# Time Trumps Quantity in the Market for Lemons *

William Fuchs, Piero Gottardi and Humberto Moreira†

September 21, 2022

**Abstract**

We consider a dynamic adverse selection model where privately informed sellers of divisible assets can choose how much of their asset to sell at each point in time to competitive buyers. With commitment, delay and lower quantities are equivalent ways to signal higher quality. Only the discounted quantity traded is pinned down in equilibrium. With spot contracts and observable past trades, there is a unique and fully separating path of trades in equilibrium. Irrespective of the horizon and the frequency of trades, the same welfare is attained by each seller type as in the commitment case. When trades can take place continuously over time, each type trades all of its assets at a unique point in time. Thus, only delay is used to signal higher quality. When past trades are not observable, the equilibrium only coincides with the one with public histories when trading can take place continuously over time.

*We thank Andres Almazan, Diego Moreno, Andy Skrzypacz, Tom Wiseman and Pavel Zryumov for comments and suggestions. We also thank seminar audiences at Bocconi, Durham, UC3M, UT Austin, Finance Theory Webinar and BSE Summer Forum. Moreira acknowledges FAPERJ and CNPq for financial support. This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001. Fuchs gratefully acknowledges the support from ERC Gran 681575, Grant PGC2018-096159-B-I00 financed by MCIN/ AEI/10.13039/501100011033, and Comunidad de Madrid (Spain), grant EPUC3M11 (V PRICIT) and grant H2019/HUM-5891

†Fuchs: University of Texas at Austin and Universidad Carlos III de Madrid. Gottardi: University of Essex and University of Venezia. Moreira: FGV EPGE Brazilian School of Economics and Finance.

# 1 Introduction

In adverse selection settings, it is well understood that the seller of an asset can use higher retained ownership or delay its trades to signal higher quality when only one of such instruments is available. How does signaling (or screening) take place when both instruments are available? We show that with linear preferences, under long-term commitment to a path of trades, the only relevant consideration is the discounted value of the total quantity sold. Either delay or fractional trade (or a combination of both) can equivalently be used to achieve separation. Instead, when only spot contracts are allowed, in equilibrium delay is exclusively used to signal quality. Time trumps quantity as a signal because the irreversibility of time provides a form of endogenous commitment which cannot be achieved with retained ownership. Once I have waited for a given amount of time to sell my asset I cannot go back and sell it earlier. Instead, an implicit promise not to sell more of the asset at a future date can always be broken.

Concretely, we study a model with several privately informed sellers of a perfectly divisible asset facing a competitive fringe of buyers who have a higher value for the asset. The valuations for the asset are linear in quantity for both buyers and sellers. Trading can take place at a pre-specified set of times. We first consider a benchmark case in which there is commitment to future trades and exclusive dealings. We show that all that really matters is the discounted quantity sold and the average per period price obtained. Thus, although all equilibria are separating and there is a unique present discounted total quantity $Q^*(c)$ sold by each seller type $c$ in any equilibrium, the exact path of trades is not pinned down (see Theorem 1). Importantly, the value of $Q^*(c)$ and the welfare of each type of seller is the same regardless of the frequency with which sellers can trade. This is in contrast to the findings of Fuchs and Skrzypacz (2019) (henceforth FS) which study the case of an indivisible asset and show that the frequency of trade can have important welfare implications. The reason for the differing results is that, when the asset is non-divisible, the set of values that $Q(c)$ can take is determined by how frequently you can trade. With the cardinality of set increasing as the interval between trades shrink. With a restricted set of quantities available to signal with, types cannot fully separate and some pooling ensues. Instead, with a fully divisible asset and commitment, the change in frequency of trade has no impact on the set of possible discounted quantities you can offer and thus there are no welfare effects on equilibrium.

When trade occurs via exclusive spot contracts without commitment, we must consider two cases: 1) Public trading: all past trades by any seller are observable and thus prices at a given date depend not just on the current quantity traded but also on the observed history of trades and 2) Private trading: buyers do not observe past trades and thus prices only depend on the quantity being traded in the current period.

With public trading, when the market is open at discrete times $0, \Delta, 2\Delta, \ldots, T$ there is a unique equilibrium, where each type of seller trades in at most two consecutive periods, exhausting its supply (except for those only trading at $T$). The equilibrium is again separating and, for all $T$ and $\Delta$, the present discounted quantity traded, $Q(c)$ coincides with $Q^*(c)$ from the commitment benchmark. Thus, sellers' welfare is also the same for any trading frequency (as with commitment and in contrast to the case where the asset is indivisible). As trading frequency increases, i.e., $\Delta \to 0$, we get that the measure of types trading at a given time goes to zero. That is, trade can be characterized in the continuous trading limit by a function $\tau(c)$ in which each type $c$ trades its full supply at date $\tau(c)$ such that $Q^*(c) = exp(-r\tau(c))$, where $r$ is the discount factor. Thus, time and not quantity is used to signal the quality of the seller's asset.

With private trades, the frequency of trade is relevant again. While we can still sustain the same equilibrium as with public trading when agents can trade continuously, this is no longer true when we consider a set of discrete trading times. The reason is that when trading is continuous, when a type trades, it trades its full supply. Thus, on path, every type has either a full unit or nothing left. Instead, with discrete trading, partial trades also need to be used to attain $Q^*(c)$. As a result, at any given trading time, with public trades, there will be two prices for any given quantity (one price for a seller that has not sold any quantity in the previous period and a different price for the seller that also traded in the previous period). With private trades this conditioning on past trades is not possible. Thus, the same equilibrium cannot be sustained. Note that the quantity traded at any given date is in principle a sufficiently rich instrument to screen sellers' types, but requires commitment (or the observability of trades at earlier dates); without commitment it proves ineffective. Hence the reason why the set of trading dates matters is quite different from the one pointed out by FS. Another way to understand our result is that we get an additional source of private information, the residual supply.

The shape of the residual supply, which evolves endogenously, is critical because it affects which trades a given type can enter. To see this, suppose the residual supply were strictly upward sloping at some given time. That is, higher quality types have strictly more of the asset to sell. In such a situation, trade could take place efficiently. Although low types would like to pretend to be higher types to get a better price, they do not have the necessary quantity to imitate the higher types. In general, when past trades are not observable the supply varies endogenously over time across types, at each date we have so a market with multidimensional private information. Identifying which are the binding incentive constraints and characterizing the properties of the equilibrium with private trades in a discrete setting is therefore quite challenging. Despite this we are able to characterize the properties of equilibria when there are two trading dates (See Proposition 3). The equilibria with unobservable past trades may exhibit pooling and we show that welfare is not generally ranked across the two information regimes (See Corollary 2).

## Related literature

Our work intersects several strands of the literature on trade with adverse selection. First we have the dynamic models of a market for an indivisible asset that build on the static model of Akerlof (1970). Janssen and Roy (2002) presented the first analysis of the dynamic case and FS provided a comprehensive analysis of this case, discussed in the previous section. Second, and closely related are the models with an indivisible asset (or worker) which build on Spence (1973). This literature (which is focused on the two type case) is concerned with the assumption in Spence's model that the student can commit to go to school for some given amount of time. This is important because if only good types go to school (and assuming schooling is not productive), after the first day of class, firms would potentially want to make offers to this student. While Nöldeke and van Damme (1990) show that the commitment equilibrium can be implemented with public offers and uniquely so under a refinement, Swinkels (1999) shows that this no longer holds true when the offers are private. With private offers, the unique equilibrium has immediate trade with both types and thus is more efficient. In our setting the payoffs of the commitment equilibrium are always obtained when trades are publicly observable, and also when they are privately observable provided trade is continuous. Additional contributions to the public vs. private offer case analysis include Hörner and Vieille (2009) and Fuchs et al. (2016). The latter shows that private offers are more efficient since buyers will be more willing to make higher offers when rejecting them cannot be used by sellers to obtain higher offers in the future. The privacy of previous traded quantities, which we study in this paper, works quite differently. It creates a two dimensional private information problem which, might lead to pooling when trades cannot take place continuously.

The dynamic bargaining literature for an indivisible durable asset is also related. In these models, the uninformed party starts by making unattractive offers and screens the informed party by only slowly improving the offers as time goes by. The time between offers plays an important role since it represents the power of commitment of the uninformed party. In the limit, as offers can be made continuously (i.e. no commitment power) the Coase conjecture forces kick-in (Coase (1972)) and the uninformed party loses any ability to profitably screen. When seller and buyer valuations of the asset are independent of each other, the limit of the unique stationary equilibrium exhibits immediate trade with all types pooling. Instead, with correlated values (as in this paper) there is slow screening over time.[1] Gerardi et al. (2022) study the role of asset divisibility in this context with two types. They show divisibility matters when there are decreasing gains

[1]See Ausubel et al. (2002) for an excellent survey of the earlier literature and Fuchs and Skrzypacz (2022) for an overview of more recent work.

from trade in the quantity traded of the asset, not when gains from trade are constant, the situation considered in this paper. We show that when the asset is divisible the commitment equilibrium is obtained also with spot trades, regardless of the frequency of trading (when trades are public).

There is also a literature that studies adverse selection in the context of directed search models (see Gale (1992) and Guerrieri et al. (2010)). These models show, in the context of markets for an indivisible object and a single trading date, that perfect separation occurs via the probability of trade associated to the different price offers made by buyers. High price markets attract few buyers relative to sellers and thus for a given seller there is a low probability of trade. Conversely, low price markets have many buyers relative to sellers so the sellers can sell with very high probability.[2] Williams (2021) (see also Guerrieri et al. (2010)) allows for divisibility of the asset in this environment and shows that separation occurs exclusively via probability of trade, not quantity traded. The rationale is based on the fact that using the trading probability allows to save on trading costs. Relative costs do not play a role in our result that only time of trade, not quantity traded, is used to separate seller types. Instead, the key factor we highlight is the differential commitment embedded in these instruments.

The seminal work by Leland and Pyle (1977) showed in a static setting with a divisible asset that the seller can retain of part of the asset to signal his higher type and obtain a higher price. This idea has lead to a large body of work in finance. In particular, it was extended by DeMarzo and Duffie (1999) to a security design problem, where the firm can choose not only the fraction of its cash-flow it retains, but how this is split in the different contingencies between the firm and outside investors. Yet, implicit in this literature is the idea that the owner is committed not to access the market again. As we show, this is very important since the temptation to sell the rest of the asset undermines the ability to signal via retention.[3]

[2]The properties of the dynamics of trade obtained in random search models with adverse selection are shaped by rather different forces, given by the interaction between bargaining and competition (see, e.g., Moreno and Wooders (2016)).

[3]See Li (2022) for some related work along these lines.

Finally, the unobservability of past trades introduces an element of non-exclusivity in the contracts being traded and may prevent perfect separation in equilibrium. In this regard our paper is also related to Attar et al. (2011), Kurlat (2016), Auster et al. (2021) and Asriyan and Vanasco (2021). While these papers focus on non-exclusivity within a period, we discuss non-exclusivity across periods. Thus, the time between periods can also be interpreted as a measure of commitment to exclusivity.

# 2 Model

There is unit mass of sellers. Sellers are indexed by the quality, $c \in [0,1]$, of the asset they own. Sellers are fully informed about the quality of their asset, while buyers are uninformed and only know that the quality of each asset is independently drawn from a distribution $F(c)$ with density $f(c) > 0$. Assets are perfectly divisible and each seller holds one unit of it.[4] An asset of quality $c$ is worth $c$ per unit to the sellers and $v(c)$ to any potential buyer. We assume $v(c) > c$ for all $c \in [0,1)$, $v(1) = 1$, $v(\cdot)$ is continuously differentiable and $v'(c) > 0$. That is, there are strict gains from trade with all seller types with the exception of the highest and types are ordered so that higher types have assets that are better for both sellers and the buyers.[5]

Buyers and sellers can trade at an exogenously given set of trading dates, given by $\varsigma \subseteq [0,\infty]$, at which the market is open. We use this notation to be able to capture different possibilities, from one shot, static trade $\varsigma = \{0\}$ to continuous trading $\varsigma = [0,\infty]$. At each trading date $t \in \varsigma$ a large mass of buyers is present. Each buyer is only active in the market at a single trading date.[6] A seller's strategy (defined more carefully later) is a choice over the path of trades to carry out.

All players are risk neutral, have linear preferences over the quantity of the asset they

[4] We consider the case where sellers are endowed with different amounts of assets in Section 6.

[5] The assumption $v(1) = 1$ can be easily relaxed. The fact that the highest type has no benefits from trade helps to simplify the analysis.

[6] This assumption only allows to simplify notation when there is commitment to future deliveries or past trades are observable. When instead past trades are not observable it precludes buyers from conditioning their offers on their own private trade histories. See Lee (2021) for a study of the case where buyers are long lived in a simpler setting.

hold and discount the benefits and costs of future trades at the rate $r$. For sellers this means that, if they expect to sell an amount $q(t)$ at date $t$ with per unit price $p(t)$, their utility from that trade, in present expected terms, is:

$$e^{-rt}q(t)\left(p(t)-c\right).$$

Similarly, the utility of buyers for such trade would be:

$$q(t)E\left[v(c)-p(t)\right],$$

where the expectation is taken according to the buyers' belief described below.

We will study different market arrangements in terms of information available to buyers over past trades and commitment of the sellers regarding future trades. Formal definitions of equilibrium for each case will be provided in the sections below.

In all the cases we will assume that buyers and sellers trade in competitive markets and face a given set of prices for all feasible contracts. We will thus not model buyers as strategic players. Instead, the optimizing behavior of buyers in a competitive equilibrium implies that: a) buyers earn an expected payoff equal to zero on all contracts that are actively traded, and b) contracts that are not traded in equilibrium must not be profitable for buyers to trade. To assess the profitability of each contract the equilibrium must also specify buyers' beliefs about the composition of sellers trading each contract. We assume that, at any given trading date, a seller can enter at most one contract. Every seller, taking prices as given, chooses then at which dates to trade and which contract to enter. There are no search frictions in the market, thus every seller can always trade the contract selected.

In particular, we will consider the following cases:

$(i)$ *Commitment to long term contracts*: buyers and sellers enter at date $0$ an exclusive contract that precludes any further trades and specifies the amount of the asset to be transferred from the seller to the buyer at every date $t \in \varsigma$ and the price to be paid.

$(ii)$ *Short term (spot) contracts with observable trades*: at any trading date $t \in \varsigma$ buyers and sellers can enter a spot contract, which specifies the quantity to be exchanged at that date and the price to be paid. All past trading activity by every seller is observed by the buyers present at $t$, hence the terms of trade at that date can be made contingent on past trades.[7]

$(iii)$ *Short term (spot) contracts with unobservable trades*: buyers and sellers can again enter a spot contract at any date $t \in \varsigma$. However, in this case all past trading activity by the sellers is unobservable to buyers present at $t$ and hence is not contractible.

## 3 A Benchmark: Long term Contracts with Commitment

If traders at $t = 0$ can fully commit to long term contracts and cannot trade additional contracts at subsequent dates after entering a contract, then effectively we can study the problem as one in which contracts are only traded at $t = 0$. All transactions (current and future) are determined by the contract entered at the initial date. Let us denote by $\omega$ a generic contract parties can enter at $t = 0$. We say the contract is *feasible* for a seller if it specifies a total amount to be delivered that is compatible with the seller's endowment: $\int_{t\in\varsigma} dq(t;\omega) \leq 1$, where the integral is with respect to the positive measure $q(\cdot;\omega)$ on $\varsigma$. We assume sellers can only enter into feasible contracts. Let $\Omega$ denote the space of feasible contracts.

Given the linearity of preferences, we can focus without loss of generality on the case where the price of a generic contract $\omega$ is paid by the buyer at the initial date $t = 0$. Regarding the transactions that are prescribed, all that matters for the utility of the sellers and the buyers is the total discounted amount to be delivered $Q(\omega) = \int_{t\in\varsigma} e^{-rt} dq(t;\omega)$. The per unit price of that contract is denoted by $P(\omega)$. The utility gain of entering contract $\omega$ at price $P(\omega)$ for a seller of type $c$, $W(c,\omega)$, is then:

$$Q(\omega)\left(P(\omega) - c\right) = W\left(c,\omega\right).$$

Let $U(c)$ denote the maximal payoff attainable by a type $c$ seller when trading in the

[7]For example we can have two distinct contracts for half a unit available for trade at $t$, at different prices, one for sellers that have not sold anything yet and one for sellers that have sold a given positive amount at some earlier date.

market, given the set of feasible contracts and the set of associated prices:

$$U(c) = \max_{\omega \in \Omega} W(c, \omega). \tag{1}$$

In a competitive equilibrium, taking as given the price of each contract in $\Omega$, every type of seller chooses the contract that gives it the highest utility, buyers optimally choose which contract to trade and markets clear. We will use $K(\omega)$ to denote the set of seller types choosing contract $\omega$. More formally:

**Definition 1** (Competitive Equilibrium with Commitment). *A competitive equilibrium in the market with commitment is given by: a price $P(\omega)$ for each contract $\omega \in \Omega$, a set of contracts $\Omega^a \subset \Omega$ that are actively traded in equilibrium, a system of buyers' beliefs specifying for each contract $\omega \in \Omega$ and each seller type $c$ the probability measure $\mu(c;\omega)$ that contract $\omega$ is traded by type $c$, and a contract choice*[8] *for each seller type $c$, $\omega^*(c) \in \Omega^a$, such that:*
*(i) Sellers optimize: (1) holds, for all $c$, i.e., the payoff of a type $c$ seller when it chooses contract $\omega^*(c)$ is greater or equal than its payoff for all other contracts in $\Omega$;*
*(ii) On-path beliefs and prices: for all $\omega \in \Omega^a$ buyers' beliefs are formed using Bayes rule and prices are such that they break even: $E[v(c)|c \in K(\omega)] = P(\omega)$.*
*(iii) Off-path beliefs and prices: for each non traded contract $\omega' \in \Omega/\Omega^a$, the belief $\mu(\cdot;\omega')$ gives positive weight only to type*[9] *$c' = \arg\min_c \{U(c) + cQ(\omega')\}$, the price is $P(\omega') = \min_c \{U(c) + cQ(\omega')\}$ and is such that, given these beliefs, trading contract $\omega'$ is not profitable for buyers.*

Conditions (i) and (ii) are standard competitive equilibrium conditions and say that traders optimize, markets clear and beliefs for contracts actively traded are pinned down by Bayes rule. Condition (iii) says that, for any contract $\omega'$ that is not traded in equilibrium, buyers believe that, if they were to enter such a contract, the type of seller this contract would attract is the type most willing to trade such contract. At the equilibrium price for $\omega'$, $P(\omega') = \min_c \{U(c) + cQ(\omega')\}$, only the type of seller $c'$ for whom $\mu(c';\omega') > 0$

[8] We write for simplicity the definition for the case where each type of seller chooses one contract, though in equilibrium it is possible that sellers of a given type select different contracts, when they are indifferent among them. This is not relevant with a continuum of types.

[9] Uniqueness of such type follows from the single crossing property.

is indifferent between trading or not this contract, while all other types prefer not to trade it. This implies that buyers' beliefs are correct also out of equilibrium, should buyers deviate and trade contract $\omega'$: the seller type given positive weight is the only type of seller willing to trade the contract at the equilibrium price $P(\omega')$. This specification of the beliefs for non-traded contracts is analogous to the conditions imposed in other competitive models with adverse selection, such as Azevedo and Gottlieb (2017), Bisin and Gottardi (2006), Dubey and Geanakoplos (2002), as well as in competitive search equilibria, see Guerrieri et al. (2010) and Eeckhout and Kircher (2010).[10] Given these off equilibrium beliefs, the equilibrium then requires that buyers are not willing to trade contract $\omega'$ at $P(\omega')$.

We characterize next the equilibria in this case, an important benchmark result.

**Theorem 1.** *When sellers can commit to long term contracts, for all $\varsigma$ all competitive equilibria are perfectly separating: that is, for any contract that is traded in equilibrium, there is a unique seller type who enters it. The utility level attained by each seller type $c$, $U^*(c)$, is the same across all these equilibria, and so is the total discounted quantity traded by this type, $Q^*(\omega(c))$. More specifically, we have:*

$$Q^*(\omega(c)) = \exp\left[-\int_0^c \frac{v'(x)}{v(x)-x}dx\right]$$

*strictly decreasing in $c$, and the per-unit equilibrium price is $P^*(\omega(c)) = v(c)$, for all $c$.*

The formal proof can be found in the Appendix. The idea is the following. The single crossing property implies that if two types $c'$ and $c''$ choose the same contract $\omega$ then all types $c \in [c', c'']$ must also choose contract $\omega$. Then one can show by contradiction that there cannot be any pooling in equilibrium. This follows since the highest type in the pool, $c''$, would find it attractive to deviate and trade a slightly lower quantity. If this

[10] It is worth noting that the "No Deals" condition used in some models (see for example Daley and Green (2012) with non-divisible assets can be problematic in the divisible case because it may lead to non-existence of equilibria. This is related to the non-existence results first pointed out by Rothschild and Stiglitz (1976) and discussed in Section 4.2 of Attar et al. (2009) for the divisible case with linear preferences and a continuum of types. Note however that, in the equilibrium outcomes characterized in our main results for the continuous time case, Theorems 2 and 3, there would be no profitable deviations for buyers even when they can act strategically, as long as the contracts offered are observable. Yet, we prefer the competitive formulation to avoid existence problems in the static or discrete time case.

alternative quantity is on-path, it must be traded by a seller of a higher type than that of any seller in the pool and hence its price must be strictly higher given the buyers' break even condition, so the deviation is profitable for seller $c''$. Otherwise, if this quantity were off-path, condition (iii) would imply that the off-path belief must be that the quantity is traded by a type equal or higher than $c''$ and its equilibrium price is lower than $v(c'')$. This however would yield a contradiction since buyers would then find this contract strictly profitable.

Given that the equilibrium is separating we have $P^*(\omega(c)) = v(c)$ and we use the IC constraints of the sellers to obtain a differential equation which, together with the fact that the lowest type must trade all its supply, allows us to uniquely determine $Q^*(\omega(c))$.

**Remark 1.** *Note that the equilibrium only pins down the value of $Q\,(c)$, but any contract specifying a sequence of trades such that the total discounted amount is equal to $Q^*\,(c)$ constitutes a contract that may be traded by type $c$ sellers in equilibrium. Hence the amount $Q^*\,(c)$ could be obtained by trading the full unit at a specific time or a smaller amount only at $t = 0$. With commitment there is no sense in which screening via different quantities traded by different types at a given date or via the same quantities traded by various types at different dates prevails. We should add that, if we also allowed for contracts with stochastic deliveries, probabilities of trade would also yield the same properties as they could be used to generate the same discounted, expected amount of total trades. In search models, the probability of trade induced by market tightness indeed plays a similar role to delay or partial trade. The equivalence also relies on all types having the same supply. We illustrate this in Section 6.*

**Example 1.** *To further illustrate Remark 1, consider the case in which $v(c) = 1/2 + c/2$. In this case the equilibrium level of total discounted trades of an arbitrary seller type $c$ is:*

$$Q^*(\omega^*(c)) = \exp\left[-\int_0^c \frac{v'(x)}{v(x)-x}dx\right] = 1 - c.$$

*Thus, for any $\varsigma$ we have an equilibrium allocation where there is only trade in the first trading date and all the separation is via quantities: $q(0;\omega^*(c)) = 1 - c$ and $q(t;\omega^*(c)) = 0$. Letting $\varsigma = [0,\infty]$ we can also have a payoff equivalent equilibrium allocation in which all types sell* *their full unit when they trade, and separation is achieved because they trade at different times:* $q(t;\omega^*(c)) = 1 \quad if \quad e^{-rt} = 1 - c$ *and 0 otherwise.*

*We could also have cases in which both time and quantity are used. Letting* $\varsigma = \{0, \Delta\}$*, we can have:*

$$q(0;\omega(c)) = \begin{cases} 1 - \frac{c}{1-e^{-r\Delta}} & if \quad c \leq e^{-r\Delta} \\ 0 & if \quad c > e^{-r\Delta} \end{cases} \qquad q(\Delta;\omega^*(c)) = \begin{cases} \frac{c}{1-e^{-r\Delta}} & if \quad c \leq e^{-r\Delta} \\ \frac{1-c}{1-e^{-r\Delta}} & if \quad c > e^{-r\Delta} \end{cases}$$

*and for all other* $t \in \varsigma$, $q(t;\omega^*(c)) = 0$.

*Note in particular that in the last specification we have two contracts that prescribe the delivery of the same quantity at* $t = \Delta$ *but have different (unit) prices since they feature the delivery of different quantities at* $t = 0$*. A similar property can obtain even without commitment as long as past trades are observable, as we will show next.*[11] *Instead, when past trades are not observable this will no longer be possible. This case will be analyzed in Section 5.*

# 4 Spot Contracts with Observable Trades

We turn next to the case in which there is no commitment to future transactions but buyers at any given time $t$ can observe and condition the price of contracts on all the previous trades of a particular seller. We first analyze this case in discrete time. That is, we restrict $\varsigma$ to be $\varsigma = \{0, \Delta, 2\Delta, .., T\}$, where $T \leq \infty$. It will be also convenient to use $q_t$ to indicate the trade $q(t)$ at time $t$.[12] The important difference with respect to the commitment case is that we need to specify a price for each feasible contract for all $t \in \varsigma$. Any spot contract at $t$ that implies a trade at that date that is not greater than the residual supply of a seller, resulting from all the trades at earlier dates, is feasible. Sellers can only enter one contract at a given $t$. Importantly, as we said, the per unit prices for the contracts at time $t$ are allowed to depend on all the past history of trades $q^{t-\Delta} = \{q_{0,}q_{\Delta}, ...q_{t-\Delta}\}$ in addition to

[11] Without commitment, the seller is still free to choose its trades at future times $s > t$. Thus, the key difference is that prices at $t$ can be conditioned on past quantities but not conditioned on future quantities.

[12] For most results we do not need trades to be equally spaced but this assumption helps us to reduce notation.

the current quantity $q_t$. We will denote them $p\left(q^{t-\Delta}, q_t\right)$ or, in short, $p\left(q^t\right)$.

Since sellers cannot commit to a sequence of trades, their choice of trades must be optimal at every $t$. We must therefore consider the whole sequence of traded quantities $\{q_t\}, t \in \varsigma$. At all times $t$, after having traded the sequence of quantities $q^{t-\Delta}$, the problem of a type $c$ seller, can be written recursively as follows:

$$U\left(c, q^{t-\Delta}\right) = \max_{q_t} \; q_t\left(p\left(q^t\right) - c\right) + e^{-r\Delta} U\left(c, q^t\right) \tag{2}$$

$$s.t. \quad \sum_{s=0}^{t} q_s \leq 1.$$

Let $q_t^*(c)$ denote the on-path optimal quantity choice of type $c$ at time $t$, after the sequence of trades $q^{t-\Delta}$. For a given contract $q^t$ let $K\left(q^t\right)$ denote the set of types that optimally choose to trade that contract on-path.

**Definition 2** (Competitive Equilibrium with Observable Trades)**.** *Given $\varsigma$, a competitive equilibrium in the market with observable trades is given by: prices $p\left(q^t\right)$ for each feasible contract at every $t \in \varsigma$, a set of contracts that are actively traded in equilibrium at each $t \in \varsigma$, the sequence of contracts traded over time by each type $c$ and a system of buyers' beliefs specifying for each contract a probability measure over the types trading that contract at every date $t$, such that:*
*(i) Sellers optimize: (2) holds, for all $c$ and at every $t$;*
*(ii) For all traded contracts buyers' beliefs are determined by Bayes rule and prices satisfy the break even condition: for all $q^t$ such that $K\left(q^t\right) \neq \emptyset$, $p\left(q^t\right) = E\left[v\left(c\right) | c \in K\left(q^t\right)\right]$;*
*(iii) Off-path beliefs and prices: for all $q^t$ such that $K\left(q^t\right) = \emptyset$, buyers' beliefs give positive weight only to the type $c' \in [0,1]$ that is most willing to trade contract $q^t$. The price of this contract and of the optimally chosen continuation sequence of trades is such that type $c'$ is indifferent between this deviation and its equilibrium trades while all other types are worse off and buyers, given these beliefs, find the contract not profitable.*

It is worth noting that in $(iii)$ we consider the whole set of sellers' types when considering who might have deviated. That is, we take a time zero perspective and consider which type of seller would lose the least from taking the sequence of trade $q^t$ together with some optimally chosen continuation sequence of trades.[13] The logic of this condition is analogous to that of $(iii)$ in Definition 2, with the complication that here, to assess the profitability of the deviation to $q^t$ for the different types of sellers we need to find for each of them the optimal continuation trades and the lowest price at which they are willing to carry out those trades.

## 4.1 Characterization of Equilibria

We show next that, with observable trades, there is a unique equilibrium for any $\Delta$ and $T$. We begin by establishing two important properties. First, Lemma 1 shows that every individual trade must be separating and thus every traded contract is priced by the unique type trading that contract. Second, Lemma 2 shows that there is also full separation with regard to the present discounted value of the total quantity traded $Q(c)$.

**Lemma 1** (Separation in q)**.** *For every $\Delta$ and $T$, in all observable trade equilibria, every on-path sequence of nonzero trades from date $0$ to date $t$, for all $t$, must be fully separating.*

*Proof.* We proceed by contradiction, supposing some pooling trade occurs. Consider the highest type $c''$ that ever participates in a pooling trade[14] and let $t$ be the date at which this happens. Let $Q(c'')$ be the total discounted quantity traded by $c''$ in equilibrium and $P(c'')$ be the average unit price paid. Note that $P(c'') < v(c'')$ since $c''$ pools with lower types for at least some trade, never pools with higher types, and buyers must break even.

Consider instead a deviation where $c''$ trades the first time quantity $q_\tau$ at time $\tau$ and all its residual supply immediately afterwards so that:

$$\begin{aligned} e^{-r\tau}q_\tau + e^{-r(\tau+\Delta)}(1-q_\tau) &= Q(c'') - \varepsilon \quad if\ \tau \leq T-\Delta \\ e^{-rT}q_T &= Q(c'') - \varepsilon \quad otherwise \end{aligned}$$

for some small $\varepsilon > 0$. Suppose first that these trades were on path. Then, since they imply

[13]Our results would also hold if we considered only the types who had traded the on-path portion of $q^t$.

[14]For ease of exposition we assume the set of types pooling includes its supreme but a similar argument can be constructed even if this property did not hold.

a smaller total discounted quantity, by the single crossing property they must be carried out by some type $c' > c''$ and their unit prices have to be strictly higher than $v(c'')$. But then, since $\varepsilon$ can be arbitrarily small and the change in the average price switching from $Q(c'')$ to $Q(c'') - \varepsilon$ is discrete, choosing these contracts would be a profitable deviation for type $c''$, a contradiction.

Suppose next that contract $q_\tau$ (following no trade at earlier dates) were not on the equilibrium path. By the same argument as above it follows that, for $\varepsilon$ low enough, the lowest (average) price at which type $c''$ would be willing to deviate and trade this contract, followed by the trade of $1 - q_\tau$ at $\tau + \Delta$, is $P < v(c'')$. Incentive compatibility requires then that such price is also lower than the price at which types lower than $c''$ would be willing to carry out those trades. Hence, the lowest average price at which a seller is willing to make those trades is less or equal than $P$ and the seller's type is greater or equal than $c''$. Hence at this price, given buyers' belief that the seller type is greater or equal than $c''$, buyers have a profitable deviation.[15] □

The intuition for this result is that, by choosing a path of trades where total discounted trades are lower and trades are postponed to the maximal extent possible, no lower type of seller would find it profitable to follow part of this path of trades. Thus, it is possible to credibly signal your type. Importantly, this trading strategy is robust to negative beliefs. The worse that can be thought of you if you are selling $q$ at $t$ is that at $t + \Delta$ you will sell your residual supply. Since the seller is using the time already past without trading, which is observable, instead of a simple promise not to sell in the future, this is always credible and buyers must be convinced that it is indeed a higher type. So, by indeed doing this, sellers can effectively signal their types and separate from others. As we will see below in Proposition 1 this same force shapes the trading pattern prevailing in the unique equilibrium which exists with this market arrangement.

[15] To conclude the argument observe that the lowest price at which $c''$ is willing to trade $q_\tau$ at $\tau$ followed by the trade of a smaller quantity than its residual supply at $t + \Delta$, or when subsequent trades occur at later dates, is higher than $P$ (since the total level of trade in such case would be $Q < Q(c'') - \varepsilon$). The same is true for all other types. Hence the equilibrium price of the off equilibrium trade $q_\tau$ at $\tau$ is indeed $P$.

**Lemma 2** (Separation in Q)**.** *For every* $\Delta$ *and* $T$*, in all observable trade equilibria, the total discounted quantity* $Q(c)$ *traded by any type* $c$ *must be strictly decreasing in* $c$*.*

*Proof.* Simple IC prevents $Q(c)$ from being increasing in $c$. Since we showed that all trades are separating we know that $P(c) = v(c)$ but then $Q(c)$ cannot be constant. To see this, consider a set of types $[c', c'']$ with the same total discounted trades $\bar{Q}$. Then all of them would clearly want to choose the set of trades entered by type $c''$ which feature higher prices and yield so a strict improvement, thus a contradiction. □

Using the two lemmas above we can now show there is a unique path of trades compatible with equilibrium in which, as we anticipated, each seller postpones its trades as much as possible.

**Proposition 1** (Uniqueness with Observable Trades)**.** *For every* $\Delta$ *and* $T$ *there is a unique observable trade equilibrium. The equilibrium is fully separating, all sellers trade in at most two consecutive periods and, if a seller's first trade occurs before* $T$*, its total sales exhaust its supply. Furthermore, for all* $c$ *the total discounted quantity* $Q(c)$ *and payoff* $U(c)$ *are the same as in the commitment equilibrium.*

*Proof.* The fact that the discounted quantity $Q(c)$ and payoff $U(c)$ for each seller type is the same as in the commitment case follows directly from Lemmas 1 and 2, since the seller's time 0 problem is equivalent to the one in the commitment case. To establish the unique pattern of trade we will proceed by contradiction. Suppose that some type $c$ made its first trade at some date $t < T$, for an amount $\bar{q}_t < 1$, and that its trade at the next available date is $\bar{q}_{t+\Delta} < 1 - \bar{q}_t$. Next consider the type $\tilde{c}$ that in equilibrium is supposed to trade $Q(\tilde{c}) = e^{-rt}\bar{q}_t + e^{-r(t+\Delta)}(1 - \bar{q}_t)$. Note that $c > \tilde{c}$ since $Q(\tilde{c}) > Q(c)$. In this situation, if $\tilde{c}$ were to imitate $c$'s trade at time $t$ and trade then its residual supply at $t + \Delta$, the latter trade would be off-path since $c$ does not make that trade and no other type was supposed to have traded $\bar{q}_t$ as all trades are separating. Note that the lowest average price at which type $\tilde{c}$ is willing to carry out the trades of $\bar{q}_t$ at $t$ and $1 - \bar{q}_t$ at $t + \Delta$ is $P = v(\tilde{c})$. At this value of $P$ type $\tilde{c}$ is indifferent between deviating and not, while all other types strictly prefer not to deviate (which simply follows from the incentive compatibility property of trades in the candidate equilibrium). Since the average price is $P = v(\tilde{c})$, and $\bar{q}_t$ is traded at $v(c) > v(\tilde{c})$, the price of $1 - \bar{q}_t$ must be less than $v(\tilde{c})$ and so buyers have a profitable deviation given the belief determined above that the type making those trades is $\tilde{c}$. □

To illustrate our results, consider again the situation considered in Example 1 where $v(c) = \frac{1+c}{2}$ and, as we saw, we have:

$$Q^*(c) = \exp\left[-\int_0^c \frac{v'(x)}{v(x) - x} dx\right] = 1 - c.$$

With two trading dates, $\varsigma = \{0, \Delta\}$, with observable trades the unique equilibrium pattern of trades is the following:

$$q_0(c) = \begin{cases} 1 - \frac{c}{1-e^{-r\Delta}} & c \leq 1 - e^{-r\Delta} \\ 0 & c > 1 - e^{-r\Delta} \end{cases}$$

$$q_\Delta(c) = \begin{cases} \frac{c}{1-e^{-r\Delta}} & c \leq 1 - e^{-r\Delta} \\ \frac{(1-c)}{e^{-r\Delta}} & c > 1 - e^{-r\Delta}. \end{cases}$$

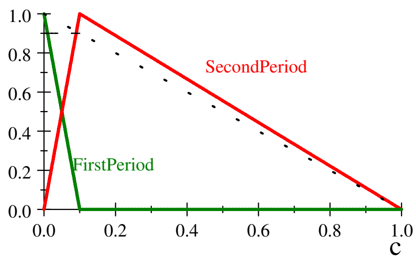


Figure 1: Two period trade for $e^{-r\Delta} = 0.9$.

When $e^{-r\Delta} = 0.9$, in the first period only types $c \in [0, 0.1]$ trade (as described by the green line). These types then trade their residual supply at $t = \Delta$ while types $c > 0.1$ only trade $\frac{1-c}{0.9}$ in the second period (red line). Importantly, at $t = \Delta$ there are two contracts being traded for any given quantity in $[0, 1]$, one by types that had not traded in the first period which will command a high price and one by types that had traded already and which will be priced lower: $p(0, q) > p(1 - q, q)$.[16] Note that, although we started with a uniform supply for all sellers, after $t = 0$ we endogenously get a non-uniform residual supply. This creates difficulties in the unobservable case, where we cannot condition prices on previous trades.

It is worth pointing out that the equilibrium payoffs are independent of the values of $\Delta$ and $T$. This is in stark contrast to the non-divisible asset case studied by FS. With

[16]As established in Proposition 1, this property is true for general $T, \Delta$, for all $t > 0$.

non-divisible assets, the set of attainable values of total discounted quantity traded $Q$ is constrained to $\{0, e^{-r\Delta}, e^{-2r\Delta}, ..., e^{-Tr\Delta}\}$ and thus depends on $\Delta$ and $T$. Once we allow for $q_t$ to take any value in $[0,1]$ this limitation is circumvented and the set of attainable values of $Q$ is $[0,1]$, for all $\Delta$ and $T$.

Although for every $\Delta > 0$ all quantities $q \in [0,1]$ can be traded at every date $t$, in equilibrium each type trades at most twice and the measure of types trading at a given $t < T$ is decreasing in $\Delta$. Indeed, as we show next, in the limit as $\Delta$ goes to $0$ the measure of types trading at a given date $t < T$ goes to $0$ and effectively only the time of trade is used by sellers to signal their type.

**Theorem 2.** *For $T = \infty$, the limit as $\Delta \to 0$ of the sequence of observable trade equilibrium allocations converges to an allocation where each type $c$ trades all its supply at a unique and distinct moment in time $\tau(c)$, satisfying $e^{-r\tau(c)} = Q^*(c)$, where $Q^*(c)$ is the commitment equilibrium value.*

Note that the above result is also true for finite $T$. In this case, it is still true that every type trades only in one date, but only the seller types $c$ such that $\tau(c) \leq T$ trade their entire supply. Every other type of seller (for whom $\tau(c) > T$) would trade at the final date $T$ a different quantity $q_T(c) < 1$ such that $e^{-rT} q_T(c) = Q(c)$.

## 4.2 Equilibria with Continuous Trading

We show in this section that the result established in Theorem 2 can also be obtained if we work directly in the limit, with trading occurring in continuous time, i.e., $\varsigma = [0, \infty]$. Thus we can say there is no discontinuity in the limit as $\Delta \to 0$. When the market is continuously open to trade, conditional on trading at $t$, the subsequent dates at which a trader can operate in the market is an open set. This poses some technical problems and so we will assume that each seller can only trade in a sequence of discrete points in time. This assumption implies that a trading strategy for a type $c$ seller consists of a choice of a countable set $R(c) \subset \varsigma$, denoting the set of times in which type $c$ trades, and of a quantity $q(t,c) \in (0,1]$ for each $t \in R(c)$, specifying the amount traded at time $t$ by type

$c$. A trading strategy, which we will denote by $\{R(c), q(c)\}$, is feasible if the total traded amount is less than the initial supply: $\sum_{t \in R} q(\cdot) \leq 1$.

Note first that Lemmas 1 and 2 can easily be extended to this case. Thus, the equilibrium value of the total discounted quantity traded $Q(c)$ is still unique and coincides with the value when trading is discrete. Indeed, there is a unique equilibrium with the same allocation as in the limit described in Theorem 2, in which each type trades its full unit at a unique moment in time $\tau(c)$ such that $e^{-r\tau(c)} = Q^*(c)$, with $Q^*(c)$ as in the commitment equilibrium.

**Corollary 1.** *When $\varsigma = [0, \infty]$ there exists a unique equilibrium where each seller type trades all its supply at a unique moment in time $\tau(c)$ satisfying: $e^{-r\tau(c)} = Q(c)$.*

It is also worth pointing out that the equilibrium coincides with the one described by FS for the case of a non-divisible asset and continuous trading. Thus, in equilibrium time trumps quantity as a way to signal when both instruments are available.

# 5 Spot Contracts with Unobservable Trades

As we observed in the example with two trading dates, in the equilibrium with observable trades, two different contracts are traded for the same spot quantity in the second period with different prices. A low price contract for those who also traded in the previous period and a high price contract for those that had not. When past trades are not observable by buyers, this differentiation is not possible as prices can only be contingent on the spot quantity being traded. We show in this section (see Proposition 2) that with discrete trading such restriction on contracts makes it impossible to replicate when trades are private the equilibrium payoffs obtained with commitment as well as with public trading. The fact that past trades are unobservable introduces a second dimension of private information, the seller's residual supply. This makes an explicit construction of equilibria for arbitrary $T$ and $\Delta > 0$ quite hard. For the case with only two trading dates, we can explicitly characterize an equilibrium, and show it might feature pooling.

When $T = \infty$ and $\Delta = 0$ we are able to establish the existence of a unique equilibrium and to characterize it. This equilibrium is separating and coincides with the equilibrium with observable trades and continuous trading described in Corollary 1, where each type trades its full unit at a single point in time. Thus, sellers again use only the time of trade, not quantity to signal their type.

## 5.1 Equilibrium Definition

We present here a definition of equilibrium that allows for the possibility that trade occurs at any point in time, i.e., that $\varsigma = [0, \infty]$. As in Section 4.2, a seller's strategy $\{R(c), q(c)\}$ is given by any sequence of discrete trading dates and quantities traded such that $R(c) \subset \varsigma$ and $\sum_{t \in R} q(\cdot) \leq 1$.

Importantly, when past trades are not observable prices can only depend on the quantity $q$ being traded at the current trading time $t$, i.e., $p(q, t)$. Given prices, we say a trading strategy $\{R(c), q(c)\}$ is optimal for type $c$ if:

$$\sum_{t \in R(c)} e^{-rt} q(t, c) \left(p\left(q(t, c), t\right) - c\right) \geq \sum_{t \in \tilde{R}} e^{-rt} \tilde{q}(t) \left(p\left(\tilde{q}(t), t\right) - c\right) \tag{3}$$

for all admissible trading strategies $\left\{\tilde{R}, \tilde{q}\right\}$. The associated payoff is again denoted $U(c)$.

Let $\{R(c), q(c); c \in [0,1]\}$ denote a profile of trading strategies for all types. Given such a profile we can then define $K(q, t)$ as the set of types that trade quantity $q$ at time $t$.

**Definition 3** (Competitive Equilibrium with Unobservable Trades). *When $\varsigma \subseteq [0, \infty]$ a competitive equilibrium with unobservable past trades is given by a profile of strategies $\{R(c), q(c); c \in [0,1]\}$, prices $p(q, t)$ for all $q$ and $t$ and buyers' beliefs such that:*

$(i)$ *Sellers optimize: (3) is satisfied, for all $c$.*

$(ii)$ *On-path beliefs and prices: for actively traded quantities, i.e., $q, t$ such that $K(q, t) \neq \varnothing$, buyers' beliefs are determined using Bayes' rule and prices are such that buyers break even, $p(q, t) = E[v(c) | c \in K(q, t)]$.*

$(iii)$ *Off-path beliefs and prices: for all $q$ and $t$ such that $K(q, t) = \varnothing$, buyers' beliefs give* *positive weight only to the type $c' \in [0,1]$ that is most willing to trade $q$ at date $t$, together with some optimally chosen additional set of trades at other dates. The price of this contract and of any other off-path trade in this set is such that type $c'$ is indifferent between this deviation and its equilibrium trades while all other types are worse off and buyers, given these beliefs, find the contract not profitable.*

## 5.2 Non-equivalence

For any value of $\Delta$ and $T$, that is, for any market with discrete trading dates, the value of total discounted equilibrium trades $Q^*(c)$ with observable trades cannot be attained as an equilibrium with unobservable trades. We will use the case where there are two trading dates $\varsigma = \{0, \Delta\}$ to illustrate this claim. As we will show below the problem, relative to the observable case, is that at $t = \Delta$ the price for quantity $q_\Delta$ can only depend on $q_\Delta$. Thus we can no longer separate the types that are trading $\{0, q_\Delta\}$ from the ones that are trading $\{1 - q_\Delta, q_\Delta\}$. Since in the separating equilibrium with observable trades all quantities $q \in [0,1]$ are traded at $t = 0$ by some low type seller and their entire residual supply, also spanning $(0,1)$, is traded at $\Delta$, there is no possibility for high types to attain the same level of trades as at $Q^*(c)$ without pooling with lower types.

Let $q_0^l(c)$ denote the lowest amount type $c$ can trade at $t = 0$ and still attain a total discounted level of trades equal to $Q^*(c)$. This is obtained in particular for the types with lower values of $c$ from the following expression: $Q^*(c) = q_0^l(c) + e^{-r\Delta}\left(1 - q_0^l(c)\right)$. We show next that for type $c$ to attain the same payoff as at the equilibrium with public trades, this type must trade $q_0^l(c) > 0$ at $t = 0$.

**Lemma 3.** *For $\varsigma = \{0, \Delta\}$, suppose the equilibrium with unobservable trades is fully separating, featuring a level of total discounted trades $Q^*(c)$ for each $c$ as when trades are observable. Then, for all types $c$ such that $Q^*(c) > e^{-r\Delta}$, it must be that $q(0,c) = q_0^l(c)$.*

*Proof.* In search of a contradiction, suppose $q(0,c) > q_0^l(c)$ for some $c$. Then there exists another type $c' < c$ such that $q(0,c) = q_0^l(c')$. This implies that $c'$ can imitate $c$ at $t = 0$ and sell quantity $q(0,c)$ for a higher price since $v(c) > v(c')$. Importantly, type $c'$ can then trade its whole residual supply $1 - q_0^l(c')$ at $t = \Delta$ thus achieving the same value $Q(c')$. Furthermore, if the trade $1 - q_0^l(c')$ at $t = \Delta$ were on path, it must hold that $p\left(1 - q_0^l(c'), \Delta\right) \geq v(c')$ since no type lower than $c'$ chooses trades of total amount $Q(c')$. Hence, in such a situation, $c'$ can profitably deviate by imitating $c$ at $t = 0$, a contradiction. If instead $1 - q_0^l(c')$ were off path, the type of seller most willing to carry out those trades is again $c'$ who would combine them with the trade of $q(0, c) = q_0^l(c')$. The lowest price at which it is willing to do so is some $p < v(c')$ (since $q(0, c)$ trades at a price higher than $v(c')$. Thus buyers, given this belief, would profit by deviating to trading $1 - q_0^l(c')$ at $p$, again a contradiction. □

Building on this result we can show that the equivalence result established in Proposition 1 when trades are observable no longer holds.

**Proposition 2.** *The (unique) level of discounted trades in the equilibrium with observable trades cannot be attained in an equilibrium with unobservable trades when $\varsigma = \{0, \Delta\}$ and $c \in [0, 1]$.*

*Proof.* We know the equilibrium with observable trades is separating and the values of total discounted trades $Q^*(c)$ by different seller types span the whole interval $[0, 1]$. To support the same level of total discounted trades in the unobservable case, given Lemma 3, all quantities in $[0, 1]$ must be traded at $t = 0$ and at $t = \Delta$ by types $c < \tilde{c}$ where $\tilde{c}$ is such that $Q^*(\tilde{c}) = e^{-r\Delta}$. But then types $c > \tilde{c}$ cannot trade in either period without pooling with some lower type, hence cannot attain the level of trades $Q^*(c)$ at the price $v(c)$. □

The non-equivalence result extends to an arbitrary number of trading dates as long as $\Delta > 0$. The issue is that, in the unique equilibrium allocation with observable trades characterized in Proposition 1, at any $t > 0$, two distinct types trade the same quantity. Since these two types traded different amounts in the past and we can condition the current terms of trade on the history of past trades, we can write separate contracts with different prices for each of them. But, when past trades are unobservable, this conditioning is not possible, and hence the same pattern of trades can no longer be supported in equilibrium.

Proposition 2 shows that with private trades there can be no equilibrium yielding the same payoffs as when trades are observable. We establish next the existence of an equilibrium when there are two trading dates and characterize its properties: with private trades we have pooling, each seller type carries out at least part of its trades at the same price as other types.

**Proposition 3.** *Suppose that, for some $\alpha \in (0,1)$, $v(c) \geq \alpha(c-1)+1$ in a neighborhood of* $1$.[17] *Then there exists an equilibrium with unobservable trades where the pattern of transactions is as follows. There are two threshold types $0 \leq c_0 < c_1 < 1$ such that $Q^*(c_0) > e^{-r\Delta}$ and:*

*(i) low types $c \in [0, c_0])$ trade as in the equilibrium with public trades: $q_0(c) = \frac{Q^*(c)-e^{-r\Delta}}{1-e^{-r\Delta}}$ and $q_\Delta(c) = 1 - q_0(c)$ at prices $p_0(c) = p_\Delta(c) = v(c)$;*

*(ii) intermediate types $c \in [c_0, c_1])$ pool on the trade of the entire supply in the second period at the price $\overline{p}_\Delta = E\left[v(c)|c \in [c_0, c_1]\right] = c_1$;*

*(iii) high types $c \in [c_1, 1])$ do not trade.*

*Moreover, the beliefs regarding the trade of any off-path quantity in the first and the second periods are the Dirac measure concentrated on type $c_0$.*

The formal proof of this proposition can be found in the Appendix. Comparing Propositions 1 and 3 we see that the patterns of trade at the equilibria with observable and with private trades are quite different. It is then relatively easy to construct examples in which one information environment yields higher expected welfare in equilibrium than the other and vice versa:

**Corollary 2.** *The expected welfare with observable trades and unobservable trades is not generally ranked.*

To establish the result we again revisit the specification used for Example 1 with $v\left(c\right) = \frac{1+c}{2}$ and trading only at $0$ and $\Delta$. When types are uniformly distributed and for $e^{-r\Delta} \geq 3/4$ the equilibrium with unobservable trades features no trading in the first period and pooling in the second: types $c \in [0, 2/3]$ trade their entire supply in the second period (i.e. $c_0 = 0$ and $c_1 = 2/3$). Prices are $p_0\left(q\right) = e^{-r\Delta}\frac{2}{3}$ and $p_\Delta\left(q\right) = 2/3$ for all $q$. One can

[17]This assumption precludes that the function $v(c)$ is tangent to the $45^0$ degree line at $c = 1$. This is always true if $v(c)$ is a concave function.

then compute the expected welfare of sellers, equal to $e^{-r\Delta}\frac{4}{18}$ and compare its value with the one for the equilibrium with observable trades characterized in Example 1, equal to $\frac{1}{6}$. Recall that buyers always make zero expected profits in equilibrium, hence expected welfare only depends on the sellers' payoff.

We see that when $e^{-r\Delta} = 3/4$ welfare in both information regimes coincides. Noticing that welfare in the observable trades case is independent of $e^{-r\Delta}$ and the cost of delaying trades to the second period is decreasing in $e^{-r\Delta}$, for any $e^{-r\Delta} > 3/4$ we get that sellers achieve a higher expected payoff at the unobservable trade equilibrium.

In general the trade-off that determines the welfare comparison is that (when $e^{-r\Delta} \geq 3/4$) with unobservable trades types between $c = 1 - e^{-r\Delta}$ and $c = 2/3$ trade more while types $c > 2/3$ do not trade at all. This observation allows us to find other specifications, where some mass is moved from lower types to types close to $1$, for which observable trading is preferable. For example, let

$$f(c) = \begin{cases} 0.9 & for\ c < 0.9 \\ 1.9 & for\ c \geq 0.9 \end{cases}.$$

Note that this change does not modify the pattern of trade in either equilibrium but it reduces the expected welfare of the unobservable trading case relative to the observable trade one. Hence it is easy to verify that for $e^{-r\Delta} = 0.75$ welfare is now higher when trades are observable.

## 5.3 Characterization of Equilibria with Continuous Trading

As shown in Theorem 2 and Corollary 1, when $T = \infty$, in the limit $\Delta \to 0$ with observable trades each type sells its full unit at a unique point in time. This suggests that the difficulties described in Proposition 2 might not arise when trading can take place continuously over an infinite horizon, i.e., when $\varsigma = [0, \infty]$.[18] We explore this case next.

[18]Another, less interesting case in which these difficulties do not arise is when the type space is restricted to the interval $[0, \tilde{c}]$, where $\tilde{c}$ is such that $Q^*(\tilde{c}) = e^{-r\Delta}$, with $Q^*(\cdot)$ describing the level of total trades at the equilibrium with observable trades. In this case the equilibrium is unique and coincides with the one with

Similarly to what we did for the observable case, we start by showing that, when $\varsigma = [0, \infty]$, in any equilibrium with unobservable trades: (1) each individual trade must be fully separating; and (2) we must have full separation in terms of the total discounted quantity traded $Q(c)$. The force inducing separation is the same we highlighted in the commitment case. Since the seller's preferences satisfy the single crossing property, higher types would be willing to signal their type by committing to trading a lower $Q$. Of course, without commitment, and in particular when past trades are not observable and thus prices cannot condition on past trades, this can be hard to replicate. As highlighted in the previous section, when time is discrete, the need to rely on both quantities and time to separate makes it impossible to replicate the equilibrium with observable trades once trades are unobservable. Yet, when $\varsigma = [0, \infty]$ we can generate any $Q \in (0, 1)$ by trading the full unit with the right amount of delay. The crucial aspect is that then, when the seller trades, it trades its full unit. Thus, it does not require buyers to trust the seller will not sell (or potentially have sold) any additional amounts. Therefore, delay combined with full trade provides a credible way to signal your type that is robust to the unbservability of past trades, in the sense that there is no room for buyers to form beliefs about what might happen with the rest of the seller's supply.

In other words, an endogenous commitment can still be generated if, whenever a type trades, it trades its full supply. This effectively amounts to a commitment never to trade at any other date. This is the key why time trumps quantity as the way to signal sellers' types in a dynamic setting.

To formalize our result we begin by showing that the properties established in Lemmas 1 and 2 for the observable trade case extend to the situation where past trades are not observable.

**Lemma 4** (Separation in q, Private). *For $\varsigma = [0, \infty]$, in an equilibrium with unobservable trades all transactions with $q > 0$ must be fully separating.*

*Proof.* The proof proceeds along similar steps of that of Lemma 1. Consider the high-

observable trades.

est type $c''$ that ever participates in a pooling trade.[19] Let $Q(c'')$ be the total discounted quantity traded in equilibrium by type $c''$ and $P(c'')$ the average price paid. Note that $P(c'') < v(c'')$ since $c''$ pools with lower types for at least some trade, never pools with higher types, and buyers must break even. Consider instead a trade of the entire supply at time $\tau$ such that:

$$e^{-r\tau} = Q(c'') - \varepsilon,$$

for some small $\varepsilon > 0$. If such contract were on the equilibrium path, it must be traded by types higher than $c''$ since, by the single crossing property, if type $c''$ does not prefer this contract to its equilibrium trade no type below it would. Thus, the price of the contract would need to be greater than $v(c'')$. But then, since $\varepsilon$ can be arbitrarily small, and the change in prices is discrete, this would be a profitable deviation for type $c''$. If instead such contract were not on the equilibrium path, by a very similar argument as in Lemma 1 we can show that the seller of type $c''$ is the one that is most willing to trade it and is willing to do it at a price $p < v(c'')$ and, at such price and belief, buyers would have a profitable deviation. Thus, all trades must be separating. □

**Lemma 5** (Separation in Q, Private)**.** *For $\varsigma = [0, \infty]$, in any equilibrium with unobservable trades the total discounted quantity $Q(c)$ traded by type $c$ must be strictly decreasing in $c$.*

The proof follows the same steps as the proof of Lemma 2 and is thus omitted.

**Proposition 4.** *For $\varsigma = [0, \infty]$, with private trading there is a unique equilibrium level of total discounted trades $Q(c)$ for each seller type $c$ and this level coincides with the one of the equilibrium with commitment.*

*Proof.* Lemmas 4 and 5 imply that the seller's problem can be written as:

$$U(c) = \max_{\tilde{c}} Q(\tilde{c})\left(v(\tilde{c}) - c\right)$$

and we require that $Q(\tilde{c})$ is such that choosing $\tilde{c} = c$ is optimal for the type $c$ seller, for all $c$. This problem is the same as the one we had in the equilibrium with commitment. Like

[19] We assume for convenience that the set of types participating in some pooling trade includes its supremum. A similar argument can be constructed when this is not the case.

in that case, from the first-order condition for the seller we obtain a differential equation characterizing $Q(c)$. This implies that $Q(c)$ is uniquely pinned down up to a constant. But since we must have $Q(0) = 1$, $Q(c)$ is then uniquely pinned down. □

The last remaining step is to characterize the unique path of trades that implements the equilibrium $Q^*(c)$. For this, recall $\tau(c)$ denotes the time such that, if a type were to trade its full unit at this date, it would attain the commitment equilibrium value $Q^*(c)$. Formally:

$$e^{-r\tau(c)} = Q^*(c).$$

Since $Q^*(c)$ is unique and strictly decreasing, $\tau(c)$ is strictly increasing and we can define $C(t)$ as the inverse of $\tau(c)$. To understand why only using time to signal is an equilibrium let us follow a guess and verify argument. Suppose that indeed each seller type $c$ trades its full unit at $\tau(c)$ and let the supporting prices be $p(q, \tau(c)) = v(c)$ for all $q \in (0, 1]$. With these prices the seller's solution of when and how much to sell is clear-cut: at any date it will be optimal to sell the full unit or nothing at all. Since the trading date $\tau(c)$ solves the sellers' problem with commitment it must be optimal also in the present environment. The off-equilibrium part of the prices is sustained by the belief that type $c$ such that $\tau(c) = t$ is the type of seller who is most willing to deviate to trading a partial amount at $t$. At the above prices this type of seller is indifferent between this deviation, together with the trade of the rest of its supply the next instant and its equilibrium trades.

**Theorem 3.** *For $\varsigma = [0, \infty]$, there exists an equilibrium with private trade where the present discounted value of total trades for each seller is the same as in the commitment equilibrium, prices and beliefs are $p(q, t) = v(C(t))$ and $\mu(c|q, t) = \delta_{C(t)}$, for all $(q, t)$, where $\delta_c$ is the Dirac measure concentrated at $c$. Moreover, in this equilibrium there is a unique pattern of trade where type $c$ sells the entire supply at time $\tau(c)$.*

The proof of Theorem 3 is in the Appendix. Finally, we should note that the presence of an infinite horizon $T = \infty$ plays an important role with unobservable trades. This is in contrast to the observable case, where the equilibrium values of $Q^*(c), P^*(c)$ can also be supported for any finite $T$, with every type $c$ such that $\tau(c) > T$ trading a fractional amount at $T$. The binding time horizon implies these types trade sooner, but trade a smaller quantity so as to achieve the same discounted quantity $Q^*(c)$. This equilibrium cannot be replicated with unobservable trades because the high types that should only trade a fraction of their supply at $T$ have an incentive to deviate and trade also some of their residual supply (which should not be traded to achieve $Q^*(c)$) at some earlier date before $T$. When trades are observable, carrying out those earlier trades affects the price at date $T$ so these deviations can be deterred, but that is not the case when trades are unobservable.

# 6 Non-Uniform Supply

The assumption of a uniform supply across all seller types is natural in many situations. For example, the one considered by Leland and Pyle (1977) where an entrepreneur is selling part of its firm. This is the case at the time in which the trading process starts and also along this process as long as past trades are observable. As already discussed, when instead past trades are unobservable, the residual supply available at any date $t > 0$ is endogenous as it depends on past trades and hence may no longer be uniform and varies with the seller types. In other cases it is even natural to assume the supply is different for different types. For example, if we consider informed hedge funds trading a particular asset, it might be impossible to know the exact amount of the asset they hold. In those situations we face a problem of multidimensional private information. A full analysis of the non-uniform supply case is beyond the scope of this paper. In what follows we illustrate how some of our results may change when the supply is non-uniform.

When the amount of the supply available to each type $c$ at the beginning of the trading process is given by $s(c) \in [0, 1]$ and varies with $c$, the characterization of the equilibria with observable trades provided in Proposition 1 and Theorem 2 no longer holds true. The key to understand this is that with uniform supply the relevant IC constraints that limit the trades occurring in the equilibrium are the local upward constraints: that is, the ones ensuring that any given type $c \in [0, 1)$ does not wish to imitate the trades of type $c + \varepsilon$ for a small $\varepsilon > 0$. As shown, in equilibrium, separation is achieved by having higher types trade less, albeit at higher prices. But when $s(c)$ is not constant, different types may not be able to trade the same contract. Hence, a local upward constraint might not be binding because type $c$ might not be able to supply the quantity needed to imitate type $c + \varepsilon$. This is best illustrated by the fact that, despite the presence of adverse selection, efficient trade may be attainable with a non-uniform supply.

**Proposition 5** (First Best)**.** *Suppose $s(c)$ is strictly increasing. Then, for any set of trading dates $\varsigma$, regardless of whether past trades are observable or not, or of whether there is commitment, the unique equilibrium features $q_0(c) = s(c)$ for all $c$. That is, we have full, instantaneous trade and the first-best welfare level is thus attained.*

*Proof.* Since only types $c' \geq c$ can trade the contract $s(c)$ this contract must be priced at least $v(c)$. Since this is true for all quantities, each type obviously chooses to trade immediately all its supply. Thus trades will be perfectly separating and efficient. □

It is also worth highlighting that the result obtained in the previous sections, that both with commitment or with observable trades equilibrium payoffs are independent of the set of trading opportunities $\varsigma$, may no longer hold when $s(c)$ is not uniform. To illustrate this point, consider the supply function described in Figure 2.

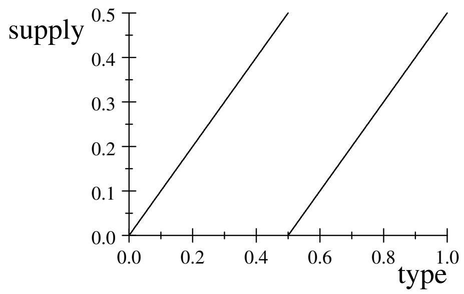


Figure 2: Non-uniform supply.

If we had only one trading date, $\varsigma = \{0\}$, types $c \geq 0.5$ would not be able to trade without pooling with the lower types with their same supply and they may then prefer to forgo trading altogether. This happens, for instance, when $v(c) = \frac{1}{3} + \frac{2}{3}c$, since at the pooling price $\frac{1}{2}v(c)+\frac{1}{2}v(c+\frac{1}{2}) = \frac{1}{2}+\frac{2}{3}c < c+1/2$ for all $c \in [0, 0.5]$. Thus, type $c+\frac{1}{2}$, who has the same supply as $c$, prefers not to trade at this price. The situation has clear analogies with the one encountered in the previous section when past trades are not observable: seller types are multidimensional, hence the quantity traded at any point in time is not sufficient to screen them.

Instead, with a sufficiently larger set of trading dates, sellers have an additional instrument to signal their type, the time of trading. For instance, if $\varsigma = [0, \infty]$, separating equilibria again exist, in which types $c < 0.5$ trade their entire supply $s(c)$ at $t = 0$ and types $c \geq 0.5$ wait until some time $\tau$ and only then trade $q_\tau(c) = s(c)$.[20] The value of $\tau$ must be sufficiently large that each type $c < 0.5$ prefers to trade at the lower price $v(c)$ at $t = 0$ rather than wait until $\tau$ to trade at the higher price $v(c + 0.5)$.

Thus, with non-uniform supply the equilibrium payoffs depend on the available trading dates $\varsigma$ even with observable trades. Furthermore, in the situation described above the surplus generated by equilibrium trades is clearly lower when $\varsigma = \{0\}$ than when $\varsigma = [0, \infty]$. This is because in the latter case there is a higher level of (discounted) trades for all types, since also high types trade fully, albeit with delay $\tau$. Note however that, for a different $v(c)$, we could also have that the ex-ante expected surplus generated in equilibrium when $\varsigma = \{0\}$ is higher than when $\varsigma = [0, \infty]$. For this to happen, the gains from trade would have to be sufficiently large so that high types are willing to pool with low types with the same supply when $\varsigma = \{0\}$, yet they separate when $\varsigma = [0, \infty]$. Pooling then occurs with $\varsigma = \{0\}$. Signalling by lowering the amount traded is not attractive since the upward sloping supply implies that lower quantities actually trade at lower prices. As in Corollary 2 in Section 5, depending on parameter values the separating equilibrium may feature a higher or lower welfare than the equilibrium with some pooling trades.

[20] It is easy to verify that the beliefs for off equilibrium trades at earlier dates $t < \tau$ put all weight on lower types $c < 0.5$. given those beliefs and the associated prices, buyers do not want to trade at those dates.

**Remark 2.** *When $s(c)$ does not take the same value for all $c$, allowing contracts to use probabilities of trade between 0 and 1 for any given quantity (as in the equilibrium of markets with directed search) offers an additional instrument, besides the quantity traded, to screen seller types. In the situation discussed above, where sellers' supply is as in Figure 2 and $\varsigma = \{0\}$, the equilibrium is again fully separating if contracts can specify not only the quantity traded but also the probability of trade, with two active markets for each quantity: a low price market with probability $1$ of trade and a high price market with a probability of trade $\pi^* = e^{-r\tau} < 1$.[21] The value of $\pi^*$ is sufficiently low that the types $c < 1/2$ prefer to trade for sure at a low price than with probability $\pi^*$ at a high price.[22] As discussed above, the welfare effects of enlarging the contract space are ambiguous.*

# 7 Conclusions

We have shown in this paper that time dominates quantity as an instrument to signal high quality. Low types trade early while high types trade at later times to signal the higher quality of the asset they are selling. The basic intuition for this result is straightforward. Once you have waited a certain amount of time to trade your full supply, you do not need to further restrict your future actions in any way to convince your counterparties the quality of your asset is high. Instead, if you only make a partial trade, in order to credibly signal you have a high quality asset, it must be believed that you won't trade your residual supply at some time in the future. Since there continue to be gains from trade, it is not possible in equilibrium to avoid succumbing to the temptation to sell the residual quantity. Thus, without commitment, a promise not to sell more in the future is not credible. We also showed that, even though the set of available trading dates does not matter when past trades are publicly observable, this is no longer true with unobservable trades. If trade happens at discrete intervals, sellers are no longer able to perfectly signal their types.

[21] In a directed search equilibrium, these probabilities of trade for sellers are generated by having relatively few sellers and many buyers in the first market and vice-versa in the second market.

[22] Williams (2021) shows in fact, in a directed search model, that when the privately known type of the seller is two dimensional, perfect screening can be achieved in equilibrium using jointly the quantity traded and the probability of trade.

There are several promising avenues for future research. Our analysis highlighted the importance of the consequences for equilibrium outcomes of the presence of two dimensional private information, where both the quality of the asset and the amount of the asset held are only known to the seller. A characterization of the equilibria in that case could offer novel, interesting insights. In those environments, or in the situations with unobservable trade considered in Section 5, where private information over the residual supply endogenously arises, one could also explore the consequences of allowing for stochastic deliveries (in a model with search or by adding this feature in the specification of contracts). This would expand the set of available trade opportunities, beyond quantities and time of trade, and so provide sellers with an additional instrument with which to signal their types. Another interesting question is how the introduction of curvature in agents' utilities, so that the size of the gains from trade depends on the total quantity traded, affects the properties of equilibria with dynamic trading.

# Appendix

*Proof of Theorem 1.* Fix a commitment equilibrium and let $U(c)$ be the seller's value function. By Lemma 7 in the Online Appendix, $U(c)$ is a convex function. Hence, we can partition $[0,1]$ into three subsets:

(i) $c \in S$ iff the contract chosen by $c$ is unique and is different from the choice of all other types. In this case, $U'(c) = -Q(c)$ is the contract chosen by type $c$ and $P(Q(c)) = v(c)$ is its equilibrium price;

(ii) $c \in \Pi$ iff there is a non-degenerate maximal interval $I_c \subset [0,1]$ of types containing $c$ that make the same choice. In this case, $Q(\tilde{c}) = \bar{Q} = -U'(\tilde{c})$ is the optimal contract and $P(\bar{Q}) = \frac{\int_{I_c} v(x)f(x)dx}{\int_{I_c} f(x)dx}$ is the equilibrium price for all $\tilde{c} \in I_c$;

(iii) $c \in K$ iff there is a non-degenerate interval of optimal choices for type $c$. In this case, each $Q \in -\partial U(c)$ is an optimal contract and $p(Q) = v(c)$ is the equilibrium price.

By Lemmas 6, 7 and 8 in the Online Appendix, $\Pi$ is a union of a countable number of disjoint intervals and $K$ is a countable set. Let $c \in \Pi$ and define $I_c = [c_-, c_+]$. Then,

$c_-, c_+ \in S \cup K$ and

$$\frac{\int_{c_-}^{c_+} v(x)f(x)dx}{\int_{c_-}^{c_+} f(x)dx} = v(c_-) = v(c_+),$$

for all $Q_- \in -\partial U(c_-)$ and $Q_+ \in -\partial U(c_+)$. Indeed, let $Q_- \in -\partial U(c_-)$. If $Q_- \in \Omega^b$, then $P(Q_-) = \frac{U(c_-)}{Q_-} + c_- = v(c_-)$. If $Q_- \notin \Omega^b$, then $P(Q_-) = \min\left\{\frac{U(c)}{Q_-} + c; c \in [0,1]\right\}$. The first-order condition of the minimization problem is equivalent to $Q_- \in -\partial U(c)$, which implies that $c = c_-$, i.e., $P(Q_-) = \frac{U(c_-)}{Q_-} + c_-$. Analogously for $Q_+ \in -\partial U(c_+)$. Since $Q(c) \in -\partial U(c_-) \cap -\partial U(c_+)$, we can take $Q_- = Q_+ = Q(c)$. If $Q(c) \neq 0$, then we get

$$\frac{\int_{c_-}^{c_+} v(x)f(x)dx}{\int_{c_-}^{c_+} f(x)dx} = v(c_-) = v(c_+),$$

which is a contradiction since $v$ is strictly increasing. Therefore, $\Pi \subset \{c \in [0,1]; Q(c) = 0\}$.

By the envelope theorem,

$$U'(c) = -Q(c),$$

for all $c \in S$. Hence,

$$U(c) = \int_c^1 Q(x)dx = [P(Q(c)) - c]\, Q(c) = (v(c) - c)Q(c)$$

since $U(1) = (v(1) - 1)\, Q(1) = 0$. Taking the derivative we have

$$-Q(c) = (v'(c) - 1)Q(c) + (v(c) - c)Q'(c)$$

or

$$(v(c) - c)Q'(c) + v'(c)Q(c) = 0. \tag{4}$$

Suppose that $Q(0) < 1$. Then, $(Q(0), 1] \subset \{Q(\omega) : \omega \in \Omega \backslash \Omega^b\}$. Using the definition of the off-path equilibrium price, we get

$$P(Q) = \min\left\{\frac{U(c)}{Q} + c; c \in [0,1]\right\},$$

for all $Q \in (Q(0), 1]$. Since the derivative of $\frac{U(c)}{Q_-} + c$ is $\frac{Q(c)}{Q} + 1 > 0$ (because $Q(c) \leq Q(0) < Q$, we have that $P(Q)Q = U(0) = P(0)Q(0)$. Hence, $P(Q) < P(0) = v(0)$ which implies that buyers would make positive profits by purchasing quantity $Q$, which is a contradiction. Therefore,

$$Q(0) = 1. \tag{5}$$

Solving the ODE (4) with initial condition (5), we get the solution in the statement of the Theorem.

By the assumptions on $v\,(c)$,[23] we must have

$$Q(1) = \exp\left[-\int_0^1 \frac{v'(x)}{v(x) - x} dx\right] = 0.$$

Indeed, notice that

$$\int_{1-\epsilon}^1 \frac{v'(x)}{v(x) - x} dx \geq \int_{1-\epsilon}^1 \frac{v'(x)}{1 - x} dx \geq -\gamma \ln(1-x)|_{1-\epsilon}^1 = \infty,$$

since $v'(x) \geq \gamma > 0$ and $v(x) \in [x, 1]$, for all $x \in [1-\epsilon, 1]$ and for some $\gamma > 0$.

We now check that this unique equilibrium candidate is an equilibrium, i.e., the seller's and buyer's optimality conditions. First notice that since this candidate is separating, the Bayesian update is trivial. Therefore, we need to check that

$$Q(c) = \arg\max_{q \in \Omega^b} p(Q)Q - cQ,$$

$$P(Q(c)) = v(c),$$

for all $c \in [0, 1]$. The last condition is trivial by construction. The first condition is equiva-

[23]It the conditions stated in the assumption are not satisfied (for instance, $v(1) > 1$ or $v'(1) = 0$), we may have $Q(1) > 0$. In this case we need to specify the prices of the off-path equilibrium allocation, i.e., for $Q \in [0, Q(1))$. Again,

$$P(Q) = \min\left\{\frac{U(c)}{Q} + c; c \in [0, 1]\right\}$$

for all $Q \in [0, Q(1))$. Since the derivative of $\frac{U(c)}{Q} + c$ is $-\frac{Q(c)}{Q} + 1 < 0$ (since $Q(c) > Q$), we have that $P(Q) = \frac{U(1)}{Q} + 1$.

lent to

$$U(c) = P(Q(c))Q(c) - cQ(c) \geq P(Q(\tilde{c}))Q(\tilde{c}) - cQ(\tilde{c})$$

for all $c, \tilde{c} \in [0, 1]$, if and only if

$$U(c) - U(\tilde{c}) \geq -Q(\tilde{c})(c - \tilde{c})$$

if and only if $-Q(\tilde{c}) = \partial U(\tilde{c})$, which is true. $\square$

*Proof of Proposition 3.* For each $c \in [0, 1]$, let $\varphi(c)$ be the implicit solution of the equation

$$E[v(\widetilde{c})|\widetilde{c} \in [c, \varphi]] - \varphi = 0. \tag{6}$$

Lemma 9 in the Online Appendix shows that $\varphi$ is well defined and provides its basic properties. Let $\overline{c}_1 = \varphi(0)$ and consider the following cases:

(1) $\delta\overline{c}_1 < v(0)$. We claim that $c_0 \in (0, 1)$. Indeed, type $c_0$ must be indifferent between trading the public trade allocation or the whole unit in the second period, i.e., must satisfy the following indifference condition:

$$e^{-r\Delta}(\varphi(c) - c) - U(c) = 0,$$

which has at least the trivial solution $c = 1$. Notice that the left hand side (l.h.s.) of the previous equation becomes $e^{-r\Delta}\overline{c}_1 - v(0) < 0$ at $c = 0$. Hence, in order to obtain a non-trivial solution for the equation we need to show that the derivative of the l.h.s. is negative at $c = 1$, which is equivalent to

$$e^{-r\Delta}\left(\varphi'(1) - 1\right) + Q(1) = \delta(\varphi'(1) - 1) < 0 \text{ or } \varphi'(1) < 1,$$

which holds by the assumption of the proposition and Lemma 10 in the Online Appendix. Therefore, there exists $c_0 \in (0, 1)$ such that

$$e^{-r\Delta}(\varphi(c_0) - c_0) = (v(c_0) - c_0)Q(c_0).$$

In particular, we have that $\overline{p}_\Delta = c_1 = \varphi(c_0)$. Since $\varphi(c_0) > v(c_0)$, we must have that $Q(c_0) > \delta$.

Let $V(c)$ be the utility that type $c$ gets at the proposal equilibrium allocation, i.e.,

$$V(c) = \begin{cases} U(c) & \text{if } c \in [0, c_0] \\ U(c_0) + \delta(c_0 - c) = \delta(c_1 - c) & \text{if } c \in [c_0, c_1] \\ 0 & \text{if } c \in [c_1, 1] \end{cases} .$$

Notice that $V$ is the continuous pasting from the convex function $U$ and a piecewise linear function with non-decreasing slopes, since $\dot{U}(c_0) = -Q(c_0) < -e^{-r\Delta}$. Hence, $V$ is a convex function.

To complete the construction of the equilibrium, we need to specify the off-path equilibrium prices $p_0(\cdot)$ on $[0, q_0(c_0)]$ and $p_\Delta(\cdot)$ on $[q_\Delta(c_0), 1]$. They must satisfy

$$p_0(q_0)q_0 = \min_{c, \widetilde{q}_1} \left\{ V(c) + cq_0 - e^{-r\Delta}(p_\Delta(\widetilde{q}_1) - c)\widetilde{q}_1; c \in [0,1] \text{ and } \widetilde{q}_1 \in [0, 1 - q_0] \right\} \quad (7)$$

and

$$\delta p_\Delta(q_1)q_1 = \min_{c, \widetilde{q}_0} \left\{ V(c) + e^{-r\Delta}cq_1 - (p_0(\widetilde{q}_0) - c)\widetilde{q}_0; c \in [0,1] \text{ and } \widetilde{q}_0 \in [0, 1 - q_1] \right\}. \quad (8)$$

The first-order conditions of these minimization problems w.r.t. $c$ are

$$-V'_+(\widetilde{c}_0) \leq q_0 + e^{-r\Delta}\widetilde{q}_1^* \leq -V'_-(\widetilde{c}_0)$$

and

$$-V'_+(\widetilde{c}_1) \leq \widetilde{q}_0^* + e^{-r\Delta}q_1 \leq -V'_+(\widetilde{c}_1),$$

where $q_0^*$ and $q_1^*$ are optimal solutions in the quantity variable.[24] Notice that

$$V'_-(c) = V'_+(c) = \begin{cases} -Q(c) & \text{if } c \in [0, c_0) \\ -e^{-r\Delta} & \text{if } c \in (c_0, c_1) \\ 0 & \text{if } c \in (c_1, 1] \end{cases},$$

$$V'_-(c_0) = -Q(c_0), V'_+(c_0) = -e^{-r\Delta}, V'_-(c_1) = -e^{-r\Delta} \text{ and } V'_+(c_1) = 0.$$

Hence, by the convexity of $V$, $\widetilde{c}_0 = \widetilde{c}_1 = c_0$ if and only if

$$e^{-r\Delta} \leq q_0 + e^{-r\Delta}\widetilde{q}_1^* \leq Q(c_0) \text{ and } e^{-r\Delta} \leq \widetilde{q}_0^* + e^{-r\Delta}q_1 \leq Q(c_0).$$

These conditions are trivially satisfied if $\widetilde{q}_1^* = 1 - q_0$ and $\widetilde{q}_0^* = 1 - q_1$.

We need to construct $p_0$ and $p_1$ that solve (7) and (8) at $\widetilde{q}_1^* = 1 - q_0$ and $\widetilde{q}_0^* = 1 - q_1$, respectively. If this is the case, we have

$$p_0(q_0)q_0 = (v(c_0) - c_0)Q(c_0) + c_0q_0 - e^{-r\Delta}(p_\Delta(1 - q_0) - c_0)(1 - q_0) \tag{9}$$

and

$$e^{-r\Delta}p_\Delta(q_1)q_1 = (v(c_0) - c_0)Q(c_0) + e^{-r\Delta}c_0q_1 - (p_0(1 - q_1) - c_0)(1 - q_1),$$

for each $q_0 \in [0, q_0(c_0)]$ and $q_1 \in [q_\Delta(c_0), 1]$. Notice that, substituting $q_0$ by $1 - q_1$ in the first equation, we get the second equation, i.e., they are equivalent.

If $q_0 = 0$, then $p_\Delta(1) = \overline{p}_\Delta$ and if $q_0 = q_0(c_0)$, then $p_0(q_0(c_0)) = v(c_0) = p_\Delta(q_\Delta(c_0))$. Hence, let us set $p_\Delta(\cdot)$ on $[q_\Delta(c_0), 1]$ as the linear interpolation between these two extreme point, i.e.,

$$p_\Delta(q_1) = \begin{cases} v(c_0) + (\overline{p}_\Delta - v(c_0))\frac{q_1 + q_0(c_0) - 1}{q_0(c_0)} & \text{if } q_1 \geq q_\Delta(c_0) \\ v(q_\Delta^{-1}(q_1)) & \text{if } q_1 \leq q_\Delta(c_0) \end{cases}$$

and $p_0(q_0)$ is defined such that (9) holds.

Since $p_\Delta(\cdot)$ is an increasing linear function on $[q_\Delta(c_0), 1]$, the term on the right hand side

[24] We are implicitly assuming that these solutions exist for the conjectured price functions. Indeed, for the constructed functions, this will be the case.

of (9) is a quadratic concave function of $q_0$, i.e., $p_0(q_0)q_0$ is a quadratic concave function of $q_0$. Therefore, $p_0(q_0)$ is a decreasing linear function on $[0, q_0(c_0)]$ with $p_0(q_0(c_0)) = v(c_0)$, which implies that $p_0(q_0) \geq v(c_0)$, for all $q_0 \in [0, q_0(c_0)]$. Therefore,

$$p_0(q_0) = \begin{cases} v(c_0) - (\overline{p}_1 - v(c_0))\frac{q_0 - q_0(c_0)}{q_0(c_0)} & \text{if } q_0 \leq q_0(c_0) \\ v(q_0^{-1}(q_0)) & \text{if } q_0 \geq q_0(c_0) \end{cases}.$$

Moreover, the objective function of the minimization problem (7) is concave in $\widetilde{q}_1$ and, therefore, we must have a corner solution, i.e., the optimal is either $\widetilde{q}_1^* = 0$ or $\widetilde{q}_1^* = 1 - q_0$. However, the value of the term $\delta(p_\Delta(\widetilde{q}_1) - c)\widetilde{q}_1$ is zero at $\widetilde{q}_1 = 0$ and positive at $\widetilde{q}_1 = 1 - q_0$ and $c = c_0$, because $p_\Delta(1 - q_0) > v(c_0) > c_0$, for all $q_0 \in [0, q_0(c_0)]$. Hence, $\widetilde{q}_1^* = 1 - q_0$.

As we remarked before, problems (7) and (8) are equivalent. Hence, the solution of problem (8) must be again at the high corner, i.e., $\widetilde{q}_0^* = 1 - q_1$.

Finally, if the off-path beliefs are concentrated at type $c = c_0$, the buyer's no deviation conditions boil down to verify that $p_0(\cdot) \geq v(c_0)$ and $p_\Delta(\cdot) \geq v(c_0)$ on the off-path quantities, which are true by construction.

(2) $\delta\overline{c}_1 \geq v(0)$. Thus, at $c_0 = 0$, we have

$$e^{-r\Delta}E[v(c)|c \in [0, c_1]] \geq v(0),$$

where $c_1 = \varphi(0) = \overline{c}_1$.

Let $V(c)$ be the utility that type $c$ gets at the proposal equilibrium allocation, i.e.,

$$V(c) = \begin{cases} e^{-r\Delta}(\overline{c}_1 - c) & \text{if } c \in [0, \overline{c}_1] \\ 0 & \text{if } c \in [\overline{c}_1, 1] \end{cases},$$

which is obviously a convex function.

To complete the construction of the equilibrium, we need to specify the off-path equilibrium prices of $q_0 \in [0, 1]$ in the first period and $q_1 \in [0, 1)$ in the second period. The price of these quantities, $p_0$ and $p_\Delta$, must satisfy the same conditions given by the minimization problems (7) and (8) but with this new utility function $V(c)$. The first-order conditions of these minimization problems in $c$ are:

$$-V'_+(\widetilde{c}_0) \leq q_0 + e^{-r\Delta}\widetilde{q}_1^* \leq -V'_-(\widetilde{c}_0)$$

and

$$-V'_+(\widetilde{c}_1) \leq \widetilde{q}_0^* + e^{-r\Delta}q_1 \leq -V'_-(\widetilde{c}_1).$$

Notice that

$$V'_-(c) = V'_+(c) = \begin{cases} -e^{-r\Delta} & \text{if } c \in (0, c_1) \\ 0 & \text{if } c \in (c_1, 1] \end{cases},$$

$$V'_-(0) = -\infty, V'_+(0) = -e^{-r\Delta}, V'_-(c_1) = -e^{-r\Delta} \text{ and } V'_+(c_1) = 0,$$

where $q_0^*$ and $q_1^*$ are optimal solutions in the quantity variable. By the convexity of $V$, to show that $\widetilde{c}_0 = \widetilde{c}_1 = 0$, it suffices that

$$e^{-r\Delta} \leq q_0 + e^{-r\Delta}\widetilde{q}_1^* \text{ and } e^{-r\Delta} \leq \widetilde{q}_0^* + e^{-r\Delta}q_1.$$

These conditions are trivially satisfied if $\widetilde{q}_1^* = 1 - q_0$ and $\widetilde{q}_0^* = 1 - q_1$.

We need to construct $p_0$ and $p_\Delta$ that solve (7) and (8) at $\widetilde{q}_1^* = 1 - q_0$ and $\widetilde{q}_0^* = 1 - q_1$, respectively. Since the on-path equilibrium is selling the whole unit in the second period, the construction of the off-path equilibrium prices is much simpler. Indeed, let us guess the following constant prices:

$$p_0(\cdot) = e^{-r\Delta}\overline{p}_1 \text{ and } p_\Delta(\cdot) = \overline{p}_\Delta.$$

With these constant prices, it is easier to see that the solutions of the minimization problems (7) and (8) at $c = 0$ are $\widetilde{q}_1^* = 1 - q_0$ and $\widetilde{q}_0^* = 1 - q_1$, respectively. Moreover, since $V(0) = e^{-r\Delta}\overline{p}_\Delta = e^{-r\Delta}\overline{c}_1$, (7) and (8) are equivalent to following conditions:

$$p_0(q_0)q_0 = e^{-r\Delta}\overline{p}_\Delta - e^{-r\Delta}p_\Delta(1 - q_0)(1 - q_0)$$

and

$$e^{-r\Delta}p_{\Delta}(q_1)q_1 = e^{-r\Delta}\overline{p}_{\Delta} - p_0(1-q_1)(1-q_1),$$

for each $q_0 \in [0,1]$ and $q_1 \in [0,1)$, which are trivially true by the definition of $p_0$ and $p_{\Delta}$.

Finally, if the off-path beliefs are concentrated at type $c = 0$, the buyer's no deviation conditions boil down to check that $p_0(\cdot) \geq v(0)$ and $p_{\Delta}(\cdot) \geq v(0)$ on the off-path quantities, which is true by the price definition and since $e^{-r\Delta}\overline{c}_1 \geq v(0)$. □

*Proof of Theorem 3.* In continuous time, we restrict the set of trading space to all possible discrete time opportunities:

$$\mathcal{R} = \left\{ \begin{array}{l} \{(t_1,q_1),(t_2,q_2),...,(t_k,q_k)\}; 0 \leq t_1 < t_2 < ... < t_k; \\ q_s \in [0,1] \text{ with } \sum_{s=1}^{k} q_s \leq 1, k \in \mathbb{N} \cup \{\infty\} \end{array} \right\}.$$

Let $U(c) = e^{-r\tau(c)}(v(c) - c)$ be the type $c$ utility and $Q(c) = e^{-r\tau(c)}$ the quantity of the commitment equilibrium, where $c(t)$ is the inverse function $\tau(c)$. On the equilibrium path the seller's utility and quantity are the commitment one.

A (measurable) price function $p : [0,\infty)\times[0,1] \to \mathbb{R}_+$ and belief $\nu : [0,1]\times[0,\infty)\times[0,1] \to \mathbb{R}_+$ are an equilibrium of the continuous time model if, for every time $t \in [0,\infty)$ and quantity $q \in [0,1]$:

• seller's no deviation condition: the price $p(t,q)$ is determined by the type that has the most willingness to buy quantity $q$ at time $t$

$$p(t,q)q = \min_{c\in[0,1]} \inf_{R\in\mathcal{R}} \left\{ e^{rt}U(c) + cq - \sum_{(s,q_s)\in R-\{(t,q)\}} e^{-r(s-t)}(p(s,q_s) - c)q_s \right\} \quad (10)$$

• buyer's no deviation condition: the buyer's belief $\mu(\cdot|t,q)$ is concentrated at a type $\widehat{c} \in [0,1]$ that solves the minimization problem (10) and $v(\widehat{c}) \leq p(t,q)$.

We separate the proof in two main steps: (i) the existence and characterization of equilibrium prices and (ii) the (unique) pattern of trade and beliefs.

(i) characterization: we claim that $p^*(t,q) = v(c(t))$, i.e., $b(t,q) = v(c(t))q$ satisfies (10), for all $(t,q)$. Indeed, fix $q > 0$ (the case $q = 0$ is proved by taking the limit when $q \to 0$)

and $t \in [0, \infty)$. Let us define

$$\mathcal{P} = \left\{ \; q^{-1} \left( e^{rt} U(c) + cq - \textstyle\sum_{(s,q_s) \in R - \{(t,q)\}} e^{-r(s-t)} (v(c(s)) - c) q_s \right) ; R \in \mathcal{R} \; \right\}.$$

To show that $p(t, q) = v(c(t))$ is the infimum of $\mathcal{P}$ is equivalent to show that $p(t, q)$: (a) is a lower bound of $\mathcal{P}$, i.e., $p(t, q) \leq \pi$, for all $\pi \in \mathcal{P}$; (b) is highest lower bound, i.e., for every $\epsilon > 0$, there exists $\pi \in \mathcal{P}$ such that $\pi < p(t, q) + \epsilon$.

From the commitment equilibrium, we know that

$$U(c) \geq e^{-rs}(v(c(s)) - c),$$

for any $c$, $s$. By multiplying by $q_s$ on both sides and adding up in $s$, we get

$$U(c) \geq e^{-rt}(v(c(t)) - c)q + \sum_{(s,q_s) \in R - \{(t,q)\}} e^{-st}(v(c(s)) - c)q_s,$$

for every $c \in [0, 1]$, $R \in \mathcal{R}$, which is equivalent to

$$p(t, q) \leq \pi := q^{-1} \left( e^{rt} U(c) + cq - \sum_{(s,q_s) \in R - \{(t,q)\}} e^{-r(s-t)} (v(c(s)) - c) q_s \right).$$

Therefore, condition (a) holds.

Let $\epsilon > 0$ and $R = \{(t, q), (t + \lambda, 1 - q)\} \in \mathcal{R}$, for some $\lambda > 0$. Hence, there is a unique $\widetilde{c} = \widetilde{c}(\lambda, t, q)$ such that $t < t(\widetilde{c}) < t + \lambda$ satisfying

$$e^{-rt} q + e^{-r(t+\lambda)}(1 - q) = e^{-rt(\widetilde{c})}.$$

Notice that, when $\lambda \to 0$, $\widetilde{c} \to c(t)$. By continuity, we can find $\lambda > 0$ sufficiently small such that

$$0 \leq U(\widetilde{c}) - \left[ e^{-rt}(v(c(t)) - \widetilde{c})q + e^{-r(t+\lambda)}(v(c(t + \lambda)) - \widetilde{c})(1 - q) \right] < \epsilon q$$

or $\pi = q^{-1}\left(e^{rt}U(\widetilde{c}) + \widetilde{c}q - e^{-r\lambda}(p(t+\lambda, 1-q) - \widetilde{c})(1-q)\right) \in \mathcal{P}$ satisfies

$$\pi < p(t,q) + \epsilon,$$

which shows that condition (b) holds.

(ii) (unique) pattern of trade and beliefs: notice that

$$U(c) = (v(c) - c)e^{-r\tau(c)} > (v(\widehat{c}) - c)e^{-r\tau(\widehat{c})} \tag{11}$$

for all $c \neq \widehat{c}$ (the strict inequality is a consequence of the strict convexity of function $U(\cdot)$). Hence,

$$U(c) > \sum_s q_s e^{-rs}(v(c(s)) - c) \geq \sum_s q_s e^{-rs}(p(s,q_s) - c),$$

since $p(s,q_s) = v(c(s))$, $q_s \geq 0$ and $\sum_s q_s \leq 1$ such that $q_{\tau(c)} < 1$. Therefore, type $c$ seller will strictly prefers to buy the whole unit in time $\tau(c)$ instead any other strategy that spread its trade along time. Hence, the buyer's beliefs can set as $\nu^*(c|t,q) = \delta_{c(t)}$, for all $(t,q)$. □

# References


Akerlof, G. (1970). The Market for Lemons: Quality Uncertainty and the Market Mechanism. *Quarterly Journal of Economics 84*, 488–500. 4

Asriyan, V. and V. Vanasco (2021). Security Design in Non-Exclusive Markets with Asymmetric Information. *Working paper*. 7

Attar, A., T. Mariotti, and F. Salanié (2009). Non-exclusive competition in the market for lemons. *TSE Working paper series*. 11

Attar, A., T. Mariotti, and F. Salanié (2011). Nonexclusive competition in the market for lemons. *Econometrica 79*(6), 1869–1918. 7

Auster, S., P. Gottardi, and R. Wolthoff (2021). Simultaneous Search and Adverse Selection. *Working paper*. 7

Ausubel, L., P. Cramption, and R. Deneckere (2002). Bargaining with incomplete information. *Handbook of Game Theory with Economic Applications 3*, 1897–1945. 5

Azevedo, E. and D. Gottlieb (2017). Perfect Competition in Markets With Adverse Selection. *Econometrica 85*(1), 67–105. 11

Bisin, A. and P. Gottardi (2006). Efficient Competitive Equilibria with Adverse Selection. *Journal of Political Economy 114*(3), 485–516. 11

Coase, R. (1972). Durability and Monopoly. *Journal of Law and Economics 15*(1), 143–149. 5

Daley, B. and B. Green (2012). Waiting for News in the Market for Lemons. *Econometrica 80(4)*, 1433–1504. 11

DeMarzo, P. and D. Duffie (1999). A liquidity-based model of security design. *Econometrica 67*(1), 65–99. 6

Dubey, P. and J. Geanakoplos (2002). Competitive pooling: Rothschild-stiglitz reconsidered. *The Quarterly Journal of Economics 117*(4), 1529–1570. 11

Eeckhout, J. and P. Kircher (2010). Sorting and decentralized price competition. *Econometrica 78*(2), 539–574. 11

Fuchs, W., A. Öery, and A. Skrzypacz (2016). Transparency and Distressed Sales under Asymmetric Information. *Theoretical Economics 11*(3), 1103–1144. 5

Fuchs, W. and A. Skrzypacz (2019). Costs and benefits of dynamic trading in a lemons market. *Review of Economic Dynamics 33*, 105–127. 2

Fuchs, W. and A. Skrzypacz (2022). *Dynamic Bargaining with Private Information*. Palgrave Macmillan. 5

Gale, D. (1992). A Walrasian theory of markets with adverse selection. *The Review of Economic Studies 59*, 229–255. 6

Gerardi, D., L. Maestri, and I. Monzón (2022). Bargaining over a Divisible Good in the Market for Lemons. *American Economic Review forthcoming*. 5

Guerrieri, V., R. Shimer, and R. Wright (2010). Adverse selection in competitive search equilibrium. *Econometrica 78*(6), 1823–1862. 6, 11

Hörner, J. and N. Vieille (2009). Public vs. Private Offers in the Market for Lemons. *Econometrica 77(1)*(December), 29–69. 5

Janssen, M. C. W. and S. Roy (2002). Dynamic Trading in a Durable Good Market with Asymmetric Information. *International Economic Review 43:1*, 257–282. 4

Kurlat, P. (2016). Asset markets with heterogeneous information. *Econometrica 84*(1), 33–85. 7

Lee, J. (2021). Price Experimentation in Confidential Negotiations. *Working paper*. 7

Leland, H. E. and D. H. Pyle (1977). Informational Asymmetries, Financial Structure, and Financial Intermediation. *The Journal of Finance 32(2)*, 371–387. 6, 29

Li, Q. (2022). Security Design without Verifiable Retention. *Journal of Economic Theory 200*. 6

Moreno, D. and J. Wooders (2016). Dynamic Markets for Lemons: Performance, Liquidity, and Policy Intervention. *Theoretical Economics 11*, 601–639. 6

Nöldeke, G. and E. van Damme (1990, jan). Signalling in a Dynamic Labour Market. *The Review of Economic Studies 57*(1), 1–23. 5

Rothschild, M. and J. Stiglitz (1976). Equilibrium in competitive insurance markets: An essay on the economics of imperfect information. *The Quarterly Journal of Economics 90(4)*, 629–649. 11

Spence, M. (1973). Job market signaling. *Quarterly Journal of Economics 87*, 355–374. 4

Swinkels, J. M. (1999). Education Signaling with Preemptive Offers. *Review of Economic Studies 66*, 949–970. 5

Williams, B. (2021). Search, Liquidity, and Retention: Screening Multidimensional Private Information. *Journal of Political Economy 129*(5), 1487–1507. 6, 32

# Online Appendix

Let $U : [0, 1] \to \mathbb{R}$ be any function. We say that $D$ is a subgradient of $U$ at $c$ if

$$D(c - \tilde{c}) \geq U(c) - U(\tilde{c}),$$

for all $c, \tilde{c} \in [0, 1]$. We denote $\partial U(c)$ the set of all subgradients of $U$ at $c$. The following lemma is useful.

**Lemma 6.** *Let $U : [0, 1] \to \mathbb{R}_+$ be a convex function. The following properties hold:*

*(a) $\partial U(c)$ is an interval;*

*(b) $\partial U(c) \geq \partial U(\tilde{c})$, for all $c > \tilde{c}$;*[25]

*(c) $\partial U(c)$ is singleton if and only if $U$ is differentiable at $c$;*

*(d) $\{c \in [0, 1]; \partial U(c)$ is not singleton$\}$ is countable.*

*Proof.* The proof is a consequence of classical results of Convex Analysis. □

Let $\Omega^b$ be the set of posted contracts (by the buyer). For all $\omega \in \Omega^b$, let $Q(\omega) = \sum_{t \in R} e^{-rt} q\,(t; \omega)$ denote the total discounted quantity traded and $P : \Omega^b \to \mathbb{R}_+$ the price function. Prices can be written simply as functions of $Q$, hence $P(\omega) = P(Q(\omega))$. Define the seller's value function $U : [0, 1] \to \mathbb{R}_+$ as

$$U(c) = \max_{\omega \in \Omega^b} (P(Q(\omega)) - c)Q(\omega).$$

The following lemma shows that $U$ is convex and provides its envelope condition.

**Lemma 7.** *(Seller's value function). $U(c)$ is a non-increasing convex function.*
*If $\omega(c) \in \arg\max_{\omega \in \Omega^b} (P(Q(\omega)) - c)Q(\omega)$, then $-Q(\omega(c)) \in \partial U(c)$.*[26]

*Proof.* $U(\cdot)$ is a convex function because it is the maximum of the linear functions $\varphi(c|Q) = (P(Q) - c)Q$ indexed by $Q(\omega) : \omega \in \Omega^b$. The other properties are immediate. □

[25] $\partial U(c) \geq \partial U(\tilde{c})$ means that $q \geq \tilde{q}$, for all $q \in \partial U(c)$ and $\tilde{q} \in \partial U(\tilde{c})$.
[26] The reciprocal result is true when $\Omega^b$ is a closed set.

Let us write, more compactly, $Q(c) = Q(\omega(c))$. The next lemma provides the link between the subgradient of $U$ and the optimal seller's optimal contract.

**Lemma 8.** *(Characterization). $Q(c) \in -\partial U(c)$, for all $c$, if and only if $Q(c)$ is an optimal contract for the seller with type $c$, with price function $P(Q(c)) = \frac{U(c)}{Q(c)} + c$, for all $c$.*

*Proof.* Notice that $-Q(\tilde{c}) \in \partial U(\tilde{c})$ if and only if

$$-Q(\tilde{c})(\tilde{c} - c) \geq U(\tilde{c}) - U(c),$$

or, equivalently,

$$U(c) \geq P(Q(\tilde{c}))Q(\tilde{c}) - cQ(\tilde{c}),$$

for all $c, \tilde{c} \in [0, 1]$, i.e., $Q(c)$ is an optimal contract for the type-$c$ seller with price function $P(Q(c)) = \frac{U(c)}{Q(c)} + c$. □

**Lemma 9.** *$\varphi(c)$ is a well defined and increasing function such that $\varphi(c) \in (v(c), v(\varphi(c)))$, for all $c \in [0, 1)$, and $\varphi(1) = 1$.*

*Proof.* Let $c \in [0, 1)$. If $\varphi = c$, the left hand side (l.h.s.) of (6) becomes $v(c) - c > 0$, and if $\varphi = 1$, it becomes $E[v(\widetilde{c})|\widetilde{c} \in [c, 1]] - 1 < 0$. Therefore, by the intermediate value theorem, there exists $\varphi(c) \in (c, 1)$ that solves the equation (6). Notice that the derivative of the left hand side (l.h.s.) of (6) with respect to $\varphi$ evaluated at a solution $\varphi(c)$ is

$$(v(\varphi(c)) - \varphi(c))\frac{f(\varphi(c))}{\int_c^{\varphi(c)} f(\widetilde{c})d\widetilde{c}} - 1 < 0,$$

which implies that $\varphi(c)$ is uniquely determined. Moreover, since the l.h.s. of (6) is strictly increasing in $c$, $\varphi(c)$ must be strictly increasing in $c$. Finally, $\varphi(c)$ is an average of the increasing function $v(c)$ in the interval $[c, \varphi(c)]$ and, therefore, $\varphi(c) \in (v(c), v(\varphi(c)))$, for all $c \in [0, 1)$, and $\varphi(1) = 1$. □

Since $\varphi(c) > v(c) > c$, we have that $1 - \varphi(c) \leq 1 - c$, for all $c \in [0, 1)$, which implies that $\varphi'(1) \leq 1$. The following lemma provides a condition that ensures $\varphi'(1) < 1$.

**Lemma 10.** *Suppose that there exists* $\alpha \in (0,1)$ *such that* $v(c) \geq \alpha(c-1)+1$ *in a neighborhood of* $1$. *Then* $\varphi'(1) \leq \alpha < 1$.

*Proof.* By the mean value theorem for integrals, for each $c$, there exists $\widetilde{\varphi}(c) \in (c, \varphi(c))$ such that $\varphi(c) = v(\widetilde{\varphi}(c))$. Hence, for $c$ sufficiently close to $1$, we have that

$$\varphi(c) = v(\widetilde{\varphi}(c)) \geq \alpha(\widetilde{\varphi}(c) - 1) + 1,$$

which implies that $\alpha(1 - \widetilde{\varphi}(c)) \geq 1 - \varphi(c)$. Dividing both sides by $1-c$ and making $c \to 1$, we get $\alpha\widetilde{\varphi}'(1) \geq \varphi'(1)$. Moreover, since $1 - \widetilde{\varphi}(c) \leq 1 - c$, we have that $\widetilde{\varphi}'(1) \leq 1$. Hence, $\varphi'(1) \leq \alpha < 1$. □