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Multiproduct price competition with heterogeneous consumers and nonconvex costs

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ABSTRACT

This paper extends the oligopolistic model of price competition to environments with multiple goods, heterogeneous consumers, and arbitrary continuous cost functions. A Nash equilibrium in mixed strategies with an endogenous sharing rule is proven to exist. It is also shown that, in environments with fixed costs and constant marginal costs, all (symmetric and asymmetric) equilibria exhibit price dispersion across stores. Furthermore, the paper identifies scenarios in which prices will necessarily be random. In these markets, stores keep each other guessing because, given the fixed costs, they would incur a loss if their price strategies were anticipated and beaten by competitors. This is interpreted as an important economic feature that is possibly behind random price promotions such as weekly specials.

1. Introduction

This paper extends the Bertrand model of price competition to environments with multiple goods, heterogeneous consumers, and arbitrary continuous cost functions which may differ across stores. The economy has finitely many agents, goods, and retail stores. The setup considers heterogeneous consumers with an arbitrary system of continuous demand functions. No assumption is imposed on the nature of goods, which can be complementary, substitute, or neither. Consumers are perfectly informed about prices and free to choose a store to shop from. Each store decides whether to enter the market or not and, in the first case, chooses a vector of prices to be charged to its customers. Commodity prices are linear, and the model encompasses cases with and without fixed fees (i.e., entry tickets). Attracting customers and pricing product complementarities are the two key elements driving the retailers' pricing strategies.

Existence of a Nash equilibrium is the first issue addressed in this paper. Note that, for some price arrays, consumers are indifferent to multiple stores. The stores' payoff functions are not continuous in these nodes, and therefore the usual fixed-point arguments do not apply. In order to overcome this issue, the equilibrium concept used here allows stores to hold consistent beliefs about how indifferent consumers will act. These beliefs are represented by an endogenous sharing rule, in

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the spirit of Simon and Zame (1990). An equilibrium in mixed strategies is proven to exist in this context.[1] In contrast to this view, many papers impose an exogenous sharing rule to determine the actions of indifferent consumers. The equilibrium properties typically depend on the particular tie-breaking rule adopted.[2] The equilibrium set is larger under the endogenous sharing rule concept. Therefore, in principle, this concept makes things easier for proving existence and harder for deriving the properties of the many possible symmetric and asymmetric equilibria.

The model is then used to study economies with fixed costs and constant marginal costs (which may differ across stores). Under conditions that guarantee that two or more stores enter the market with positive probability, all symmetric and asymmetric equilibria present cross-store price dispersion—that is, entrant stores will charge the same price vector with zero probability. In this scenario, any symmetric equilibrium is characterized by a mixed strategy with no mass point in the set of prices. As for asymmetric equilibria, there must be at least one store whose strategy support is infinite whenever consumers are homogeneous and stores face identical cost functions. In these environments, random pricing is necessary to avoid predatory price competition and then raise revenues to cover fixed costs. This feature of the model is interpreted as a possible explanation for random price promotions such as weekly specials. This supply-side explanation for weekly specials is novel in the literature. As is discussed in Section 6, the existing explanations for random sales rely on informational asymmetries on the consumers' side, instead of cost nonconvexities in the retail activity.

An important related reference is Sharkey and Sibley (1993). These authors study the single-good model of price competition with inelastic consumers and identical stores facing an entry cost and a constant marginal cost.[3] In that setting, it is straightforward to show that no pure-strategy equilibrium exists. Sharkey and Sibley show that any symmetric mixed-strategy equilibrium is atomless on the set of prices. The authors compute a symmetric equilibrium in which stores randomize over the entry decision and a compact set of prices. They also argue that the single-good setup is equivalent to another one in which there are multiple goods, identical stores, and a representative consumer. In that case, the game could be rewritten as one in which stores compete through profit levels. (This is not possible when either consumers or stores are heterogeneous.)

In the multiproduct setup analysed here, in which consumers and stores can possibly be heterogeneous, pure-strategy equilibrium cannot be ruled out. Asymmetric equilibria are analysed throughout the paper since, in this setting, heterogeneous stores may price goods differently in order to attract different consumers. Moreover, while equilibrium prices are always above marginal costs in single-good models, the multi-product setup presents cases with loss leading—that is, stores charging less than the marginal cost for some products in order to increase the sales of other products either through complementarity effects or by attracting more customers.[4]

The remainder of the paper is organized as follows. Section 2 presents the model, and Section 3 proves existence of an equilibrium. Economic environments displaying cross-store price dispersion and random pricing are discussed in Section 4. Next, Section 5 derives a symmetric mixed-strategy equilibrium for economies with homogeneous consumers and stores. This equilibrium is used to identify the key forces behind the possibility of loss leading in oligopoly. Section 6 interprets random pricing as weekly specials and briefly describes existing literature on random sales. Section 7 presents a brief conclusion.

2. Model

Consider an economy with multiple agents and goods, respectively indexed by $i \in \mathbb{I} = \{1, \ldots, I\}$ and $l \in \{0, 1, \ldots, L\}$, where I and L are positive integers. Each agent i is endowed with $w_i > 0$ units of good zero and has preferences represented by a utility function $u_i : \mathbb{R}_+^{L+1} \to \mathbb{R}$, which is continuous and nondecreasing in $x_i \in \mathbb{R}_+^{L+1}$ and strictly increasing in $x_{i,0}$. Good zero is the *numeraire* good; its price is normalized to one. The other L goods are offered by $N \geq 2$ retail stores, indexed by $n \in \mathbb{N} = \{1, 2, \ldots N\}$.

Each store n sets a fixed fee $r_n \in \Delta_r \subset \mathbb{R}$ and linear commodity prices $q_n \in \Delta_q \subset \mathbb{R}_+^L$ in order to maximize its expected profit. The set $\Delta_r \times \Delta_q$ is assumed to be nonempty and compact.[5] This encompasses economies with and without fixed fees, namely, $\Delta_r = [-\sum_{i \in \mathbb{I}} w_i, \sum_{i \in \mathbb{I}} w_i]$ and $\Delta_r = \{0\}$.[6]

Retailers face continuous cost functions $c_n : \mathbb{R}_+^L \to \mathbb{R}_+, \forall n$. These functions need not be convex and may display fixed costs, i.e., $c_n(\mathbf{0}) > 0$. For this reason, the point $p_{\text{out}} \notin \Delta_r \times \Delta_q$ is introduced into the strategy space in order to represent the nonentry decision (in which case the retailer makes zero profit). The strategy space of each store is then given by

$$\Gamma = \Delta_r \times \Delta_q \cup \{p_{\text{out}}\}. \tag{1}$$

[1] This same technique is used by Monteiro and Page (2008) to prove equilibrium existence for a larger class of oligopolistic models, in which stores design catalogs in a metric space in order to compete for privately informed consumers.

[2] Dastidar (1995, 1997) exemplifies how different sharing rules affect the equilibrium outcome in the Bertrand model with convex costs.

[3] See Marquez (1997) for an extension of this model in which fixed costs vary across firms.

[4] Holton (1957) documented the loss-leading practice in the supermarket sector.

[5] The set of possible prices is either a primitive of the model or is implied by endowments and preferences. For instance, an upper bound for q would be naturally defined if the agents' utility functions were quasi-concave and continuously differentiable with bounded $(\partial u_i/\partial x_l)/(\partial u_i/\partial x_0)$, $\forall l$.

[6] In practice, prices are typically linear or piecewise linear because consumers can bypass nonlinearities through reselling products to one another. Fixed fees are practiced by some cooperative supermarkets and chain stores. These fees could also be negative (for example, some stores do offer small gifts for customers).

The consumption set of each agent is $\mathbb{X} = \{x \in \mathbb{R}_+^{L+1} : ||x|| \leq \bar{x}\}$, where $|| \cdot ||$ is a norm and $\bar{x} > \sum_{i \in \mathbb{I}} w_i$ is an arbitrary upper bound for consumption.[7] Each agent can either stay at home and consume the vector $(w_i, \mathbf{0})$, or freely choose one store to shop from. The time needed to select products precludes the agents from shopping in multiple stores.[8]

The economy is represented by a two-stage game. Initially, the retailers simultaneously choose their price vectors in Γ. Next, after observing the price offers, the consumers choose whether to stay at home or shop in some store. In this second step, agents use their indirect utility functions to rank the price offers. The indirect utility and demand functions are defined as follows.

First, notice that shopping in a nonentrant store is equivalent to staying at home. In these cases, the indirect utility and demand are respectively given by

$$v_i(p_{\text{out}}) = u_i(w_i, \mathbf{0}); \tag{2}$$

$$x_i(p_{\text{out}}) = (w_i, \mathbf{0}). \tag{3}$$

Moreover, agent i will never shop in a store in which $r \geq w_i$ (thanks to u_i being strictly increasing in $x_{i,0}$). One can then replace the net income $w_i - r$ by $\max(0, w_i - r)$ in the budget constraints. This is without loss of generality as one still has $v_i(p_{\text{out}}) > v_i(p)$, for any $p = (r, q)$ such that $r \geq w_i$. Therefore, for $p \in \Delta_r \times \Delta_q$, define:

$$v_i(p) = \max\{u_i(x_i) : x_i \in \mathbb{X}, x_{i,0} + \sum_{l=1}^{L} q_l x_{i,l} \leq \max(0, w_i - r)\}; \tag{4}$$

$$x_i(p) = arg\max\{u_i(x_i) : x_i \in \mathbb{X}, x_{i,0} + \sum_{l=1}^{L} q_l x_{i,l} \leq \max(0, w_i - r)\}. \tag{5}$$

Now, assume that $x_i(p)$ is singleton, $\forall p$. Thus, continuity of u_i guarantees that v_i and x_i are both continuous on $\Delta_r \times \Delta_q$. Notice that strict quasiconcavity of u_i is a sufficient condition for $x_i(\cdot)$ being a function instead of a correspondence. However, the setup also incorporates other possibilities such as the traditional case with inelastic consumers—namely, $L = 1$, $\Delta_r = \{0\}, \Delta_q = [0, \bar{q}]$, and $u_i(x_i) = x_{0,i} + \lambda \min(x_{i,1}, 1)$, where $\lambda > \bar{q} > 0$.

The following assumptions summarize the entire setup.

Assumption 1. There are $I \geq 1$ agents endowed with $w_i > 0$ units of good zero who choose bundles in $\mathbb{X} = \left\{x \in \mathbb{R}_+^{L+1} : ||x|| \leq \bar{x}\right\}$, where $\bar{x} > \sum_{i \in \mathbb{I}} w_i$.

Assumption 2. There are $N \geq 2$ retail stores choosing prices in $\Gamma = \Delta_r \times \Delta_q \cup \{p_{\text{out}}\}$, where $\Delta_r \times \Delta_q \subset \mathbb{R} \times \mathbb{R}_+^L$ is nonempty, compact, and does not include p_{out}.

Assumption 3. For each $i \in \mathbb{I}$, one has that: (a) $u_i : \mathbb{R}_+^{L+1} \to \mathbb{R}$ is continuous and nondecreasing in $x_i \in \mathbb{R}_+^{L+1}$, and strictly increasing in $x_{i,0}$; and (b) $v_i : \Gamma \to \mathbb{R}$ and $x_i : \Gamma \to \mathbb{X}$ are continuous functions.

Assumption 4. The cost function $c_n : \mathbb{R}_+^L \to \mathbb{R}_+$ is continuous, $\forall n$.

2.1. Backward induction

2.1.1. Second stage

Define the set of options available for consumers as being $\bar{N} = \{\text{home}\} \cup \mathbb{N}$, where $p_{\{\text{home}\}} = p_{\text{out}}$. After observing the prices $\vec{p} = (p_1, \ldots, p_N) \in \Gamma^N$ resulted from the first stage of the game, the consumers choose $s_i(\vec{p}) \in \bar{N}$ such that:

$$v_i(p_{s_i}) \geq v_i(p_n), \forall n \in \bar{N}. \tag{6}$$

Some consumers might be indifferent between two or more options. Thus, there may be multiple second-stage solutions associated with each $\vec{p}$. An endogenous sharing rule (or tie-breaking rule) will be defined in Section 3 in order to determine what consumers do in case of indifference.

2.1.2. First stage

In the initial node, the N profit-maximizing retailers move simultaneously and choose their offers, $p_n \in \Gamma$. For each possible price vector chosen in the first stage, $\vec{p} = (p_1, \ldots, p_N) \in \Gamma^N$, there will be a second-stage solution, $\{s_i(\vec{p}), \forall i \in \mathbb{I}\}$, which yields the following payoff for retailer n:

$$\text{Profit}(n, \vec{p}) = [q_n \cdot y_n(\vec{p}) + r_n \mathbf{I}(n, \vec{p}) - c_n(y_n(\vec{p}))]\mathbf{1}(p_n); \tag{7}$$

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[7] This is without loss of generality when either consumers are satiated or $\Delta_q \subset \mathbb{R}_{++}^L$.

[8] In fact, saving consumers' time is the main economic motivation behind the retailing activity. Furthermore, the multiproduct model reduces to multiple single-good cases when consumers can costless shop around for the best price of each product.

where $y_n(\vec{p}) \in \mathbb{R}^L_+$ is such that $y_{n,l}(\vec{p}) = \sum_{\{i \in \mathbb{I}: s_i(\vec{p})=n\}} x_{i,l}(p_n)$, $\forall l \neq 0$; $\mathbf{I}(n, \vec{p})$ is the cardinality of $\{i \in \mathbb{I} : s_i(\vec{p}) = n\}$; and $\mathbf{1}(\cdot)$ is an indicator function such that $\mathbf{1}(p_{\text{out}}) = 0$ and $\mathbf{1}(p_n) = 1$, $\forall p_n \neq p_{\text{out}}$.

3. Equilibrium existence

In the second stage, the consumers' actions are not strategic. For each node $\vec{p} = (p_1, \ldots, p_N) \in \Gamma^N$, a second-stage solution is a list of s_i satisfying condition (6), $\forall i \in \mathbb{I}$. The retailers anticipate the possible second-stage solutions associated with each $\vec{p}$ and make their first-stage decisions.

There might be multiple second-stage solutions associated with each $\vec{p} \in \Gamma^N$. In these cases, the retailers acting in the initial node must hold beliefs about the probability of each solution occurring. Here, these beliefs are modeled as endogenous sharing rules, in the spirit of Simon and Zame (1990).[9]

First, define $\Xi(\vec{p})$ as the set of all second-stage solutions associated with each price schedule $\vec{p} \in \Gamma^N$. Formally, let $\Xi : \Gamma^N \to \bar{N}^I$ be a correspondence such that $(s_1, \ldots, s_I) \in \Xi(\vec{p})$ if and only if s_i satisfies (6), $\forall i \in \mathbb{I}$.

Second, let $\Pi : \Gamma^N \to \mathbb{R}^N$ be the correspondence of all profit payoffs that can be achieved in the second-stage solutions associated with each $\vec{p}$ in Γ^N, that is, $\Pi(\vec{p}) = \left\{\vec{\theta} \in \mathbb{R}^N : \exists (s_1, \ldots, s_I) \in \Xi(\vec{p}) \text{ such that } \theta_n = \mathit{Profit}(n, \vec{p}), \text{ as defined in (7)}, \forall n \in \mathbb{N}\right\}$.

Finally, let $\vec{\pi}$ be a Borel measurable function selected from $co\Pi$, where $co\Pi(\vec{p})$ represents the convex hull of $\Pi(\vec{p})$, for each $\vec{p}$ in Γ^N.[10] Since $\Pi(\vec{p})$ represents the set of all profit profiles associated with $\vec{p}$, $co\Pi(\vec{p})$ defines the universe of all possible expected profit payoffs associated with $\vec{p}$, and $\vec{\pi}(\vec{p})$ is a particular expected profit function reflecting the retailers' beliefs about the second stage (i.e., about the tie-breaking rule used by consumers).

Definition 1. The retailers' expected profits are given by a Borel measurable function $\vec{\pi} : \Gamma^N \to \mathbb{R}^N$ selected from the correspondence $co\Pi : \Gamma^N \to \mathbb{R}^N$.

Therefore, the reduced game in the initial node is given by $\{\mathbb{N}, \Gamma, \vec{\pi}\}$. Each retailer chooses a strategy (potentially mixed) to maximize its expected profit. Define $\mathcal{B}$ as the Borel sigma-algebra for Γ and Λ_Γ as the set of probability measures on $(\Gamma, \mathcal{B})$. Thus, the retailer n's strategy is a probability measure $\sigma_n \in \Lambda_\Gamma$ that is a best response to the other retailers' strategies. Formally, the problem faced by each retailer $n \in \mathbb{N}$ in the initial node is

$$\max_{\sigma_n \in \Lambda_\Gamma} \int \pi_n(\vec{p}) \mathrm{d}\sigma^*; \tag{8}$$

where σ^* represents the product measure $\sigma_1 \times \cdots \times \sigma_N$.

Definition 2. An equilibrium consists of a Borel measurable function selected from $co\Pi$ and pricing mixed strategies, namely $\vec{\pi} = (\pi_1, \ldots, \pi_N)$ and $\vec{\sigma} = (\sigma_1, \ldots, \sigma_N)$, such that σ_n solves (8), given $(\sigma_1, \ldots, \sigma_{n-1}, \sigma_{n+1}, \ldots \sigma_N)$, for all $n \in \mathbb{N}$.

Proposition 1. *Under* Assumptions 1–4, *there exists an equilibrium for this economy*.

Proof. Since Γ is a compact metric space, one needs only to prove that $co\Pi$ is bounded and upper hemicontinuous with nonempty, convex, and compact values. The existence result then follows from the main theorem in Simon and Zame (1990, p. 865).

Note first that $\Pi : \Gamma^N \to \mathbb{R}^N$ is nonempty and bounded since: (i) there is always a second-stage solution associated with each $\vec{p} \in \Gamma^N$ as the number of retail stores is finite; and (ii) profits and losses are bounded since both prices and quantities lie in bounded sets and cost functions are continuous. Moreover, Π has a closed graph, thus it is upper hemicontinuous and compact valued. To see the closed-graph property, take any two sequences $\{\vec{p}_k\} \to \vec{p}$ and $\{\vec{\theta}_k\} \to \vec{\theta}$ such that $\vec{p}_k \in \Gamma^N$ and $\vec{\theta}_k \in \Pi(\vec{p}_k), \forall k \in \{1, \ldots, \infty\}$. From the definition of $\Pi(\vec{p}_k)$, there exists $(s_{1,k}, \ldots, s_{I,k}) \in \Xi(\vec{p}_k)$ such that $\theta_{n,k} = \text{Profit}(n, \vec{p}_k)$, as defined in (7), $\forall n, k$. The sequence $\{(s_{1,k}, \ldots, s_{I,k})\}_{k=1}^{\infty}$ is bounded, then there is a subsequence converging to $(s_1, \ldots, s_I)$. Since $s_{i,k} \in \bar{N} = \{\text{home}\} \cup \{1, \ldots, N\}$, this subsequence must be constant for k sufficiently large. Note also that v_i is continuous on $\Delta_r \times \Delta_q$, and there is no sequence of prices in $\Delta_r \times \Delta_q$ converging to p_{out} (and vice versa). Therefore, passing through that subsequence, one has $v_i(p_{s_i}) = \lim_{k\to\infty} v_i(p_{s_{i,k}}) \geq \lim_{k\to\infty} v_i(p_{n,k}) = v_i(p_n)$, $\forall n, i$. Thus, $(s_1, \ldots, s_I) \in \Xi(\vec{p})$. Since c_n and x_i are continuous, then Profit $(n, \vec{p}_k) \to \mathit{Profit}(n, \vec{p})$ and $\vec{\theta} = \lim_{k\to\infty} \vec{\theta}_k \in \Pi(\vec{p})$.

Therefore, $co\Pi$ is bounded and compact valued since Π is bounded and compact valued; upper hemicontinuous since Π is compact valued and upper hemicontinuous; nonempty since Π is nonempty; and convex valued from the definition of convex hull. □

4. Price dispersion

The applied literature on price competition has primarily focused on models with constant marginal costs. Although it is helpful to know that an equilibrium exists for a much more general setup, it is difficult to understand the driving forces

[9] Braido (2005) uses a similar approach in a financial innovation context.

[10] The measurable selection $\vec{\pi}$ always exists since $co\Pi$ is an upper hemicontinuous correspondence, as it will be shown in the proof of Proposition 1.

behind this problem without additional structure. For instance, under general continuous cost functions, one cannot assure that stores are always willing to attract new customers—which is considered a central feature of price competition. Therefore, the following assumptions are introduced.

Assumption 5. Δ_q is a convex set with nonempty interior such that $\mathbf{0} \in \Delta_q$; and Δ_r is either $\{0\}$ or a convex set with nonempty interior such that $0 \in \Delta_r$.

Assumption 6. Each store n faces a cost function given by

$$c_n(y_n) = c_{n,\text{fix}} + \sum_{l=1}^{L} c_{n,l} y_{n,l}; \tag{9}$$

where $y_n = (y_{n,1}, \ldots, y_{n,L})$ represents the amount of products delivered; $c_{n,\text{fix}} > 0$ is the fixed cost; and $(c_{n,1}, \ldots, c_{n,L}) \geq \mathbf{0}$ is the vector of constant marginal costs.

Up to this point, the set of prices could have an empty interior—for instance, $\Delta_r \times \Delta_q$ could be finite. Assumption 5 is needed in order to allow stores to threaten each other with arbitrarily small price cuts. The setup still encompasses models with and without the fixed fees. Assumption 6 imposes constant marginal costs, but still allows cost functions to differ across stores. Under these two additional assumptions, one can prove that the equilibrium strategies cannot have a common mass point in $\Delta_r \times \Delta_q$. In other words, if there are two or more entrant stores, they will charge the same price vector with zero probability. The following lemma will be used to prove this result.

Lemma 1. *Consider a price $p \in \Delta_r \times \Delta_q$ such that $q \cdot x_i(p) + r > 0$, for some i. Under Assumptions 1–5, there exists $p^\varepsilon \in \Delta_r \times \Delta_q$ such that $\|p - p^\varepsilon\| < \varepsilon$ and $v_i(p^\varepsilon) > v_i(p)$, $\forall \varepsilon > 0$. (In other words, there are price cuts that make consumer i strictly better off.)*

Proof. Since $q \cdot x_i(p) + r > 0$, one must have either $r > 0$ or $q_l x_{i,l}(p) > 0$, for some $l \neq 0$. Thus, take p and make an infinitesimal reduction either in r or in q_l (which is possible thanks to Assumption 5). Since u_i is strictly increasing in $x_{i,0}$, this change in prices strictly increases agent i's indirect utility function. □

Proposition 2. *Let $(\vec{\pi}, \vec{\sigma})$ be an equilibrium (either symmetric or asymmetric) for an economy satisfying Assumptions 1–6. Then, for any $p \in \Delta_r \times \Delta_q$, $\sigma_m(\{p\}) > 0$ implies $\sigma_n(\{p\}) = 0$, $\forall n \neq m$. (In other words, there is no equilibrium in which two or more stores set the same price vector with positive mass.)*

Proof. Suppose by contradiction that there was a price vector $p' \in \Delta_r \times \Delta_q$ such that $\sigma_m(\{p'\}) > 0$ for a subset of retailers, say $\mathbb{M} \subset \mathbb{N}$, containing two or more stores. Since stores can make no loss by choosing p_{out}, it must be the case that, for each $m \in \mathbb{M}$, the price p' should raise enough revenues to cover fixed and variable costs when confronted with prices in a non-null subset of $\times_{n \in \mathbb{N} \setminus \{m\}} \text{supp}(\sigma_n)$. But since $\times_{m \in \mathbb{M}} \sigma_m(\{p'\}) > 0$, the stores in $\mathbb{M}$ should tie in a non-null set of nodes $\vec{p}$, say Ω. In other words, there should be a non-null set of nodes $\vec{p}$ in which $p_m = p'$, $\forall m \in \mathbb{M}$, and at least one consumer i chooses p' over the other prices available. Therefore, regardless of the sharing rule embedded in $\vec{\pi}$, there would be some store in $\mathbb{M}$ interested in shifting mass to a price vector in a ε-neighborhood of p'. From Lemma 1, this deviation would attract consumers from other stores in all nodes $\vec{p} \in \Omega$ (which increases profits thanks to the fixed costs) and would cause (at worst) an infinitesimal reduction in profits in the remaining nodes. □

A corollary of Proposition 2 is that any symmetric equilibrium strategy must have no mass point in $\Delta_r \times \Delta_q$. Note, however, that Assumptions 1–6 do not rule out equilibria with a single or no entrant store. In fact, the structure imposed on the set of prices and on cost functions need not guarantee positive profits. Additional conditions are needed for this. Let us then assume that, given $\Delta_r \times \Delta_q$, the cost parameters are small enough such that the monopoly profit is positive for some store. In this case, there will always be at least one store whose strategy is not degenerated in p_{out}. Moreover, if the cost parameters are also identical across stores, then there will be at least two entrant stores (as it will be shown in Proposition 3).

Let us define a function representing the profit that store n could make when it is alone in the market and all indifferent consumers shop at n. Formally, for each $p_n \in \Delta_r \times \Delta_q$ and $n \in \mathbb{N}$, define:

$$P_n(\text{single}, p_n) = \sum_{\{i \in \mathbb{I} : v_i(p_n) \geq v_i(p_{\text{out}})\}} (q_n - c_n) \cdot x_i(p_n) + r_n \mathbf{I}(p_n) - c_{n,\text{fix}}; \tag{10}$$

where $\mathbf{I}(p_n)$ represents the cardinality of $\{i \in \mathbb{I} : v_i(p_n) \geq v_i(p_{\text{out}})\}$.

Assumption 7. The monopoly profit $\pi_n^* = \max_{p_n \in \Delta_r \times \Delta_q} P_n(\text{single}, p_n)$ is strictly positive for at least one store $n \in \mathbb{N}$.

Notice that π_n^* is well-defined, thanks to compacity of $\Delta_r \times \Delta_q$ and continuity of x_i and v_i.[11] It corresponds to the highest possible monopoly profit, when the sharing rule leads consumers to shop at store n whenever they are indifferent between shopping and staying at home.

[11] The number of subsets of $\mathbb{I}$ is finite and, for each subset $\mathbb{I}^* \subset \mathbb{I}$ containing $\mathbf{I}^*$ agents, there is a solution for the problem: $\max_{p_n \in \Delta_r \times \Delta_q} \sum_{i \in \mathbb{I}^*} \sum_{l=1}^{L} (q_{n,l} - c_{n,l}) x_{i,l}(p_n) + r_n \mathbf{I}^*$ subject to $v_i(p_n) \geq v_i(p_{\text{out}})$, $\forall i \in \mathbb{I}^*$.

Proposition 3. *Let $(\vec{\pi}, \vec{\sigma})$ be an equilibrium (symmetric or asymmetric) for an economy satisfying Assumptions 1–7. Then, there must be at least one store whose strategy is not degenerated in p_{out}. Moreover, if stores are identical and Assumptions 1–7 hold, then there must be at least two entrants.*

Proof. The first part follows directly from Assumption 7. For any sharing rule, there is at least one store that could make a profit slightly smaller than $\max(\pi_1^*, \ldots, \pi_N^*) > 0$ by setting a price vector slightly smaller than its respective monopoly level. Moreover, when stores are identical, any nonentrant store could threaten the monopolist (by entering in the market with a slightly smaller price vector). Thus, at least two stores must enter the market with positive probability in any equilibrium. □

Propositions 2 and 3 predict cross-store price dispersion whenever stores are homogeneous. Therefore, identical stores shall set random prices (with no mass point) in any symmetric equilibrium. As for asymmetric equilibria, the next proposition shows that there must be at least one store whose strategy support is infinite whenever consumers and stores are homogeneous.

Proposition 4. *Let $(\vec{\pi}, \vec{\sigma})$ be an equilibrium for an economy satisfying Assumptions 1–7 in which consumers are homogeneous and stores face identical cost functions. Then, $\cup_{n=1}^{N} \text{supp}(\sigma_n) \cap \Delta_r \times \Delta_q$ is infinite.*

Proof. From Proposition 3, there must be at least two stores whose strategies are not degenerated in p_{out}. Suppose by contradiction that $\cup_{n=1}^{N} \text{supp}(\sigma_n) \cap \Delta_r \times \Delta_q$ was finite. Then, define $p_{n_1}^*$ as one of the most attractive price vector – from the perspective of the homogeneous consumers – among all prices in $\cup_{n=1}^{N} \text{supp}(\sigma_n)$. Formally, $p_{n_1}^*$ must be such that $p_{n_1}^* \in \text{supp}(\sigma_{n_1})$ and

$$v(p_{n_1}^*) \geq v(p_n), \quad \forall p_n \in \text{supp}(\sigma_n), n \in \bar{N}. \tag{11}$$

Similarly, let $p_{n_2}^*$ be such that $p_{n_2}^* \in \text{supp}(\sigma_{n_2})$ and

$$v(p_{n_2}^*) \geq v(p_n), \quad \forall p_n \in \text{supp}(\sigma_n), n \in \bar{N} \setminus \{n_1\}. \tag{12}$$

In other words, $p_{n_2}^*$ is one of the most attractive price vector among those charged with positive probability by all stores except n_1.

Notice that the number of nodes $\vec{p} \in \times_{n=1}^{N} \text{supp}(\sigma_n)$ should also be finite. Moreover, Lemma 1 applies for all prices in $\cup_{n=1}^{N} \text{supp}(\sigma_n) \cap \Delta_r \times \Delta_q$; that is, $p_n \in \text{supp}(\sigma_n)$ implies $q \cdot x(p_n) + r > 0$ since, otherwise, p_n would be dominated by p_{out}. Now, take some node $\vec{p}^*$ in which $p_{n_1}^*$ and $p_{n_2}^*$ are both played. Since $\text{supp}(\sigma_n)$ was assumed to be finite for all n, the node $\vec{p}^*$ should have positive mass. One could then derive contradictions for each of the following two scenarios.

First, consider the case in which $v(p_{n_1}^*) = v(p_{n_2}^*)$, and notice that there will be some store (either n_1 or n_2 or another equally attractive one) that could profit by shifting mass to a slightly smaller price vector, thereby attracting all consumers in the economy (see Lemma 1).

In the second possible scenario, in which $v(p_{n_1}^*) > v(p_{n_2}^*)$, the store n_1 would serve the entire market in the node $\vec{p}^*$ and $\pi_{n_2}(\vec{p}^*) < 0$. On the one hand, if $\pi_{n_1}(\vec{p}^*) < P(\text{single}, p_{n_2}^*)$, then the store n_1 could increase its profit by shifting mass to an appropriated price vector in a neighborhood of $p_{n_2}^*$. From the definition of $p_{n_2}^*$, the new price vector would still beat all prices played with positive probability by the other stores. Then, store n_1's expected profit would increase—thanks to the fact that x is continuous on $\Delta_r \times \Delta_q$. On the other hand, if $\pi_{n_1}(\vec{p}^*) \geq P(\text{single}, p_{n_2}^*)$, then the store n_2 – which bears a loss in the node $\vec{p}^*$ – could profit by shifting mass to a price vector slightly smaller than $p_{n_1}^*$ (the argument is analogous). □

5. Computing a symmetric equilibrium

A symmetric equilibrium can be derived when consumers and stores are homogeneous. In this case, as noted by Sharkey and Sibley (1993), the multi-dimensional pricing game becomes analogous to one in which stores compete through profit levels. The problem then reduces to one dimension, and an equilibrium can be computed as follows.

In order to attract the homogeneous consumers and collect a certain profit level α, each store solves the following problem:

$$\max_{p \in \Delta_r \times \Delta_q} v(p) \ \text{ s.t. } \ P(\text{single}, p) = \alpha; \tag{13}$$

where $P(\text{single}, p)$ is defined in (10).[12]

Lemma 2. *Consider an environment with homogeneous consumers in which Assumptions 1–7 hold. Then, problem (13) is well-defined, for any $\alpha \in [0, \pi^*]$.*

Proof. In this environment, problem (13) is equivalent to:

$$\max_{p \in \Delta_r \times \Delta_q} v(p) \tag{14}$$

[12] The sharing rule implicit in this equilibrium is such that consumers always choose to shop when they are indifferent between shopping and staying at home.

$$\text{s.t. } v(p) \geq v(p_{\text{out}}); \tag{15}$$

$$I(q-c)\cdot x(p) + Ir - c_{\text{fix}} = \alpha. \tag{16}$$

The proof follows then from two facts: (i) v is continuous; and (ii) the subset of $\Delta_r \times \Delta_q$ such that conditions (15)–(16) hold is nonempty and compact, for any $\alpha \in [0, \pi^*]$. The nonemptiness property is not straightforward. To see this, notice from Assumption 7 that there exists $p^* \in \Delta_r \times \Delta_q$ satisfying conditions (15)–(16) for $\alpha = \pi^*$. Moreover, it must be the case that $P(\text{single}, \mathbf{0}) < 0$, since $c_{\text{fix}} > 0$ and $x(\cdot)$ is bounded (as $\mathbb{X}$ is compact). Hence, any p^* that implements the monopoly profit π^* must be different from $\mathbf{0}$. Notice that $v(\cdot)$ is decreasing (since u_i is strictly increasing in $x_{i,0}$), and then condition (15) holds for any $p \leq p^*$. Moreover, from continuity of $x(\cdot)$, there exists $p = (r, q) \leq p^*$ such that condition (16) holds, for any $\alpha \in [0, \pi^*]$. □

Naturally, any price strategy such that $\alpha < 0$ is dominated by the nonentry strategy. Moreover, there is a maximum profit level that can be achieved, namely $\pi^* > 0$ (see Assumption 7). Consider then an equilibrium strategy (in terms of profit levels) given by: (i) an entry probability $\beta > 0$ and (ii) a continuous distribution function $F : [0, \pi^*] \to [0, 1]$.

The expected profit of a store choosing $\alpha \in [0, \pi^*]$ is given by

$$[1-\beta F(\alpha)]^{N-1}\alpha - [1-(1-\beta F(\alpha))^{N-1}]c_{\text{fix}}; \tag{17}$$

where $[1-\beta F(\alpha)]^{N-1}$ is the probability that this retailer will serve the entire market.

Notice that, if the strategy described by β and F yields the same expected profit for each pure strategy in its support, then it will constitute a Nash equilibrium for this game. Since the nonentry strategy delivers zero profit, let us make expression (17) equals zero, such that:

$$[1-\beta F(\alpha)]^{N-1}(\alpha + c_{\text{fix}}) = c_{\text{fix}}, \quad \forall \alpha \in [0, \pi^*]. \tag{18}$$

Since $F(\pi^*) = 1$, one must have

$$[1-\beta]^{N-1}(\pi^* + c_{\text{fix}}) = c_{\text{fix}}. \tag{19}$$

From (19), the entry probability is

$$\beta = 1 - \left(\frac{c_{\text{fix}}}{\pi^* + c_{\text{fix}}}\right)^{\frac{1}{N-1}}. \tag{20}$$

Then, from (18) and (20), one has

$$F(\alpha) = \frac{1-[c_{\text{fix}}/(\alpha + c_{\text{fix}})]^{1/(N-1)}}{1-[c_{\text{fix}}/(\pi^* + c_{\text{fix}})]^{1/(N-1)}}. \tag{21}$$

In this equilibrium, all stores will enter the market with the same positive probability $\beta \in [0, 1]$, and their expected profits will be zero. One can now use the equilibrium strategy defined over profit levels to derive pricing implications for this economy. Let us divide the analysis in two cases.

5.1. Model with fixed fees

Consider the case with fixed fees, that is $\Delta_r = [-Iw, Iw]$. In this case, the Pareto efficient solution displays a single entrant store charging $p = (c_{\text{fix}}/I, c_1, \ldots, c_L)$. Naturally, this is not an equilibrium since this store would profit by deviating to the monopoly price.

In this game, setting prices different than marginal costs is a dominated strategy. The stores will maximize consumer's surplus – by setting prices equal marginal costs – and extract part of it through the fixed fee. Therefore, the equilibrium strategy (in terms of prices) will display $\sigma(\{p_{\text{out}}\}) = 1 - [c_{\text{fix}}/(\pi^* + c_{\text{fix}})]^{1/(N-1)}$, and the set $\text{supp}(\sigma_n) \cap \Delta_r \times \Delta_q$ will consist of points $p = ((\alpha + c_{\text{fix}})/I, c_1, \ldots, c_L)$, where α is distributed according to F in (21).

5.2. Model without fixed fees

The link between profit levels and price vectors is established by the maximization problem defined in (13). The participation constraint $v(p) \geq v(p_{\text{out}})$ always holds true in a model without fixed fees since, at worst, consumers can buy nothing and keep their endowments. Thus, problem (13) reduces to:

$$\max_{p \in \Delta_r \times \Delta_q} v(p) \ \text{ s.t. } \ I(q-c)\cdot x(p) + Ir - c_{\text{fix}} = \alpha. \tag{22}$$

The price vectors associated with each profit level $\alpha \in [0, \pi^*]$ will necessarily be distorted since, otherwise, the revenues will not cover the fixed costs. The logic behind this is very standard. For each profit level, the stores set prices that maximize surplus (and then attract consumers). Second-best distortions will appear just as in the Ramsey-taxation problem. The exact shape of price distortions will depend on cross effects embedded in the indirect utility and demand functions. The possibility of loss leading is related to negative cross effects in the demand system.

This result is well-known for public monopolies subject to budgetary constrains (see Boiteux, 1956). It is also valid in a competitive setting with fixed costs, where the equilibrium allocation is Pareto optimal, and therefore maximizes consumer's surplus subject to a zero-profit condition (see Bliss, 1988, p. 382). It is interesting to notice that the exact same forces act in price-setting oligopolies with fixed costs.

6. Random prices and weekly specials

Weekly flyers announce special prices of competing retail stores – such as supermarkets and drugstores – in most parts of the world. These promotional prices differ across stores with similar costs, located in the same region, and competing for the same consumers. In many cases, prices seem to evolve randomly over the weeks. Interestingly, the results in Section 4 shows that prices will necessarily be random when stores play a symmetric equilibrium or when they face identical cost functions and serve homogeneous consumers.

Most of the existing literature relies on imperfections on the consumer's side – namely, transportation costs or imperfect information about prices – to explain price dispersion across identical stores. Shilony (1977) modeled an environment where the retail stores are located in different neighborhoods. The consumers are perfectly informed and can either shop from the local store or make a costly trip to some other neighborhood whose store offers a better price. The possibility of extracting surplus from local consumers implies that no deterministic price vector can be a Nash equilibrium for this game. In a second approach, Butters (1977) considered a model where each store can send multiple price offers to uninformed consumers. The price offers reach each consumer randomly. In equilibrium, it is optimal for stores to announce multiple prices, which generates price dispersion.

A third line of argument was developed by Salop and Stiglitz (1977), who studied an economy where each consumer pays a different fixed cost to gather information about prices practiced by each store. In a similar vein, Varian (1980) explored the case where an exogenous fraction of uninformed consumers coexist with perfectly informed agents. In both cases, the equilibrium must be in mixed strategies. There are also papers that use uninformed consumers facing search costs to generate equilibrium with random price dispersion—e.g., Reinganum (1979); Burdett and Judd (1983); Robb (1985); Stahl (1989); McAfee (1995).[13]

The model developed here abstracts from those market imperfections and focus on a key supply-side characteristic of the retailing activity: the presence of fixed costs. In practice, retail stores add services to the original goods, leading the consumers to purchase these goods from them rather than directly from the producers. For instance, retailers deal with different producers and transport all products to the same place, reducing the shopping time spent by each consumer. They also provide quality assurance by searching for the best products and producers. Moreover, they bargain for better prices in sectors where the wholesale market is not competitive, acting on behalf of a large number of individuals who separately would have low bargaining power. All such activities are subject to nonconvex transaction costs, such as the investments needed for transporting different goods and the time spent on searching and bargaining. This technological nonconvexity restricts competition and has important pricing implications in this sector.

Weekly specials are viewed here as the strategic outcome of an oligopolistic model of price competition, where the consumers are rational and well-informed, and the stores face fixed costs. In these environments, random pricing avoids predatory competition and, then, raises revenues to cover the fixed costs. To the best of my knowledge, this supply-side explanation for weekly specials is novel in the literature.

7. Conclusion

This paper analyses a general version of the classical Bertrand model of price competition. The economy has finitely many agents, goods, and retail stores. Consumers are possibly heterogeneous and their utility and demand functions are continuous. Goods can be complementary, substitute, or none of these. Retail stores face continuous cost functions that need not be identical. Commodity prices are linear, and the model encompasses cases with and without fixed fees (i.e., entry tickets). The retailers act first and simultaneously announce a vector of prices to be charged to consumers. (They are also allowed to stay out of the market.) Next, after observing the price offers, the consumers choose whether to stay at home or shop at a store. In this setting, Proposition 1 shows that there is a Nash equilibrium with an endogenous sharing rule.

The applied results focus on economies with fixed costs and constant marginal costs. Propositions 2 and 3 state that prices will be different across stores with probability one. The presence of fixed costs imply that stores cannot enter the market with identical pricing strategies unless they are random (i.e., unpredictable). It is possible to have stores adopting different pure strategies when consumers and stores are heterogeneous. In practice, this corresponds to cases in which stores specialize in serving different consumers. However, as shown in Proposition 4, prices must be random (with infinite support) in environments with homogeneous consumers and identical cost functions. This is interpreted as a possible explanation for weekly specials. Unlike most of the existing literature, random prices are not used here to fool uninformed consumers. Instead, stores use mixed strategies in order to be unpredictable and then avoid predatory price competition.

[13] Baye and Morgan (2001) studied random price dispersion on the internet by means of a model in which a profit maximizing gatekeeper charges fees to sellers that advertise prices and to consumers who access the list of advertised prices. Bounded rationality among sellers (e.g., Baye and Morgan, 2004) and consumers (e.g., Anderson and de Palma, 2005) have also been used to justify price dispersion.

The paper also computes a symmetric equilibrium for economies with homogeneous consumers and stores. When entry tickets are not allowed, prices in the support of the equilibrium strategy will be distorted. This mimics the Ramsey–Boiteux price distortions and, thus, loss leading may appear in equilibrium.

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