On the Impossibility of an Exact Imperfect Monitoring Folk Theorem*

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Abstract

We generalize techniques used by Athey and Bagwell (2001) in a specific game to characterize efficient equilibria of almost every two player game with imperfect public monitoring. Using this characterization, we show that an exact folk theorem can only hold for a measure zero set of such games. In particular, the classic folk theorem is not robust to small ammounts of noise in the observation of other players' actions. In another application, we provide easy to check conditions that guarantee that a repeated game has no efficient equilibria. We also obtain a new inefficiency result for the imperfect monitoring prisoner's dilemma.

1 Introduction

It is well known that repeated games with imperfect monitoring admit an approximate folk theorem (Fudenberg, Levine and Maskin, 1994). More precisely, for a large class of such games, any feasible individually rational payoff may be approximated by perfect public equilibria (PPEs)[1], as players become infinitely patient. However, this theorem is silent as to whether such a game has any efficient equilibria at all. Therefore, an interesting question is for what class of games the approximate result can be extended to an exact folk theorem.

In this paper, we show that any such class would have to be very restrictive, encompassing only a measure zero set of games. More precisely, we show that almost every game of imperfect monitoring has feasible individually rational payoffs which are not PPE values, for any discount factor. Therefore, an exact folk theorem cannot hold for a generic game.

To prove this result, we generalize the techniques of Athey and Bagwell (2001). Their paper focuses on a specific duopoly game and is the first to

*Part of this research was done when the first author was a student at the Getulio Vargas Foundation. We would like to thank Aloisio Araujo, Susan Athey, Vinicius Carrasco, Eduardo Faingold, Drew Fudenberg, Johannes Hörner, Vijay Krishna, Andreu Mas-Colell, Stephen Morris and seminar participants at IMPA, Getulio Vargas Foundation Graduate School of Economics and Game Theory Festival, Stony Brook (2007) for comments and suggestions. Any remaining mistakes are our own.

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[1] Throughout the paper, we will use equilibria interchangeably with PPEs.

describe efficient PPEs of a repeated game with imperfect monitoring. We extend their method to have a complete description of the set of efficient PPEs for almost every two-player game. Our characterization is in terms of a static contracting problem, which considerably simplifies the derivation of equilibria in this and similar games.

Another related work is Fudenberg, Levine and Takahashi (2007). They present an algorithm for determining the limit set of efficient PPEs. Although their techniques apply to the n-player case, they only characterize the limit set as players become infinitely patient. In contrast, our characterization, while being restricted to the two-player case, gives the set of efficient PPEs for fixed values of impatience factor less than one.

The characterization will also be used to obtain a simple condition that guarantees that a repeated game has no efficient equilibria. As an immediate corollary we obtain an inefficiency result for the repeated imperfect monitoring prisoner's dilemma. This result is new to the literature, and valid for general perturbations of payoffs and information structures.

We now illustrate our results in an example.

1.1 The prisoner's dilemma

Consider a repeated prisoner's dilemma with imperfect monitoring, usually described as a partnership problem. Two players ($i = 1, 2$) own a firm and operate it. Each period, the effort levels of player i are[2] $s_i \in \{c, d\}$, i.e., he may either cooperate (c) or defect (d). Although the firm's profit $y \in Y \subset \mathbb{R}$ is observed, it is randomly distributed according to $\pi(\cdot|s)$, where $s = (s_1, s_2)$. While effort is not directly observable, it affects the distribution of y. Assuming that the effort levels represents also their costs, player i's payoff is $r_i(s_i, y) = y/2 - s_i$.

The average payoff of player i is defined by

$$g_i(s) = E[r_i|s] = \int r_i(s, y) d\pi(y|s).$$

For the game to be interesting, they should be distributed as in a prisoners' dilemma, indicated in Figure 1:

$$\begin{aligned} g_1(c,d) &< g_1(d,d) < g_1(c,c) < g_1(d,c) \\ g_2(d,c) &< g_2(d,d) < g_2(c,c) < g_2(c,d). \end{aligned}$$

This stage game is repeated every period and players discount future utility by some impatience factor $0 < \delta < 1$.

It is well known that, in this example, the approximate folk theorem generically holds if Y has at least three elements (Fudenberg, Levine and Maskin, 1994). That is, any feasible individually rational payoff may be approximated by PPEs as δ approaches to 1. Yet, we will show that generically the prisoner's

[2]We use the subindex to denote the players' identity which is also the coordinate of the payoff vector as well.

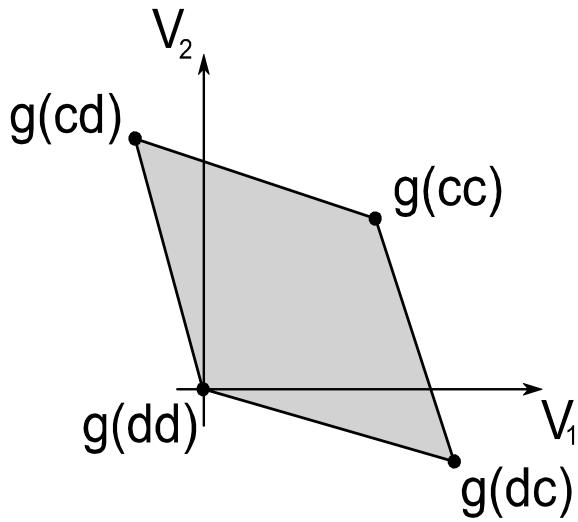

Figure 1: The feasible payoffs in the prisoner's dilemma.

dilemma has no efficient equilibria, for any fixed discount factor. This is an immediate consequence of **Theorem 1**, which guarantees that games with a certain combinatorial structure do not have efficient equilibria. Because the conditions of Theorem 1 are very easy to check, this result can be used to rule the existence of efficient equilibria for a large class of games.

Also, this means that the prisoner's dilemma has feasible strictly individually rational payoffs which are not equilibrium values. So, the conclusions of the classic folk theorem fail. We will show that this phenomenon is very general. Except for some special cases, an exact folk theorem is false for almost every game with imperfect monitoring (**Theorem 2**).

However, the example above is somewhat misleading, as our anti-folk theorem is not simply an inefficiency result. Indeed, there are open sets of games which have efficient equilibria. The game in Athey and Bagwell (2001), for example, has efficient equilibria which are robust to perturbations of the game. Thus, the general case will be more complicated.

Regarding the prisoner's dilemma inefficiency, Radner, Myerson and Maskin (1986) showed that in some cases where the approximate folk theorem fails, the equilibria are bounded away from the efficient frontier. However, we are not aware of inefficiency results in cases where the approximate folk theorem holds.

We now lay down the model, give the characterization of the efficient equilibria and prove the results.

2 The Framework

2.1 Stage game and supergame

Two players play a stage game at $t = 0, 1, 2, \cdots$. Player i takes actions s_i in a *finite* set S_i. Actions are not directly observable, but they induce probability measures $\pi(\cdot|s)$ on a *finite* set of public outcomes Y (denoted by ΔY), where $s = (s_1, s_2) \in S = S_1 \times S_2$. The actual payoff $r_i(s_i, y)$ of player i depends on his own action and on the public outcome. However, it does not depend directly on the other player's action (although it does affect the distribution of y). We allow for mixed actions α_i in ΔS_i, the space of probabilities on S_i. Probabilities $\pi(\cdot|\alpha)$ and payoffs $r_i(\alpha_i, y)$ are defined as usual, for any mixed strategy profile $\alpha = (\alpha_1, \alpha_2)$.

Let $g_i(\alpha) = E[r_i(\alpha_i, y)|\alpha]$ be the average gain of player i from playing a profile α, where $E[\cdot|\cdot]$ represents the conditional expectation operator.

We may collect these elements into the following:

Definition 1 *A stage game* Γ *is a list* $(S_i, Y, r_i : S_i \times Y \longrightarrow \mathbb{R}, \pi : S \longrightarrow \Delta Y)_{i=1,2}$*, where* S_1*,* S_2 *and* Y *are finite sets.*

We also admit correlated strategies (see Aumann, 1987), i.e., every period players can condition their action on a public randomization device[3]. This makes the set of feasible payoffs a convex polygon - the convex hull of the pure strategy payoffs. Figure 2 gives then the set of feasible payoffs of a typical game with the following strategic form:[4]

$$\begin{array}{c} \\ u \\ m \\ d \end{array} \quad \begin{array}{c} \quad l \quad r \\ \begin{pmatrix} A & B \\ C & D \\ E & F \end{pmatrix} \end{array}.$$

For simplicity, we then make the following assumption, which holds for almost every game:

A1. *The set of feasible payoffs of the supergame is a polygon such that each side contains only two pure action profiles.*[5]

The public history at time t is defined as $h^t = (y^1, y^2, \cdots y^{t-1})$. Players also observe their own actions, and player i's private history is $h_t^i = (s_i^1, s_i^2, \cdots s_i^{t-1})$. A strategy for player i is a sequence of maps sequence of histories $h^t \times h_i^t$ to randomized strategies ΔS_i, for all $t = 0, ..., \infty$.

In the repeated game (*supergame*) player i maximizes the normalized average payoff:

[3] To make exposition simpler, we will not introduce explicit notation for the correlated strategies. This does not change any of the results, and is common practice in the literature (Athey and Bagwell, 2001, and Fudenber, Levine and Maskin, 1994).

[4] From now on we will use the letters $A, B, \cdots$ to denote both action profiles, such as ul, and their payoff vectors, $g(ul)$.

[5] More precisely, only payoff vectors of two pure action profiles.

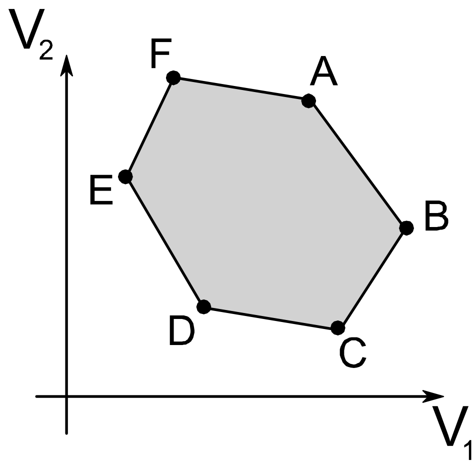

Figure 2: The set of feasible payoffs for a generic game.

$$v_i = (1-\delta)\sum \delta^t E\left[g_i^t\right]$$

for a given fixed $0 \leq \delta < 1$.

2.2 Characterization of the PPEs

Our solution concept is the perfect public equilibrium (PPE). That is, a subgame perfect equilibria in which players condition their actions only on the public history. Abreu, Pearce and Stachetti (1990) show that, as long as other players play public strategies, there is no gain in conditioning on private history.[6] Let $PPE(\delta)$ denote the set of all PPE values for a given discount factor δ.

To characterize the PPEs, we heavily rely on recursive methods, so we now state some standard definitions that will be used.[7] The key idea of the recursive approach is to factor equilibria into an action profile to be played on the current period, and continuation values to be played conditional on the public outcome y. The continuation values for each player are expressed by reward functions $u : Y \longrightarrow \mathbb{R}^2$.

Definition 2 *A reward function u δ-enforces a profile α if, for every $i = 1, 2$ and $s_i \in S_i$,*

$$\begin{aligned} v_i &= (1-\delta)g_i(\alpha) + \delta E[u_i(y)|\alpha] \\ &\geq (1-\delta)g_i(s_i, \alpha_{-i}) + \delta E[u_i(y)|s_i, \alpha_{-i}] \end{aligned} \quad (1)$$

[6] Yet, Kandori and Obara (2006) show that efficiency may be improved if all players use private strategies.

[7] Abreu, Pearce and Stachetti (1990), Fudenberg, Levine and Maskin (1994) and Mailath and Samuelson (2006) are good references.

that is,

$$E[u_i(y)|\alpha] - E[u_i(y)|s_i, \alpha_{-i}] \geq \frac{1-\delta}{\delta}(g_i(s_i, \alpha_{-i}) - g_i(\alpha)).$$

A key element of the recursive approach is the Bellman map T. The Bellman map of a set $W \subset \mathbb{R}^2$ is the set of values that may be achieved by using promises on W.

Definition 3 *The Bellman map T is defined for compact subsets of $\mathbb{R}^2$ by*

$$T(W) = \text{co}\{g(\alpha) + E[u|\alpha] : u \text{ takes values in } W \text{ and } \delta\text{-enforces } \alpha\},$$

where co is the convex hull. A set $W \subset \mathbb{R}^2$ is self-generating if $W \subset T(W)$.

In the next section we will characterize the equilibria of generic two-player games. Section 4 applies the characterization to prove Theorem 2.

3 Efficient PPEs

We will now give, for almost every two-player game, a complete description of the efficient equilibria. Since the set of feasible payoffs is a polygon, we just need a method to determine which are the equilibria on each side of the polygon. Then, by applying this method to each side of the polygon, we may find all efficient equilibria on a finite number of steps.

Consider, from now on, a side $\overline{AB}$ of the polygon of set of feasible values. For simplicity, assume $\overline{AB}$ to be negatively inclined with slope with absolute value m, as in Figure 2 (which is true generically). Also, suppose that $\overline{AB}$ contains no static equilibria. Our strategy will be to somehow restrict the difficult problem of determining the game's efficient PPEs to the one-dimensional segment $\overline{AB}$.

3.1 Mixing property

In general, restricting the problem of finding equilibria to the one-dimensional segment cannot be done, because for equilibria with payoffs in $\overline{AB}$ one may use equilibrium values outside $\overline{AB}$ as punishments. So we need a generic condition which we call *mixing*. A strategy profile is locally mixing if deviations from it cannot be detected with certainty. The stage game is said to be mixing if every profile is locally mixing. Formally:

Definition 4 *The strategy profile s is locally mixing if*[8] *$\pi(\cdot|\tilde{s}_i, s_{-i}) \ll \pi(\cdot|s)$ for every $i = 1, 2$ and $\tilde{s}_i \in S_i$. The stage game Γ is mixing if every profile $s \in S$ is locally mixing. Or equivalently, if $\pi(\cdot|s)$ are equivalent, for every $s \in S$.*

[8] That is, $\pi(\cdot|\tilde{s}_i, s_{-i})$ is absolutely continuous with respect to $\pi(\cdot|s)$.

Mixing is the key property that allows us to work out the equilibria on any given side $\overline{AB}$ of the feasible values polygon without having to compute all the game's equilibria. Because mixing implies that any public history may happen with positive probability, any prescribed punishments are carried out with positive probability in equilibrium. Thus, equilibria with values on $\overline{AB}$ may only use continuation values and strategy profiles on $\overline{AB}$.

Let $I(\delta) = \overline{AB} \cap PPE(\delta)$. In terms of the Bellman map these facts translate into the following:

Lemma 1 *If A and B are locally mixing, the closed interval $I(\delta)$ is the largest self-generating closed interval in $\overline{AB}$.*

Proof. $I(\delta)$ is a closed interval, as it is the intersection of $\overline{AB}$ with $PPE(\delta)$, which is known to be closed and convex. But every self generating interval in $\overline{AB}$ is contained in $I(\delta)$. Because of mixing, for every self-generating set $W \subset \mathbb{R}^2$, $T(W) \cap \overline{AB} \subset T(W \cap \overline{AB})$. Since $PPE(\delta)$ is self-generating, we have

$$\begin{aligned} I(\delta) &= PPE(\delta) \cap \overline{AB} = T(PPE(\delta)) \cap \overline{AB} \\ &\subset T(PPE(\delta) \cap \overline{AB}), \end{aligned}$$

that is, $I(\delta)$ is self-generating. ■

3.2 Characterization of efficient PPEs

In this subsection we will characterize the set of equilibria in $\overline{AB}$. This result is recorded as Lemma 3. By Lemma 1 the set of equilibrium payoffs in $\overline{AB}$ is just the largest self-generating interval on $\overline{AB}$, where A and B are locally mixing. We will now compute this set.

Take any interval $[a, b]$ in $\overline{AB}$. We will check what the conditions for $[a, b]$ to be self-generating are. If $[a', b'] = T([a, b]) \cap \overline{AB}$, then, by the definition of T and the mixing property,

$$a' = (1 - \delta)A + \delta E[\overline{u}|A], \tag{2}$$

where $\overline{u}$ is the promise function that enforces A with minimal value $E[u_1|A]$.

Let ε_δ be the horizontal distance between $E[\overline{u}|A]$ and a. That is, $\varepsilon_\delta = E[\overline{u}_1|A] - a_1$. By rearranging equation (2), we see that $a' \leq a$ if and only if

$$\frac{\delta}{1 - \delta}\varepsilon_\delta \leq a_1 - A_1, \tag{3}$$

that is, if $[a, b]$ is self-generating then (3) holds. Thus, whether efficient equilibria exist depends on how fast ϵ_δ grows compared to $\delta/(1 - \delta)$. We will show that, for sufficiently high patience (close to 1), the rate is the same, and the existence of equilibria will depend on a constant.

Let us now calculate ε_δ. By its definition, we know that

$$\begin{aligned} \varepsilon_\delta &= \min_u E[u_1|A] - a_1 \\ &\text{s.t.} \quad u \ \delta\text{-enforces } A \\ &\qquad\quad u(y) \in [a, b], \forall y \in Y \end{aligned} \tag{4}$$

where u is the promise function that solves program (4). Since δ-enforcing property is invariant by translation, we can show that, at least for δ large enough, it is possible to find ε_δ by the following much simpler program:

$$\begin{aligned} \widetilde{\varepsilon}_\delta &= \min_u \max_y -u_1 \\ & \quad \text{s.t.} \quad u \ \delta\text{-enforces } A \\ & \qquad\quad E[u|A] = 0 \\ & \qquad\quad u_2 = m u_1 \end{aligned} \tag{5}$$

in which u is normalized to have 0 average and m is the absolute value slope of segment $\overline{AB}$. This program is in fact very simple, because if u satisfies its constraints for some δ, then $\frac{\delta}{1-\delta}\frac{1-\delta'}{\delta'}u$ satisfies them for δ'. In particular, $\widetilde{\varepsilon}_\delta = \frac{1-\delta}{\delta}\widetilde{\varepsilon}_{1/2}$.

The next lemma shows formally that the programs (4) and (5) are equivalent for large enough δ.

Lemma 2 *If δ is large enough, then programs (4) and (5) are equivalent, and* $\epsilon_\delta = \widetilde{\varepsilon}_\delta = \frac{1-\delta}{\delta}\widetilde{\varepsilon}_{1/2}$.

Proof. First let us prove that $\widetilde{\varepsilon}_\delta \leq \varepsilon_\delta$. If u satisfies the constraints in program (4), then $u - E[u|A]$ satisfies the constraints in (5). Now, take $\widetilde{u}_{1/2}$ solving program (5) for $\delta = 1/2$ and let $u_\delta = \frac{\delta}{1-\delta}u_{1/2}$ be the solutions for other values of δ. Then, $u = \widetilde{u}_\delta + a + \varepsilon_\delta(1, m)$ satisfies all constraints of program (4), except for having $u_1(y) \leq b_1$ for all $y \in Y$. But if we take δ high enough, this restriction is also satisfied, so $\varepsilon_\delta \leq \widetilde{\varepsilon}_\delta$.[9] ■

By (3), if $[a, b]$ is self-generating, then

$$\widetilde{\varepsilon}_{1/2} \leq a_1 - A_1. \tag{6}$$

Condition (6) allows us to describe a necessary condition for a segment to be self-generating in very simple terms. Since the set of equilibria in $\overline{AB}$ is the largest self-generating interval, this also allows us to describe this set. For this, let us extend the previous procedure for any action profile. Define the amount of punishment that has to be inflicted on player i for him to play his unfavorable action α in the segment $\overline{AB}$:

$$\begin{aligned} P_i^\alpha &= \min_u \max_y -u_i \\ & \quad \text{s.t.} \quad u \ 1/2\text{-enforces } \alpha \\ & \qquad\quad E[u|\alpha] = 0 \qquad \text{(balanced expected payment)} \\ & \qquad\quad u_2 = m u_1 \qquad \text{(budget balance).} \end{aligned} \tag{7}$$

Notice that P_1^A is exactly what we called $\widetilde{\varepsilon}_{1/2}$.

We now sum up the previous discussion in a lemma that describes all the efficient equilibria in $\overline{AB}$. From (6) and (7) we immediately have the following:

[9] This proof and some computer simulations show that, in practice, δ does not have to be very high. For interesting cases we find that δ around 2/3 is enough.

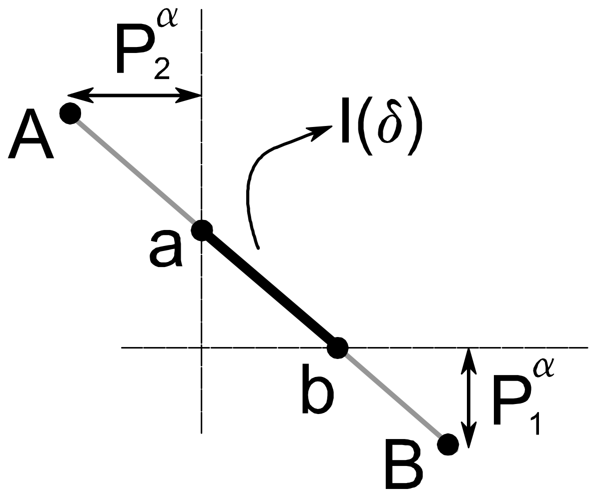

Figure 3: The set of equilibria in $\overline{AB}$.

Lemma 3 *Suppose that A and B are locally mixing. Let a and b be any points in $\overline{AB}$ with $a_1 = A_1 + P_1^A$ and $b_2 = B_2 + P_2^B$ (see Figure 3). Then, for large enough δ the set of equilibria in $\overline{AB}$ is*

- $[a, b]$ *if* $a_1 < b_1$;
- *empty, otherwise.*

Hence, the only restriction on the equilibria is that players have to get at least what they would get playing their least favored action, plus the amount of punishment necessary to enforce it.

Note that, to enforce profile A (respectively B), player 1 (respectively 2) must receive punishment strictly greater than his gain from deviating to his best response to A (respectively B). We have the following useful:

Lemma 4 *Suppose that the stage game is mixing. At any equilibrium payoff in $\overline{AB}$, player 1 (respectively 2) receives strictly more than on his best response to A (respectively B).*

Proof. Let $\widehat{A}$ be the profile where player 2 plays A_2 and player 1 plays his best response to A_2. From program (7) applied to A, since u enforces A and $E[u|A] = 0$, we must have

$$E[-u_1|\widehat{A}] \geq \widehat{A}_1 - A_1.$$

Therefore, $\max_y - u_i > \widehat{A}_1 - A_1$ and $A_1 + P_A^1 > \widehat{A}_1$. ■

4 The Anti-Folk Theorem

The inefficiency result for the prisoner's dilemma and the anti-folk theorem follow easily from the previous lemmata of the previous section. We start with the inefficiency in the prisoner's dilemma:

Theorem 1 (The impossibility of efficient PPEs) *Let* Γ *be a mixing stage game and* $\overline{AB}$ *be a segment negatively inclined contained in the Pareto frontier (as in Figure 3). If any player plays the same action on profiles* A *and* B, *then there is no PPE payoff in* $\overline{AB}$.

Proof. For simplicity suppose again that $\overline{AB}$ is negatively sloped and that A is player 1's least favored action. Suppose also it is player 2 that plays the same action on profiles A and B. Therefore, player 1's best response to A_2 dominates B_1. By Lemma 4 there is no efficient equilibrium in $\overline{AB}$. ■

For the prisoner's dilemma example of section 1.1, since the Pareto frontier is made up of segments $\overline{g(c,d), g(c,c)}$ and $\overline{g(c,c), g(d,c)}$, Theorem 1 applies immediately. Hence, for almost every prisoner's dilemma there are no efficient equilibria.

We now turn to the anti-folk theorem.[10]

Theorem 2 (Anti-Folk) *Almost every supergame with feasible payoffs strictly dominating the minimax point and without efficient static equilibria has feasible individually rational payoffs which are not equilibrium payoffs. That is, the folk theorem is generically false for such games.*

Proof. Consider a game with mixing and only two pure strategy profiles on each edge of the feasible set (this is a generic situation). Take a segment $\overline{AB}$ with a point strictly dominating the minimax value. Suppose that it is negatively sloped, and vertices labeled as in the Lemma 3.

In any equilibrium payoffs in $\overline{AB}$ player 1 gets strictly more than by playing his best response to A. But the best response to A yields at least as much utility as his minimax value. The positively sloped case is similar. This completes the proof. ■

5 Conclusions

Our main contribution is to show that, even though games with imperfect monitoring admit an approximate folk theorem (e.g., Fudenberg, Levine and Maskin, 1994), an exact folk theorem is valid only for a zero measure set of such games with two players.

This finding also shows that the classic folk theorem is extremely unstable with respect to imperfect monitoring. Given a game with perfect monitoring,

[10] The additional assumptions used in the statement are necessary because there are pathological examples in which the folk theorem is true for the stage game, so these cases have to be excluded.

if its informational structure suffers a random perturbation, the classic folk theorem will be false with probability one.

We must point out that our anti-folk theorem is not simply an inefficiency result, since there are open sets of games with efficient PPEs. Yet, the message that the prisoner's dilemma and other similar examples convey is that exact efficiency is difficult to achieve for repeated games with imperfect information.

Also, we give simple conditions that may be used to show that a repeated game has no efficient equilibria. This result implies, in particular, a new inefficiency result for the prisoner's dilemma.

Finally, the equilibrium characterization in terms of a static contracting problem simplifies the derivation of equilibria in games similar to Athey and Bagwell (2001). The characterization and the methods used here can be therefore employed in the analysis of specific games.

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