becomes available; the characteristics of B are as follows: $E\tilde{r}_B = 20\%, \sigma_B = 25\%$. Furthermore, the correlation of A's and B's return patterns is -1.

By reallocating your portfolio to include some of asset B, how much additional return could you expect to receive if you wanted to maintain your portfolio's risk at $\sigma = 10\%$. Hint: solve for $w_B$, not for $w_A$. 

Problems and Exercises - 9
5.6 You are a portfolio manager considering whether or not to allocate some of the money with which you are entrusted to the market index of Australian stocks. Your assistant provides you with the following historical return information:

<table>
<thead>
<tr>
<th></th>
<th>Your Portfolio</th>
<th>Australian Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>1992</td>
<td>54%</td>
<td>50%</td>
</tr>
<tr>
<td>1993</td>
<td>24%</td>
<td>-10%</td>
</tr>
<tr>
<td>1994</td>
<td>-6%</td>
<td>10%</td>
</tr>
<tr>
<td>1995</td>
<td>24%</td>
<td>60%</td>
</tr>
<tr>
<td>1996</td>
<td>-6%</td>
<td>-20%</td>
</tr>
<tr>
<td>1997</td>
<td>54%</td>
<td>80%</td>
</tr>
</tbody>
</table>

a. Show that the addition of the Australian index (AUS) to your portfolio (your) will reduce risk (at no loss in returns) provided

\[
\text{corr(your, AUS)} < \frac{\sigma_{\text{your}}}{\sigma_{\text{AUS}}}
\]

(assuming, as in the case, \( \sigma_{\text{AUS}} > \sigma_{\text{your}} \))

b. Based on this historical data could you receive higher returns for the same level of risk (standard deviation) by allocating some of your wealth to the Australian index?

c. Based on historical experience, would it be possible to reduce your portfolio’s risk below its current level by investing something in the Australian index?

d. What fraction of the variation in Australian stocks can be explained by variation in your portfolio’s returns?

e. Is this situation a violation of the CAPM?

Chapter 6

6.1 In the CAPM setting, it is argued that only a fraction of the total risk of a particular asset is priced. Use the CML and SML to prove this assertion.

6.2 Consider two fully isolated economies, economy 1 and economy 2. The same assets are traded in both economies, but the average investor in economy 2 is more risk averse than the average investor in economy 1. Compare the CMLs in both economies.

6.3 Consider two fully isolated economies. Asset returns in economy 2 are, in general, more positively correlated than asset returns in economy 1. Compare the CMLs in both economies.
6.4 Under the CAPM, all investors form portfolios of two assets, a risk-free asset and a risky portfolio M, irrespective of their level of wealth Y. If an investor becomes wealthier, he may want to increase or decrease the proportion of his wealth held in the risky portfolio, and, by the Cass-Stiglitz theorem, unless his preferences have a very specific form, as his wealth changes, he will want to alter the composition of the risky part of his portfolio. The CAPM does not assume such preference restrictions. Yet, the CAPM equilibrium does not seem to permit the desired changes in the composition of the agent’s risky portfolio! Is there a contradiction?

6.5 Consider the following 3 assets:

\[
\begin{bmatrix}
\tilde{r}_1 \\
\tilde{r}_2 \\
\tilde{r}_3
\end{bmatrix}
= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}
\text{ and } V = \begin{pmatrix}
1 & -1 & 0 \\
-1 & 4 & -1 \\
0 & -1 & 9
\end{pmatrix}
\]

Compute g and h from our portfolio composition characterization. Identify the MVP. Identify the zero covariance portfolio for asset 3.

Note that
\[
V^{-1} = \begin{pmatrix}
A & B & C \\
B & B & C \\
C & C & D
\end{pmatrix}
\]

where
\[
A = 1.346154 \\
B = .346154 \\
C = .0384615 \\
D = .115385
\]

6.6 Consider a two-period economy with I agents, J risky assets, and one risk-free rate asset. You can write the agent’s wealth as follows:

\[
\tilde{Y}_i = \left(Y_0^i - \sum_j x_j^i \right) (1+r_f) + \sum_j x_j^i (1+\tilde{r}_j)
\]

where
\[
Y_0^i = \text{initial wealth of agent } i, \\
r_f = \text{the risk-free interest rate}, \\
\tilde{r}_j = \text{the random rate of return on the jth risky asset}, \\
x_j^i = \text{the amount invested in the jth asset by agent } i.
\]

As usual, the individual’s choice problem is:

\[
\max_{x_j^i} E \left[ U_i (\tilde{Y}_j) \right]
\]
a. Show that the FOC can be written as follows:

\[ E\left[U_i'\left(\bar{Y}_i, \bar{r}_j - r_f\right)\right] = 0 \]  

(1)

b. Show that Equation (1) can be written as follows:

\[ E\left[U_i'\left(\bar{Y}_i\right)\right]E(\bar{r}_j - r_f) = -Cov\left[U_i'\left(\bar{Y}_i\right), \bar{r}_j\right] \]

(2)

Recall that \( \text{Cov}(x, y) = E(xy) - E(x)E(y) \).

c. Using the following property of the covariance \( \text{Cov}[g(x), y] = E[g'(x)\text{Cov}(x, y)] \), show that Equation (2) can be rewritten as follows:

\[ E\left[U_i'\left(\bar{Y}_i\right)\right]E(\bar{r}_j - r_f) = -E\left[U_i''\left(\bar{Y}_i\right)\right]\text{Cov}(\bar{Y}_i, \bar{r}_j) \]

(3)

d. Aggregating on all individuals (summing on \( i \)), show that Equation (3) can be rewritten as follows:

\[ E(\bar{r}_j - r_f) = \frac{Y_{M0}}{\sum_i R_{Ai}} \text{Cov}(\bar{r}_M, \bar{r}_j) \]

(4)

where \( R_{Ai} = \frac{-E\left[U_i''\left(\bar{Y}_i\right)\right]}{E\left[U_i'\left(\bar{Y}_i\right)\right]} \) is reminiscent of the absolute risk aversion coefficient and \( \sum_i \bar{Y}_i = Y_{M0} (1 + \bar{r}_M) \).

e. Show that Equation (4) implies:

\[ E(\bar{r}_M - r_f) = \frac{Y_{M0}}{\sum_i R_{Ai}} \text{Var}(\bar{r}_M) \]

(5)

f. Derive the traditional CAPM relationship. Hint: Combine Equations (4) and (5).

Chapter 7

7.1 Arrow-Debreu pricing: You are given the following term structure of interest rates for three periods into the future:

\[ r_1 = 0.0989 \quad r_2 = 0.1027 \quad r_3 = 0.1044 \]

a. Construct the Arrow-Debreu prices for these state-dates.
b. Assume that there are three states in each period and that the constant Arrow-Debreu state-matrix is as follows:

\[
\begin{pmatrix}
.28 & .33 & .30 \\
.27 & .34 & .31 \\
.24 & .28 & .36 \\
\end{pmatrix}
\]

Assume we are in state 1. Are these prices consistent with the term structure just given?

7.2 What is the relationship between the price of an Arrow-Debreu security and the corresponding state probability? How is the price of an Arrow-Debreu security affected by a change in the discount factor? What else affects the price of an Arrow-Debreu security?

7.3 Arrow-Debreu pricing: Consider a world with two states of nature. The following matrix provides the one-period Arrow-Debreu prices in all situations

<table>
<thead>
<tr>
<th>t\t+1</th>
<th>State 1</th>
<th>State 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>State 1</td>
<td>.53</td>
<td>.43</td>
</tr>
<tr>
<td>State 2</td>
<td>.45</td>
<td>.52</td>
</tr>
</tbody>
</table>

Describe the term structure of interest rates over three periods.

7.4 You are given the following prices of a set of coupon bonds. Construct the term structure and price of the corresponding date-contingent claim.

<table>
<thead>
<tr>
<th>Bond Prices and Coupons</th>
</tr>
</thead>
<tbody>
<tr>
<td>t = 0</td>
</tr>
<tr>
<td>bond 1</td>
</tr>
<tr>
<td>bond 2</td>
</tr>
<tr>
<td>bond 3</td>
</tr>
<tr>
<td>bond 4</td>
</tr>
<tr>
<td>bond 5</td>
</tr>
</tbody>
</table>

7.5 General equilibrium and uncertainty: Consider a two-period exchange economy with 2 agents. They have identical utility functions:

\[
U(c_1, c_2(\theta)) = \ln c_1 + \ln c_2(\theta)
\]

where: \(c_1\) is the consumption level at date 1
c₂(θ) is the consumption level at date 2 if state θ occurs.

Let us assume two possible states of nature at date 2, and the following endowment structure:

<table>
<thead>
<tr>
<th></th>
<th>t = 0</th>
<th>t = 1</th>
<th>θ = 1</th>
<th>θ = 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agent 1</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Agent 2</td>
<td>4</td>
<td>3</td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

a. Describe intuitively a Pareto-optimal allocation of resources. Is it unique?

b. Suppose only one type of security may be issued. Is it possible to achieve the Pareto-optimal allocation? If so, how?

c. Suppose the commodity in the previously described economy can be costlessly stored. No other asset is traded. Describe intuitively how this opportunity will be exploited and why. Is the utility level of the two agents increased?

d. Let us now assume agent 1 is risk neutral and there is aggregate uncertainty, that is, at date 2 the endowment structure is

<table>
<thead>
<tr>
<th></th>
<th>θ = 1</th>
<th>θ = 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

Discuss intuitively what the new Pareto-optimal allocation should look like.

Assume the existence of Arrow-Debreu securities and characterize the resulting market equilibrium.

7.6 A two-part problem:

a. Think of an economy with two agents with utility functions of the form:

\[ U(c) = c_0 + E(c_1(θ)) - 2 \text{var}(c_1(θ)) \]

where: \( c_0 \) is the consumption level at date 0
\( c_1(θ) \) is the consumption level at date 1 if state θ occurs.

Their endowments are given by
The state probabilities are 1/4 for state 1 and 3/4 for state 2.

1. Intuitively, what condition should a Pareto-optimal allocation satisfy in this particular setup?

2. Are there several Pareto optima? If so, characterize the set of Pareto optima.

3. Is the following allocation a Pareto optimum? Why?

b. Now the agents have a utility function of the form:

\[ U(c) = c_0 + E \left( \ln(c_1(\theta)) \right) \]

where: 
- \( c_0 \) is the consumption level at date 0
- \( c_1(\theta) \) is the consumption level at date 1 if state \( \theta \) occurs.

The endowments are:

<table>
<thead>
<tr>
<th></th>
<th>t = 0</th>
<th>t = 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>State 1</td>
<td>Agent 1</td>
<td>4</td>
</tr>
<tr>
<td>State 2</td>
<td>Agent 1</td>
<td>2</td>
</tr>
<tr>
<td>State 1</td>
<td>Agent 2</td>
<td>4</td>
</tr>
<tr>
<td>State 2</td>
<td>Agent 2</td>
<td>4</td>
</tr>
</tbody>
</table>

1. Describe the Pareto optimum or the set of Pareto optima for this utility function and compare it with the one under item 7.6.a.1.

2. Two Arrow-Debreu securities denoted by \( Q_1 \) (state 1) and \( Q_2 \) (state 2) are introduced. Both are in zero net supply. Compute the competitive equilibrium allocation. Is it Pareto optimal? Discuss intuitively its characteristics (the determinants of the prices of the two securities and the post-trade allocation).

3. Suppose only one Arrow-Debreu security depending on state 1 is traded. It is in zero net supply. Compute the competitive equilibrium allocation. Is it Pareto optimal?
7.7 Think of a simple exchange economy with two agents that have the following utility functions:

\[ U_1(.) = 0.25c_0 + 0.5E[\ln(c_1(\theta))] \]
\[ U_2(.) = c_0 + E[\ln(c_1(\theta))] \]

where \( \theta \) denotes the two equally likely states at date 1. The endowments are

<table>
<thead>
<tr>
<th>( t = 0 )</th>
<th>( t = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agent 1</td>
<td>2</td>
</tr>
<tr>
<td>Agent 2</td>
<td>4</td>
</tr>
</tbody>
</table>

a. Describe the Pareto optimum or the set of Pareto optima. Is it unique? Is perfect risk sharing achieved? Why or why not? Answer the same question if the state 2 endowments were 5 for agent 1, and 3 for agent 2.

b. Agents can trade with two Arrow-Debreu securities denoted by Q (with payoffs (1,0)), and R (with payoffs (0,1)). Calculate the competitive equilibrium allocation. Is it Pareto optimal? Discuss intuitively its characteristics (the determinants of the prices the two securities and the post-trade allocation).

c. Suppose that both securities can be traded at the beginning and there are no short-selling constraints. A firm has zero cost of introducing one unit of either security and it wants to maximize its profits. Which security should it introduce? For which agent is this introduction more valuable? Is the outcome Pareto optimal? Why or why not?

Chapter 8

8.1 Two-part question:

a. To what extent should we care whether the current organization of markets leads to a Pareto-optimal allocation?

b. Are markets complete? Do we care?

8.2 Options and market completeness: Remember that in certain circumstances, it is not possible to achieve market completeness with call options only (why?). Show that in the following market structure with 3 assets and 4 states, introducing a put option on the first asset with exercise price 1 is sufficient to achieve market completeness (i.e., to generate a complete set of Arrow-Debreu securities for those four states.)
8.3 Three-part question:

a. Describe intuitively the idea of an Arrow-Debreu security. Arrow-Debreu securities are not observed in real markets. Is the concept nevertheless useful? What is the link between Arrow-Debreu securities and options?

b. You observe the following assets with the corresponding state-dependent payoffs:

<table>
<thead>
<tr>
<th>Securities</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>States</td>
<td>3</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>16</td>
<td>25</td>
</tr>
</tbody>
</table>

Is this market complete?

c. Is the following market structure complete?

<table>
<thead>
<tr>
<th>Assets</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>5</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>6</td>
<td>14</td>
</tr>
<tr>
<td>States</td>
<td>3</td>
<td>12</td>
<td>16</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>8</td>
<td>20</td>
</tr>
</tbody>
</table>

If the market is not complete, introduce a derivative security to complete it.

Construct an Arrow-Debreu security by introducing calls or puts on these assets. Can you reproduce the same structure by using only puts?

Chapter 9

9.1 Consider a two-date economy where there are three states of the world at date 1. Consumption per capita at date 1 will be $5, $10, or $15. A security that pays one unit of consumption next period is worth $8 today. The risk-free security pays a gross return of 1.1. A call option on per capita consumption with an exercise price of $12 costs $1. The probabilities of the three states are 0.3, 0.4, and 0.3, respectively.

a. Are markets complete? If yes, what are the state prices?
b. Price a put option on per capita consumption with an exercise price of $8.

c. Can you derive the risk-neutral probabilities?

d. Suppose that an investor has the following expected utility function: \( \ln c_0 + \delta \mathbb{E} \ln c_0 \). Can you solve for consumption at date \( t = 0 \)?

9.2 Arrow-Debreu pricing: Consider a two-period economy similar to the one we are used to, but with the following basic data:

<table>
<thead>
<tr>
<th>Agent</th>
<th>( t = 0 ) Endowment</th>
<th>( t = 1 ) Endowment</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta = 1 )</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>( \theta = 2 )</td>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>

The agents have utility functions \( U(c_1, c_2(\theta)) = c_1 + \mathbb{E}[\ln(c_2(\theta))] \), and the state probabilities are \( \operatorname{Prob}(\theta_1) = 1/3 = 1 - \operatorname{Prob}(\theta_2) \).

Construct the risk-neutral probabilities and compute the value today of agent 2's period 2 endowment.

Chapter 10

10.1 CAPM versus Consumption CAPM:

a. Contrast the two models in words.
b. Explain why, in principle, the consumption CAPM is more satisfactory.
c. Can you think of circumstances where the two models are essentially identical?

10.2 Consider an endowment economy identical to the one considered in the discussion of the consumption CAPM. Consider an option that entitles the owner to exercise the right to buy one unit of the asset one period in the future at a fixed price \( p^* \). The price \( p^* \) is known at period \( t \), whereas the option to buy is exercised at period \( t + 1 \).

Suppose the representative agent's utility function is \( U(c) = \ln(c) \). You are asked to price the option.

Follow these steps:

a. Write the value of the option at expiration as a function of the price of the underlying asset.

b. Write the price of the asset at the later date as a function of the state at that date, \( \theta \), the total quantity of good available.
c. Given the chosen utility function, write \( q(\theta, c) \), the Arrow-Debreu price.

d. Use the Arrow-Debreu price to price the option.

10.3 Consider the setting described previously with the following modification: The asset in question requires the owner to buy the asset at a price \( p^* \) (a forward contract). Price the security.

10.4 Consider the usual representative agent economy. Suppose that the representative agent has log utility, \( U(c) = \ln(c) \). Define the wealth portfolio as a claim to all future dividends available for consumption in this economy.

a. Show that the price of this wealth portfolio is proportional to consumption itself.

b. Show that the return on this portfolio is proportional to consumption growth. Hint: Think about the definition of the return on an asset to write down the return on the wealth portfolio.

c. Finally, express the price of a cash flow paying off $100 at each date for the next two periods in terms of the return on the wealth portfolio.

10.5 Begin with the fundamental equation \( E[mR_i] = 1 \) where \( R_i \) is the gross return on asset \( i \), and \( m \) is the pricing kernel. Noting that this fundamental equation should hold for the gross returns on the riskless asset and on the market portfolio (\( R_M \)) as well, obtain a CAPM-like expression. Define a functional relation between \( m \) and \( R_M \) so that you will obtain precisely the CAPM expression.

Chapter 11

11.1 Consider an economy where the endowment is given by the following binomial process:
The representative agent has utility function\[ u(c_0, c_1(\hat{\theta}_1), c_2(\hat{\theta})) = \ln(c_0) + \delta \ln(c_1(\hat{\theta}_1)) + \delta^2 \ln(c_2(\hat{\theta})) \], and the (subjective) state probabilities are\[ \pi(\theta_{11}) = \pi(\theta_{12}) = \pi(\theta_{22}) = \frac{1}{2}, \quad \pi(\theta_{21}) = \pi(\theta_{23}) = \frac{1}{4} \]

a. Solve for the expected utility of the representative agent. Assume that \( \delta = 0.96 \).

b. Solve for Arrow-Debreu prices, risk-neutral probabilities, and the pricing kernel.

c. What is the value of the consumption stream at date 0?

d. Price a one- and a two-period bond. Can we observe a term premia? What do you conclude?

e. Price a European call on consumption at date 0. The expiration date is just immediately after the representative agent has received his endowment at date 1 (strike = 1).

f. Let us assume that you are an outsider to this economy and you do not observe the representative agent. Price the call with the binomial method presented in Chapter 11. Note that the consumption stream can be interpreted as dividends and assume that the representative agent announces the price process (!) as well as the interest rate.

g. Do not hesitate to comment on your results, (i.e., show the unique relationship between the methods).

Chapter 12

12.1 Suppose the existence of three assets with the following characteristics:

<table>
<thead>
<tr>
<th></th>
<th>( E(r_i) )</th>
<th>( b\beta_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>.07</td>
<td>0.5</td>
</tr>
<tr>
<td>B</td>
<td>.09</td>
<td>1.0</td>
</tr>
<tr>
<td>C</td>
<td>.17</td>
<td>1.5</td>
</tr>
</tbody>
</table>

Assume that the specific risk can be eliminated through diversification and that the return model can be written:

\[ r_i = E(r_i) + b_i F_1 + e_i \]

a. Plot the three assets on a \( E(r_i) \) - \( b_i \) graph.

b. Using assets A and B, construct a portfolio with no systematic risk. Do the same using assets B and C.
c. Construct an arbitrage portfolio and compute its expected return.

d. Comment and describe how prices will adjust as a result of the arbitrage opportunity.

e. Suppose now that \( E(r_A) = .06, E(r_B) = .10, \) and \( E(r_C) = .14 \). Plot the three assets in the graph in item a. Is it still possible to construct an arbitrage portfolio with positive expected return? Comment.

12.2 Assume the following market model:

\[
\tilde{r}_j = \alpha_j + \beta_{jM} \tilde{r}_M + \tilde{\epsilon}_j
\]

with

\( \tilde{r}_j = \) the rate of return on asset \( j \)
\( \alpha_j = \) a constant
\( \beta_{jM} = \) the sensitivity of asset \( j \) to fluctuations in the market return
\( \tilde{r}_M = \) the market rate of return
\( \tilde{\epsilon}_j = \) a stochastic component with \( E(\tilde{\epsilon}_j) = 0, \text{Var}(\tilde{\epsilon}_j) = \sigma_{\tilde{\epsilon}_j}^2 \) and \( \text{cov}(\tilde{\epsilon}_i, \tilde{\epsilon}_j) = 0, \forall i, j. \)

a. Prove that \( \sigma_j^2 = \beta_j^2 \sigma_M^2 + \sigma_{\tilde{\epsilon}_j}^2 \).

b. Prove that \( \sigma_{ij} = \beta_i \beta_j \sigma_M^2 \).

12.3 CAPM and APT:

a. Briefly describe these two models: Capital Asset Pricing Model and Arbitrage Pricing Theory? What are their main assumptions?

b. CAPM and APT are so-called valuation models. In fact, they permit us to compute the expected rate of return of a security. How can this be used to derive a current price?

c. Contrast the two models. Are they compatible? Are they identical under certain conditions?

12.4 Draw the parallels and underline the key differences between the Arbitrage Pricing Theory (APT) and Arrow-Debreu pricing.
12.5 The APT does not assume individuals have homogeneous beliefs concerning the random returns of the assets under consideration. True or false? Comment.

Chapter 13

13.1 General equilibrium and uncertainty: Consider a two-date economy with two agents and one good. Assume two states of nature at the second date with probabilities π and 1-π. The resources are the following:

<table>
<thead>
<tr>
<th></th>
<th>t = 0</th>
<th>t = 1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>θ = 1</td>
<td>θ = 2</td>
</tr>
<tr>
<td>Agent 1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Agent 2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Agent 1's utility function is: \( U_1 = c_0^1 + \delta E(c^1(\theta)) \)

Agent 2's utility function is: \( U_2 = \ln c_0^2 + \delta E(\log c^2(\theta)) \)

The subjective discount rate is the same for both agents, (i.e., \( \delta \)).

a. Comment on the form of the utility functions.

b. Determine the Pareto optimum.

c. Assuming markets are complete, compute the contingent prices, taking the good at date 0 as the numeraire.

d. Assume there is only one asset (a bond), which gives one unit of the good in each state of nature.


2. Find the competitive equilibrium.

3. Compute the equilibrium allocation for \( \pi = 0.5 \) and \( \delta = 1/3 \). Is this allocation Pareto optimal?

13.2 Summarize what we have learned about the Modigliani-Miller theorem in the context of incomplete markets.

13.3 General equilibrium and uncertainty: Consider a two-period, two-agent economy with preferences and endowments as follows:
Intermediate Financial Theory – Danthine and Donaldson

<table>
<thead>
<tr>
<th></th>
<th>t = 0</th>
<th>t = 1</th>
<th>Preferences</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agent 1</td>
<td>20</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>Agent 2</td>
<td>30</td>
<td>15</td>
<td>15</td>
</tr>
</tbody>
</table>

\[ U_1(.) = c_0^1 + \pi(\theta_1) \frac{1}{2} \ln c^1(\theta_1) + \pi(\theta_2) \frac{1}{2} \ln c^1(\theta_2) \]

\[ U_2(.) = c_0^2 + \pi(\theta_1) \frac{1}{2} c^2(\theta_1) + \pi(\theta_2) \frac{1}{2} c^2(\theta_2) \] (risk neutral)

The state probabilities are: \( \pi(\theta_1) = 1/3 \), \( \pi(\theta_2) = 2/3 \).

a. Imagine that in this economy only the asset \( Q=(1,0) \) is traded. Construct the financial equilibrium. Will risk sharing take place? Is the initial allocation already Pareto optimal? How would you answer if the state probabilities were equal?
b. Now we are in the initial setup where trade with \( (1,0) \) is still possible. Assume that asset \( R=(0,1) \) is also traded. What will be the price of this asset? How much of the two assets will be exchanged? What is the post-trade allocation? Is it Pareto-optimal?

13.4 Consider a two-date, two-agent economy, with the following initial endowments:

<table>
<thead>
<tr>
<th></th>
<th>t = 0</th>
<th>t = 1</th>
<th>( \theta = 1 )</th>
<th>( \theta = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agent 1</td>
<td>4</td>
<td>6</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Agent 2</td>
<td>6</td>
<td>3</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

The agents' utility functions are:

Agent 1 \[ 1/2 \cdot \ln c_0^1 + E \ln c^1_0 \]

Agent 2 \[ 1/2 \cdot c_0^2 + E \ln c^2_0 \]

The probability of each state is \( \text{Prob}(\theta = 1) = 0.4 \) and \( \text{Prob}(\theta = 2) = 0.6 \).

a. Compute the initial utility of each agent.
b. Suppose now there is a firm that generates an uncertain output at time \( t = 1 \) of \( (2; 3) \). The firm owner is interested in consuming only at date 0 and accordingly he would like to sell his claim of ownership to the output of the firm. He solicits your advice as to the type of security that he should issue. Suppose first that he issues a stock, that is, a claim to 2 units of good in state \( \theta = 1 \) and 3 units in state \( \theta = 2 \).

1. What is the price of the security/the stock market value of the firm?
2. What are the post-trade allocations?
3. What are the post-trade utilities?

Discuss intuitively how your results would have differed if the two states had been equally likely.
c. What happens if, instead of a stock claim, the firm owner issues Arrow-Debreu securities. Discuss intuitively (no calculations needed) how your answer to these same questions would need to be modified.

d. Now forget the firm. The situation is different: A foreign government would like to market a risk-free security, specifically a claim promising payment of 2 units of the good next period, irrespective of the state. How much money will this bond issue generate? What are the post-trade allocations and utilities? Are you surprised?

e. In one line, tell us how the problem would be altered if the bond issuer is not a foreign government but the local government.

13.5 Consider a two-period economy with two agents and one good. Assume two possible states of nature at the second period with the same probability. The agents only care about their second-period consumption level. They maximize their expected utility.

Agent 1's utility function is: \( U_1 = c_1 \)
Agent 2's utility function is: \( U_2 = \log(c_2) \)

a. Agents can trade in a complete contingent security market system. Show, in full generality, that the price of the contingent security is the same for the two states of nature. Show that agent 2's consumption doesn't depend on the state of nature.

b. Assume that, in addition to their endowments, the two agents each possess one-half of a firm. The firm invests two units of input in period 1. The output is available at period 2.

There are two possible technologies. With the first one, one unit of input produces one unit of good, independently of the state of nature. With the second one, one unit of input produces three units of good in the first state of nature and none in the other state.

Let \( x \in [0,1] \) be the part of input used with the second technology. The sale of the goods at the second period is the agents' only resource, so each of them receives \((x+1)\) in the first state of nature and \((1-\frac{1}{x})\) in the second.

1. Assuming that the markets are complete. Write the agents' budget constraints as a function of the parameter \( x \).

2. Determine the consumption and utility levels as a function of the parameter \( x \). Show that the agents agree about the optimal choice of \( x \). What value will they choose?

c. From now on, suppose there are no markets for contingent securities. However, the good can be traded on a spot market during the second period. Compute the equilibrium utility level attained by the agents as a function of \( x \). Determine the preferred investment policy of each agent. Show that they disagree on the optimal level of \( x \).
d. Assume that the agents can trade two securities at period 1: One security pays one unit of good independently of the state of nature, the other one is a share of the firm just described. Show that the markets are complete. What will agent 2 do in such a context? (No formal proof is required.)

Chapter 14

14.1 Assume the following speculator's program:

$$\max_{f} \mathbb{E}[\bar{c} + (p^f - p)f]$$

Is it true that $f^* > 0$ if and only if $p^f > \mathbb{E}p$?

14.2 Show that it is always better, that is, more profitable, for a producer to speculate on the futures market than on the physical market.

Assumptions:

- no uncertainty in production
- no basis risk

Definition: To speculate on the physical market means producing because an increase in price is expected, although in terms of the futures price, the production is not profitable (marginal cost is not covered).

14.3 The law of demand states that a price increase leads to a decrease in the quantity demanded. Comment on the applicability of the law of demand when the object being exchanged is a financial asset.

14.4 Provide at least one example where price and volume behavior is significantly different in a market with heterogeneously informed agents than it would be in a market where agents are homogeneously informed.

14.5 Consider a firm facing exchange rate risk for its output commodity: The production decision is made at date t and the output is sold in foreign currency at date t + 1. Assume that no currency futures market exists, however, a market for a domestic financial asset is available. You can write the profit of the firm as follows:

$$\bar{\pi} = p\bar{y} - \frac{1}{2} y^2 + z(q^f - \bar{q})$$

where $p =$ the known foreign currency price

$y =$ the output
\( \frac{1}{2} y^2 \) = the cost function
\( e \) = the exchange rate
\( z \) = the number of shares of the domestic asset sold short at date \( t \)
\( \tilde{q} \) = the nominal payoff of the domestic financial asset at date \( t+1 \)
\( q^f \) = the date \( t \) price of the domestic financial asset

It is assumed that
\( E(\tilde{q}) = q^f \), (i.e., the date \( t \) price of the financial asset is an unbiased predictor of the future price).

Suppose the firm maximizes a mean-variance utility function:

\[
\max_{\pi, x} E(\pi) - \frac{1}{2} \text{var}(\pi)
\]

• Write \( E(\pi) \) and \( \text{var}(\pi) \).
• Compute and interpret the FOCs
• Show that output is greater in the case of certainty than in the case of uncertainty.