A note on the non-convexity problem in some shopping-time and human-capital models

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Abstract

Several works in the shopping-time and in the human-capital literature, due to the non-concavity of the underlying Hamiltonian, use first-order conditions in dynamic optimization to characterize necessity, but not sufficiency, in intertemporal problems. This note selects some works in these two areas and shows that optimality can be characterized, and some results quantitatively improved, by means of an application of Arrow’s [Arrow, K. J., 1968. Applications of control theory to economic growth. In: Dantzig, G.B., Veinott Jr., A.F. (Eds.), Mathematics of the Decisions Sciences. American Mathematical Society, Providence, RI] sufficiency theorem.

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1. Introduction

Several works in the economic literature, particularly in the shopping-time (e.g., Lucas, 2000; Gillman et al., 1997; Cysne, 2003; Cysne et al., 2005)1 and in the human-capital literature (e.g., Uzawa, 1965; Lucas, 1988, 1990; Mulligan and Sala-I-Martin, 1993;

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1 See Eq. (5.3) in Lucas (2000), Eq. (3) in Cysne (2003), Eq. (10) in Cysne et al. (2005) and the terms $Ac$ and $(1 - A)c$ in the Hamiltonian in Section 2 of Gillman et al. (1997).
Caballe and Santos, 1993; Chari et al., 1995; Ladron-de-Guevara et al., 1999; Kosempel, forthcoming) use first-order conditions in dynamic optimization to directly characterize necessity, but not sufficiency, in intertemporal problems.

Seeing such questions under a perspective of optimal-control theory, the reason for the absence of sufficiency, usually either implicitly or explicitly recognized by the authors, is that the non-concavity of the associated Hamiltonian does not allow for the use of Mangasarian’s (1966) well known sufficiency conditions.

Mangasarian’s theorem states that if the Hamiltonian is (strictly) concave with respect to the control and the state variables, then the first-order conditions are also sufficient for an interior (unique) optimum. The papers cited above are some examples in the economic literature in which such conditions are not obeyed.

Arrow’s (1968) theorem, though, generalizes Mangasarian’s result, and, as we shall see, is able to provide quantitative insights and to generate sufficiency in some cases in which Mangasarian’s result is not directly applicable.

Arrow’s theorem requires another type of concavity. In words, first, the Hamiltonian is maximized with respect to the control variables, for a given value of state and costate variables. The optimum values of the control variables, as a function of the state variables and of the costate variable, are then substituted into the Hamiltonian. Call this new function (of the state and costate variables) the maximized Hamiltonian. Arrow’s main result is that if this maximized Hamiltonian is (strictly) concave with respect to the state variables, for the given values of the costate variables, then the first-order conditions characterize a (unique, concerning the state variable) optimum. 3

Of course, if the Hamiltonian is concave with respect to both the state and control variables, then the maximized Hamiltonian will be concave in the state variables. But the reverse is not true. This is the reason why Arrow’s theorem is able to generalize Mangasarian’s sufficiency conditions.

The main purpose of this note is calling the attention to the fact, and exemplifying how, in some specific cases, an application of Arrow’s theorem can yield returns at very reasonable costs in terms of the required algebrisms. As a by-product of the analysis, a complementary insight into some papers of the shopping-time and human-capital literature (the ones used as examples) is also delivered.

The plan of the note is as follows. Section 2 presents a formal version of Arrow’s theorem. In Section 3 I exemplify the use and usefulness of the theorem within the shopping-time literature and, in Section 4, within the human-capital literature. Section 5 concludes.

2. Arrow’s theorem

Following Seierstad and Sydsaeter’s (1987, p. 107 and 236), Arrow’s theorem, adapted to an infinite horizon, reads as follows4:

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2 See Eq. (15) in Uzawa (1965), Eq. (13) in Lucas (1988), Eq. (2.3) in Lucas (1990), Eqs. (2’) and (3’) in Mulligan and Sala-i-Martin (1993), Eqs. (3) and (6) in Caballe and Santos (1993), Eq. (5) in Chari et al. (1995), Eq. (2.4) in Ladron-de-Guevara et al. (1999) and Eq. (6) in Kosempel (forthcoming).

3 It is assumed that this argument applies (Lebesgue) almost-everywhere regarding the time domain in which such functions are considered, and in an open and convex neighborhood (concerning the state variable) of the candidate(s) for optimum.

4 This theorem appeared the first time in Arrow (1968).
Theorem 1 (Arrow’s sufficiency theorem). Let \((\bar{x}(t), \bar{u}(t))\) be a pair of (continuously differentiable) functions that satisfies the conditions (2) and (3) below, in the problem of finding a piecewise-continuous control vector \(u(t)\) and an associated continuously differentiable state vector variable \(x(t)\), with \(x(t)\) belonging to a given open and convex set \(A \subseteq \mathbb{R}^n\) for each \(t \geq t_0\), defined on the time interval \([t_0, \infty)\), that maximizes
\[
\int_{t_0}^{\infty} f_0(x(t), u(t), t) \, dt
\]
subject to the differential equations
\[
\dot{x}_i(t) = f_i(x(t), u(t), t), \quad i = 1, 2, \ldots, n
\]
and to the conditions
\[
\begin{align*}
&x_i(t_0) = x_i^0, \quad i = 1, 2, \ldots, n, \\
&x_i(\infty) \text{ free, } \quad i = 1, 2, \ldots, n, \\
&u(t) \in \mathbb{U} \subseteq \mathbb{R}^r.
\end{align*}
\]
Suppose, in addition, that there exists a piecewise continuously differentiable function \(p(t) = (p_1(t), \ldots, p_n(t))\) defined on \([t_0, \infty]\) such that, given the Hamiltonian function
\[
H(x(t), u(t), p(t), t) = f_0(x(t), u(t), t) + \sum_{i=1}^{n} p_i f_i(x, u, t).
\]
\(H(\bar{x}(t), \bar{u}(t), p(t), t)\) exists, and the following conditions are satisfied:
\[
\begin{align*}
&H(\bar{x}(t), \bar{u}(t), p(t), t) \geq H(\bar{x}(t), u, p(t), t), \quad \text{for all } u \in \mathbb{U}, \ t \in [t_0, \infty], \\
&\dot{p}_i(t) = -H_{x_i}(\bar{x}(t), \bar{u}(t), p(t), t), \quad i = 1, \ldots, n
\end{align*}
\]
and the transversality conditions
\[
\lim_{t \to \infty} p_i(x_i(t) - \bar{x}_i(t)) = 0 \quad i = 1, \ldots, n.
\]
Then, if the maximized Hamiltonian \(H^*(x, p(t), t) \equiv \max_{u \in U} H(x, u, p(t), t)\) exists and is a concave function in \(x\) for all \(t \geq t_0\), \((\bar{x}(t), \bar{u}(t))\) solves problem (1)–(3) above.

An intuition for the understanding of how Arrow’s sufficiency condition can be more economic than Mangasarian’s is that the latter requires more than what is effectively necessary for the solution of the problem. By (2), the state vector \(x(t)\) depends on the control vector \(u(t)\). Therefore, not all points \((x, u) \in A \times U\) are actually visitable by the state and control variables. Mangasarian’s sufficiency condition, by demanding concavity of the Hamiltonian in \((x, u) \in A \times U\), does not fully incorporate this information.

3. An application to a shopping-time model

In this section I apply Arrow’s theorem to Cysne (2003). This paper considers an economy with \(n\) different assets performing monetary functions. The first asset is money. Since the main point made here does not depend on the value of \(n\), I will work with \(n = 2\), in which case the second asset is a quasi-money. The result concerning the application of Arrow’s theorem generalizes with no change when \(n > 2\). Bonds (B) are a third asset in the economy, but are not included in \(n\) because they do not perform a monetary function.
Bonds are used only as a store of value and pay the (endogenously determined) benchmark interest rate \( r \). The monetary assets are represented by the two-dimensional vector \( X = (X_1, X_2) \), and their real quantities by the vector \( x = (X_1/P, X_2/P) = (x_1, x_2) \), \( P \) the price level. The real value of the stock of bonds is \( b = B/P \). Each asset \( x_1, x_2 \) pays an interest rate \( r_1, r_2 \). Since \( x_1 \) is money, \( r_1 = 0 \), but this point is not important here. relatively to the benchmark rate, which is the one paid by bonds, the vector of opportunity costs reads \( u = (u_1, u_2) = (r_1/C_0, r_2/C_0) \).

With \( g > 0 \) denoting a discount factor and \( c \) consumption, households are assumed to maximize

\[
\int_0^\infty e^{-gt}U(c(t)) \, dt. \tag{7}
\]

The potential product (that available when the shopping time \( (s) \) is equal to zero) \( y \) is normalized to one. The household is endowed with one unit of time so that \( y + s = 1 \). Make \( r_R = (r_1 - \pi, r_2 - \pi) \) and denote by \( \pi \) and \( h \), respectively, the rate of inflation and the lump-sum transfers from households to the government. When maximizing (7), households face the budget constraint

\[
\dot{b} + \sum_{i=1}^2 \dot{x}_i = 1 - (c + s) - h + (r - \pi)b + \langle r_R, x \rangle \tag{8}
\]

and the transacting-technology constraint

\[
c = G(x)s. \tag{9}
\]

The monetary aggregator function \( G(x) \) is differentiable, increasing in each one of the \( x \) variables, first degree homogeneous, and concave in \( x \). As in Lucas (2000), the utility function is assumed to be given by

\[
\begin{align*}
U(c) &= c^{1-\sigma}/(1-\sigma), & \sigma \neq 1, \ \sigma > 0, \\
U(c) &= \ln c \text{ (case } \sigma = 1). \tag{10}
\end{align*}
\]

Below, we shall call \( \sigma \) the coefficient relative risk aversion.\(^5\) In terms of Theorem 1, the Hamiltonian \( (\tilde{H}) \) for the problem reads

\[
\tilde{H}(s, G(x), b, \lambda) = e^{-gs}U(G(x)s) + p \left( 1 - (G(x) + 1)s - h + (r - \pi)b + \sum_{j=1}^2 (r_j - \pi)x_j \right). \tag{11}
\]

The so-called “current-value Hamiltonian” differs from the Hamiltonian above by a discount factor. Since using or not this discount factor does not imply any difference in the application of Mangasarian’s or Arrow’s sufficiency conditions, we can proceed the analysis using the current-value Hamiltonian

\(^5\) Since there is no uncertainty in the model, \( \sigma \) should actually be called “the inverse of the elasticity of intertemporal substitution”.
\[ H(s, G(x), b, \lambda) = U(G(x)s) + \lambda \left( 1 - (G(x) + 1)s - h + (r - \pi)b + \sum_{j=1}^{2} (r_j - \pi)x_j \right). \]

(12)

In order to apply Arrow’s theorem, consider \( s \) as the only control variable, and \( b \) and \( x \) as the state variables.\(^6\) The Hamiltonian given by (12) clearly is not concave in these variables because of the term \( (G(x) + 1)s \). Maximizing (12) with respect to \( s \) and substituting the optimum value of \( s \) into the Hamiltonian (12) leads to the maximized Hamiltonian

\[ H^*(G(x), b, \lambda) = \frac{\sigma}{1 - \sigma} \left( \frac{G(x)}{\lambda(1 + G(x))} \right)^{(1-\sigma)/\sigma} + \lambda \left( 1 - h + (r - \pi)b + \sum_{j=1}^{n} (r_j - \pi)x_j \right) \quad (\sigma \neq 1), \]

\[ H^*(G(x), b, \lambda) = \log \left( \frac{G(x)}{\lambda(1 + G(x))} \right) + \lambda \left( 1 - 1/\lambda - h + (r - \pi)b + \sum_{j=1}^{n} (r_j - \pi)x_j \right) \quad (\sigma = 1). \]

The next step in the application of the theorem is showing that the maximized Hamiltonian is concave with respect to the state variables. Since the term in \( b \) is linear, the only variables we have to care about are those in the vector \( x \). More precisely, those which are not in the linear term \( \sum_{j=1}^{2} (r_j - \pi)x_j \). The Hamiltonian is trivially concave in the case \( \sigma = 1 \) since \( G(x) \) is concave and increasing in \( x \) and, given \( \lambda \left( \frac{G(x)}{\lambda(1 + G(x))} \right) \geq 0 \), and taking \( G(x) \) as a variable, \( \log \left( \frac{G(x)}{\lambda(1 + G(x))} \right) \) is a composite function of two monotone increasing concave functions. When \( \sigma \neq 1 \), note that the term \( \frac{\sigma}{1 - \sigma} \left( \frac{G(x)}{\lambda(1 + G(x))} \right)^{(1-\sigma)/\sigma} \) in the maximized Hamiltonian is concave in \( x \) (by the same result that composite functions of increasing and concave functions are concave) provided that

\[ \sigma \geq \frac{1}{2}. \]

(13)

Note that condition (13) makes \( U(G(x)s) \) concave in both \( G \) (by these means, also in \( x \)) and \( s \) in the original problem, but not the original Hamiltonian (12). Therefore, Mangasarian’s (1966) sufficient conditions cannot be used even when (13) is assumed.

The extension of the solution to the non-convexity problem detailed above to Lucas (2000) or to Cysne et al. (2005) is straightforward.

The intuition for the result given by (13) is presented in Figs. 1 and 2.

Fig. 1 presents the case in which the coefficient of risk aversion (CRA) is high enough. The feasible region of maximization is determined by the level curve of the term multiplying \( \lambda \) in (12). Even though this equation (through its isoquant) determines a non-convex feasible region in the \((G(x), s)\) plane (the shadowed region in the figures), if the curvature of the utility function is high enough the non-convexity poses no problem. Fig. 2 presents the alternative case.

\(^6\) Working with \( a = b + \sum_{j=1}^{2} x_j \) as the only state variable (and \( s \), and the \( x_j \)’s as control variables) leads to the same first-order conditions as when one formulates the problem regarding only \( s \) as a control variable and the \( x_j \)’s and \( b \) as state variables. Such first-order conditions obey \(-\dot{\lambda}(t) + g\dot{\lambda}(t) = H_s\) and \(-\dot{\lambda}(t) + g\dot{\lambda}(t) = H_b\) which is the property required from the state variables in the application of the theorem.
4. Applications to human-capital models

In this section I repeat the procedure of the last section, however using a human-capital model, rather than a shopping-time model. Non-convexities in this literature usually arise because human-capital accumulation is modelled as a linear function of the stock of...
human capital. This linearity is a tool used in the attainment of a sustained balanced-growth path.

The analysis here concentrates on Lucas (1988). The reason is that the technology of human-capital accumulation (Eq. (13) in Lucas’s work) used in this paper has been used by many other authors in the literature. This note focuses solely on the optimization problem solved by the representative consumer which, in the terminology used by Lucas, leads to the competitive-equilibrium path (as opposed to the efficient path determined in the problem solved by the social planner). In this case, the representative consumer takes the average human capital \( h_a \) as a given. Further insight on the social-planner problem is provided by footnote 8.

4.1. Lucas (1988)

Preferences over consumption streams are (I omit the argument \( t \) of the functions in order to simplify the notation) given by

\[
\int_0^\infty e^{-\rho t} \frac{N}{1-\sigma} (c^{1-\sigma} - 1) \, dt, \quad \sigma \neq 1, \quad \sigma > 0, \quad 0 < \rho < 1.
\]

(14)

Human capital (\( h \)) accumulates according to

\[
\dot{h} = h \delta (1 - u).
\]

(15)

Above, \( u \) is the fraction of non-leisure time devoted to production, a control variable in the optimum path chosen by the representative consumer. \( N \) is the total number of workers and \( uN h \) is the effective workforce used in the production of the consumption good. With \( K \) standing for the level of physical capital, the technology of goods production is

\[
Nc + \dot{K} = AK^\beta (uNh)^{1-\beta} h_a^\gamma, \quad A > 0, \quad 0 < \beta < 1, \quad \gamma \geq 0.
\]

(16)

The last term in the second member of Eq. (16), \( h_a^\gamma, \gamma \geq 0 \), stands for the externality of the level of human capital in the production of the consumption good. As mentioned above, in the problem I analyze here, this term is a given.8 As explained in the example in the previous section, we can use here the current-value Hamiltonian, which in this case is then given by

\[
H(K, h, \theta_1, \theta_2, c, u; h_a) = \frac{N}{1-\sigma} (c^{1-\sigma} - 1) + \theta_1 [AK^\beta (uNh)^{1-\beta} h_a^\gamma - Nc] + \theta_2 [\delta h(1 - u)].
\]

(17)

\( \theta_1 \) and \( \theta_2 \) are multiplier functions that give the marginal value of the state variables \( K \) and \( h \), respectively, discounted back to time zero. Both \( \theta_1 \) and \( \theta_2 \) are non-negative, an implication of the fact that, in the point of optimum, \( H_c = h_u = 0 \).

This Hamiltonian is clearly non-concave in the control and state variables, due to the terms \( K^\beta (uNh)^{1-\beta} \) and \( h(1 - u) \). Therefore, Mangasarian’s conditions cannot be used here.

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7 As in Eq. (15) below.
8 If a social planner solves the same problem, then the average level of human capital is no longer taken as given. In that version, therefore, it can easily be shown that Arrow’s sufficiency condition holds if and only if \( \gamma = 0 \). This result would (with \( \sigma = 0 \), linear utility) correspond to Lucas’ version of Uzawa’s (1965) paper.
The Hamiltonian is strictly concave in \((c, u)\). Using the unique optimal values of \(c\) and \(u\) in (17), the maximized Hamiltonian reads
\[
H^*(K, h, \theta_1, \theta_2; h_a) = \frac{N}{1 - \sigma} \left( \theta_1^{\frac{\sigma}{\beta}} - 1 \right)
+ \frac{\theta_1 AN^{1-\beta} \left( \frac{\theta_2 \delta}{\theta_1 AN^{1-\beta}(1 - \beta)} \right)^{-1/\beta}}{h_a K}
- \theta_1^{\frac{\sigma}{\beta}} N + \theta_2 \delta \left[ h - \left( \frac{\theta_2 \delta}{\theta_1 AN^{1-\beta}(1 - \beta)} \right)^{-1/\beta} \right].
\]

The optimized Hamiltonian is always concave (though not strictly concave) in the state variables \(K\) and \(h\). Therefore, we can conclude that the first-order conditions derived above do represent a (not-necessarily unique) optimum for the problem solved by the representative consumer.

### 4.2. Externality in the production of human capital

In this subsection I follow the line worked out by Parente and Prescott (1994) and place the externality in the production of human capital, rather than in the production of the consumption good. The analysis is simplified by considering an economy without physical capital. The technologies and the respective Hamiltonian read
\[
y = Auh, \quad A > 0,
\]
\[
\dot{h} = \delta(1 - u) h^{1-\theta} h_a^\theta, \quad 0 < \theta < 1,
\]
\[
H(u, h, \lambda; h_a) = \frac{(Ahu)^{1-\sigma} - 1}{1 - \sigma} + \lambda \delta(1 - u) h^{1-\theta} h_a^\theta.
\]

This Hamiltonian is not jointly concave in \((u, h)\), particularly because of the term \((1 - u) h^{1-\theta}\) in (20). Using the optimal value of \(u\) we obtain the maximized Hamiltonian
\[
H^*(h, \lambda; h_a) = \frac{A^{1-\sigma} \left( \frac{\theta}{\sigma \sigma} \right)^{1/\sigma} \left( \frac{\theta}{\sigma \sigma} \right)^{1/(1-\alpha)} h^{\theta(1-\alpha)} h_a^{\theta(1-\alpha)} - 1}{1 - \sigma}
+ \lambda \delta \left[ h^{1-\theta} h_a^\theta - \left( \frac{\delta \lambda}{A^{1-\sigma}} \right)^{\frac{1}{\alpha}} h^{\theta(1-\alpha)} h_a^{\theta(1-\alpha)} \right].
\]

The maximized Hamiltonian is concave in the state variable \(h\), provided that
\[
\sigma \geq \frac{\theta}{1 + \theta}.
\]

Condition (22) has the same aspect as condition (13). It requires that the coefficient of risk aversion provides a curvature great enough to the utility function. Again, the conclusion that the first-order conditions characterize optimality could not have followed from Mangasarian’s sufficient conditions.
5. Conclusion

In this work I have chosen some papers in the shopping-time and in the human-capital literature in which the usual Mangasarian’s sufficiency conditions for optimality in control-theory problems are not satisfied. In such cases, Pontryagin et al.’s (1962) Maximum Principle cannot tell us if a point satisfying the first-order conditions represents an optimum or not.

Next, I have shown, in each case, that quantitative insights can be provided and that optimality can be characterized by means of an application of Arrow’s (1968) sufficiency theorem.

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