Supplement to "Output contingent securities and efficient investment by firms"

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Abstract

We complement the analysis in Braido and Martins-da-Rocha (Forthcoming) by showing the existence of a competitive equilibrium in a model with a continuum of identical firms and perfectly correlated success-or-failure shocks.

Consider an economy with a single firm which chooses an investment in the set $A := [0,1]$ and faces a success-or-failure production function. Two production levels are possible, $y_{\mathrm{L}} > 0$ and $y_{\mathrm{H}} > y_{\mathrm{L}}$. The transition $Q(y,a)$ stands for the probability of producing y when the firm invests a. Since there are only two possible output levels, we can simplify the notation by setting $Q_{\mathrm{H}}(a) := Q(y_{\mathrm{H}}, a)$. We make the assumption that higher efforts increase the likelihood of success, i.e., $a \mapsto Q_{\mathrm{H}}(a)$ is strictly increasing.

There is a canonical state-of-nature representation of this technology. We take the set Ω to be the interval $[0,1]$ and the probability P to be the uniform measure. For every investment level a, we define $\omega(a) := 1 - Q_{\mathrm{H}}(a)$ and pose

$$f(\omega, a) := \begin{cases} y_{\mathrm{L}} & \text{if} \quad \omega \leqslant \omega(a) \\ y_{\mathrm{H}} & \text{if} \quad \omega > \omega(a). \end{cases} \tag{1}$$

Since $a \mapsto \omega(a)$ is strictly decreasing, we denote by $\omega \mapsto a(\omega)$ its inverse mapping from $[\omega(1), \omega(0)]$ to $[0,1]$. For each state of nature $\omega \in [\omega(1), \omega(0)]$, the firm obtains the output

$$f(\omega, a) = \begin{cases} y_{\mathrm{L}} & \text{if} \quad a \leqslant a(\omega) \\ y_{\mathrm{H}} & \text{if} \quad a > a(\omega). \end{cases} \tag{2}$$

For states $\omega < \omega(1)$ and $\omega > \omega(0)$, the realized outputs are respectively $y_{\rm L}$ and $y_{\rm H}$ regardless of the initial investment a.

This production function $a \mapsto f(\omega, a)$ is not concave. Then, for some specifications of f, there is no financial equilibrium in which the firm maximizes the competitive market value. To overcome the non-convexity of the success-or-failure technology, we propose to model perfect competition of the productive sector by considering the extreme case with a continuum $K := [0, 1]$ of identical firms facing success-or-failure shocks that are perfectly correlated. This assumption is imposed to simplify the presentation.[1]

We pose a few remarks before proceeding. We first notice that the presence of a continuum of firms is consistent with our behavioral assumption that agents are convinced that a change in the investment of each firm does not affect the probability over the aggregate output. We also stress that shocks are not independent across firms. Independence would reduce the model to the case without aggregate uncertainty, where the choice of the firms' objective is not anymore an issue. The assumption that shocks are perfectly correlated allows us to keep output variability at equilibrium, which resembles the case with a single firm. Finally, shocks are not necessarily observable and contractible even when they affect all firms identically.[2]

We abuse notation and do not index firm-specific variables with the firm's name $k \in [0, 1]$. Since the production function of each firm is non-convex, we may have multiple solutions to the "representative" firm's maximization problem. In particular, ex-ante identical firms having different names may choose different investment levels. Therefore, we opt to represent firms' investment decisions using a probability measure α on the Borel σ-algebra of the set of investment levels $A = [0, 1]$. The interpretation is that $\alpha(B)$ is the fraction of firms choosing investment in a Borel set B of A. The corresponding (average) aggregate production contingent on the exogenous state of nature ω is denoted by $\mathbb{E}_\alpha[f(\omega)]$. It follows

[1] At the cost of notational complexity, we could have considered a slightly more general model allowing for different firms with imperfectly correlated production levels. For existence of an equilibrium, what matters to deal with non-convex production technologies is that we have a non-atomic measure space of firms.

[2] Examples of aggregate shocks include changes in a government's macroeconomic policy such as taxes and social security contributions on labor. Another possible aggregate shock is a general increase in labor productivity because of an easily accessible improvement in technological knowledge. Political instabilities in the Middle East that lead to changes in oil production or technological innovations in solar energy production are also examples of shocks affecting all firms. We hardly see contracts contingent on events like these.

from the production function represented in Equation (2) that

$$\mathbb{E}_\alpha[f(\omega)] = \int_A f(\omega, a)\alpha(da) = y_{\mathrm{L}}\alpha([0, a(\omega)]) + y_{\mathrm{H}}(1 - \alpha([0, a(\omega)])). \tag{3}$$

Since we have infinitely many possible primitive states of nature ω, the set Z of aggregate production levels is now described by the interval $[y_{\mathrm{L}}, y_{\mathrm{H}}]$.[3]

Given a distribution α of investment, we can define the distribution μ_α over the (average) aggregate production as follows

$$\mu_\alpha(B) := P(\{\omega \in \Omega \,:\, \mathbb{E}_\alpha[f(\omega)] \in B\}),$$

for every Borel set $B \subseteq [y_{\mathrm{L}}, y_{\mathrm{H}}]$. Having infinitely many firms, we need to consider infinitely many consumers. We assume that there is also a continuum $I := [0, 1]$ of identical consumers, each one having the full ownership of a single firm. We also skip using names i to index consumer-specific variables.

Fix some equilibrium investment distribution $\bar{\alpha}$. Identical firms must have the same equilibrium initial value $\overline{V}$. We can assume without loss of generality that agents pool their asset holdings and make consumption plans $x_0 \in \mathbb{R}_+$ and $c_1 : Z \to \mathbb{R}_+$ in order to satisfy the following reduced-form budget constraint

$$x_0 + \int_Z c_1(z)\bar{\rho}(dz) \leqslant e_0 + \overline{V}, \tag{4}$$

where $\bar{\rho}$ is the equilibrium measure representing output-contingent prices. Each agent' problem has a unique optimal solution $(\bar{x}_0, \bar{c}_1)$ in which $\bar{c}_1(z) = z$ and $\bar{x}_0 = e_0 - \bar{a}$.

In order to simplify the exposition, we assume hereafter that u_0 is a linear function with $u_0' = 1$. The equilibrium stochastic discount factor becomes $\bar{\chi}(z) = u_1'(z)$. Firms maximize the same competitive market value function

$$V_{\bar{\alpha}}(a) := \int_Z \widetilde{y}_{\bar{\alpha}}(a|z)\bar{\rho}(dz) - a,$$

where

$$\widetilde{y}_{\bar{\alpha}}(a|z) := \int_\Omega f^k(\omega, a)P(d\omega|\mathbb{E}_{\bar{\alpha}}[f] = z)$$

[3] This illustrates that, when firms' outputs are not independent, considering a continuum of firms does not remove aggregate uncertainty. In fact, here, it potentially increases the set of possible aggregate outcomes.

is the conditional expected production under the investment distribution $\bar{\alpha}$ given an (average) aggregate output z. In equilibrium, firms may choose different investment levels, but they will all have the same market value $\overline{V}$. Formally, if we denote by $\operatorname{supp}(\bar{\alpha})$ the support of the equilibrium investment distribution $\bar{\alpha}$, then we have $V(a) = \overline{V}$, for every investment $a \in \operatorname{supp}(\bar{\alpha})$ and $V(a) \leqslant \overline{V}$, for $a \notin \operatorname{supp}(\bar{\alpha})$. We can then write firms' competitive market value as follows:

$$
\begin{aligned}
V_{\bar{\alpha}}(a) &= \int_Z \bar{\chi}(z) \int_\Omega f^k(\omega, a) P(d\omega | \mathbb{E}_{\bar{\alpha}}[f] = z) \mu_{\bar{\alpha}}(dz) - a \\
&= \int_\Omega \bar{\chi}\left(\mathbb{E}_{\bar{\alpha}}[f(\omega)]\right) f(\omega, a) P(d\omega) - a.
\end{aligned}
$$

Given the production function represented in Equation (1), we obtain

$$
V_{\bar{\alpha}}(a) = y_{\mathrm{L}} \int_0^{\omega(a)} \bar{\chi}\left(\mathbb{E}_{\bar{\alpha}}[f(\omega)]\right) P(d\omega) + y_{\mathrm{H}} \int_{\omega(a)}^1 \bar{\chi}\left(\mathbb{E}_{\bar{\alpha}}[f(\omega)]\right) P(d\omega) - a. \tag{5}
$$

Smooth probabilities Let us now compute an equilibrium distribution $\bar{\alpha}$ for the production technology in which $a \mapsto Q_{\mathrm{H}}(a)$ is decreasing, continuously differentiable and satisfies $Q'_{\mathrm{H}}(0) = \infty$ and $Q'_{\mathrm{H}}(1) = 0$.[4] Recall that $\omega(a) := 1 - Q_{\mathrm{H}}(a)$ and notice that, for any $a \in (0, 1)$, we have

$$
V'_{\bar{\alpha}}(a) = Q'_{\mathrm{H}}(a) \bar{\chi}\left(\mathbb{E}_{\bar{\alpha}}[f(\omega(a))]\right) \Delta y - 1,
$$

where $\Delta y := y_{\mathrm{H}} - y_{\mathrm{L}}$.

Since Q'_{H} is continuous, there are limits $\underline{b}$ and $\bar{b}$ with $0 < \underline{b} < \bar{b} < 1$ such that

$$
Q'_{\mathrm{H}}(\underline{b}) \bar{\chi}(y_{\mathrm{H}}) \Delta y = 1 \quad \text{and} \quad Q'_{\mathrm{H}}(\bar{b}) \bar{\chi}(y_{\mathrm{L}}) \Delta y = 1. \tag{6}
$$

Moreover, since $\bar{\chi}(z) = u'_1(z)$ is continuously decreasing, there is a continuously decreasing function $a \mapsto \varphi(a)$ such that

$$
\forall a \in [\underline{b}, \bar{b}], \quad Q'_{\mathrm{H}}(a) \bar{\chi}(\varphi(a)) \Delta y = 1.
$$

Naturally, we have $\varphi(\underline{b}) = y_{\mathrm{H}}$ and $\varphi(\bar{b}) = y_{\mathrm{L}}$.

[4] This corresponds to the production technology for which we do not have existence with a single firm.

Let us define the distribution $\bar{\alpha}$ to be such that

$$\varphi(a) = y_{\rm L}\bar{\alpha}([0,a]) + y_{\rm H}(1-\bar{\alpha}([0,a])).$$

Since the function φ is continuous, the distribution $\bar{\alpha}$ is non-atomic. Indeed, for every $a \in [\underline{b}, \bar{b}]$, we have

$$\bar{\alpha}[0,a] = (y_{\rm H} - \varphi(a))/\Delta y,$$

where $\varphi(a) = \bar{\chi}^{-1}\left(\frac{-1}{Q'_{\rm H}(a)\Delta y}\right)$. No firm chooses investment levels lower than $\underline{b}$ or higher than $\bar{b}$. It is easy to see that the distribution $\bar{\alpha}$ has been constructed in order to set $V'_{\bar{\alpha}}(a) = 0$, for all $a \in [\underline{b}, \bar{b}]$. Notice also that $V'_{\bar{\alpha}}(a) > 0$, for $a < \underline{b}$ and $V'_{\bar{\alpha}}(a) < 0$, for $a > \bar{b}$. This concludes our argument and proves that $\bar{\alpha}$ is a competitive equilibrium investment profile for this economy.

Remark. Notice from Equation (6) that the smaller the distance between $\bar{\chi}(y_{\rm H})$ and $\bar{\chi}(y_{\rm L})$, the narrower the interval $[\underline{b}, \bar{b}]$. In particular, as u_1 approaches a linear function, we find $\underline{b}$ converging to $\bar{b}$ and $\bar{\alpha}$ converging to a Dirac measure (symmetric equilibrium). Aggregate uncertainty in this situation closely approximates the case with a single firm—in which the (average) aggregate output is either $y_{\rm L}$ or $y_{\rm H}$.

General existence theorem We can relax the assumptions on the transition probability $Q_{\rm H}$ and still obtain the existence result for economies with a continuum of firms facing perfectly correlated shocks. The reasoning is somewhat more technical.

Let $\mathcal{M}(A)$ be the vector space of signed Borel measures on $A = [0,1]$. An investment decision is a distribution α in $\mathcal{M}^1_+(A)$ the set of all positive measures with total mass 1. We make explicit the relation between α and each firm's competitive market value $V_\alpha(a)$ by defining

$$V_\alpha(a) := \int_\Omega \bar{m}_\alpha(\omega) f(\omega, a) P(d\omega) - a,$$

where $\bar{m}_\alpha(\omega) := \bar{\chi}(\mathbb{E}_\alpha[f(\omega)])$. We also denote by $G(\alpha)$ the set of optimal investment levels

$$G(\alpha) := \text{argmax}\{V_\alpha(a) \ : \ a \in A\}.$$

A distribution of investment $\bar{\alpha}$ corresponds to an equilibrium in which firms maximize the competitive market value when it only puts mass on optimal investment levels, i.e., when $\bar{\alpha}(G(\bar{\alpha})) = 1$.

Theorem. *There exists a competitive equilibrium distribution of investments.*

Proof. Let $F : \mathcal{M}^1_+(A) \to \mathcal{M}^1_+(A)$ be the correspondence defined by

$$F(\alpha) := \{\hat{\alpha} \in \mathcal{M}^1_+(A) \;:\; \hat{\alpha}(G(\alpha)) = 1\}.$$

A competitive equilibrium is a distribution $\bar{\alpha}$ of investment levels that is a fixed point of F, i.e., $\bar{\alpha} \in F(\bar{\alpha})$.

We propose to apply Kakutani's Fixed-Point Theorem. The convex set $\mathcal{M}^1_+(A)$ is endowed with the weak-star topology of the duality $\langle \mathcal{M}(A), \mathcal{C}(A)\rangle$, where $\mathcal{C}(A)$ is the space of continuous real-valued functions defined on A. Since $\mathcal{C}(A)$ endowed with the sup-norm is separable and since $\mathcal{M}(A)$ is the topological dual of $\mathcal{C}(A)$, we get that $\mathcal{M}^1_+(A)$ is a compact metrizable space.

Lemma 1. *The correspondence $G : \mathcal{M}^1_+(A) \to A$ is upper semi-continuous for the weak-star topology.*

Proof of Lemma 1. Following Berge's Maximum Theorem, it is sufficient to show that $(a, \alpha) \mapsto V_\alpha(a)$ is continuous. Let $(a_n, \alpha_n)_{n\in\mathbb{N}}$ be a sequence in $A \times \mathcal{M}^1_+(A)$ converging to $(a, \alpha) \in A \times \mathcal{M}^1_+(A)$. We first show that $\lim_{n\to\infty} \mathbb{E}_{\alpha_n}[f(\omega)] = \mathbb{E}_\alpha[f(\omega)]$, for P-almost every state ω. Notice that

$$\mathbb{E}_{\alpha_n}[f(\omega)] = \alpha_n(\omega) y_{\mathrm{L}} + (1 - \alpha_n(\omega)) y_{\mathrm{H}} = y_{\mathrm{L}} + [y_{\mathrm{H}} - y_{\mathrm{L}}](1 - \alpha_n(\omega)),$$

where $\alpha_n(\omega) = \alpha_n([0, a(\omega)])$ is the measure of the interval $[0, a(\omega)]$. Since $(\alpha_n)_{n\in\mathbb{N}}$ converges for the weak-star topology to α, we have

$$\lim_{n\to\infty} \alpha_n([0, a]) = \alpha([0, a]),$$

for every $a \in A$ that is not an atom of α, i.e., for every a such that $\alpha(\{a\}) = 0$. Since there are at most countably many atoms of α and since $\omega \mapsto a(\omega)$ is strictly increasing, we obtain $\lim_{n\to\infty} \alpha_n(\omega) = \alpha(\omega)$, for P-almost every ω. This implies that $\lim_{n\to\infty} \mathbb{E}_{\alpha_n}[f(\omega)] = \mathbb{E}_\alpha[f(\omega)]$. By continuity of u_1', we find

$$\lim_{n\to\infty} m_{\alpha_n}(\omega) = m_\alpha(\omega),$$

for P-almost every state ω.

We now show that $\lim_{n\to\infty} V_{\alpha_n}(a_n) = V_\alpha(a)$. Recall that

$$V_{\alpha_n}(a_n) = -a_n + \int_\Omega m_{\alpha_n}(\omega) f(\omega, a_n) P(d\omega).$$

Since $\lim_{n\to\infty} f(\omega, a_n) = f(\omega, a)$, for P-almost every ω, we can apply the Lebesgue Dominated Convergence Theorem to obtain the desired result.[5] $\square$

Lemma 2. *The correspondence F is upper semi-continuous for the weak-star topology.*

Proof of Lemma 2. Since $\mathcal{M}_+^1(A)$ is compact, it is sufficient to show that F has a closed graph. Let $(\alpha'_n, \alpha_n)_{n\in\mathbb{N}}$ be a sequence converging to (α', α) and satisfying $\alpha'_n \in F(\alpha_n)$ for each n. Since $G(\alpha)$ is compact, there exists an open set K and compact set $\bar{K}$ such that

$$G(\alpha) \subseteq K \subseteq \bar{K}.$$

Since G is upper semi-continuous, there exists N large enough such that for each $n \geqslant N$, we have $G(\alpha_n) \subseteq K$. In particular, $\alpha'_n(\bar{K}) = 1$. Since $(\alpha'_n)_{n\in\mathbb{N}}$ converges for the weak-star topology to α' we get that $\alpha'(\bar{K}) \geqslant \limsup_n \alpha'_n(\bar{K}) = 1$. We have thus proven that $\alpha'(\bar{K}) = 1$. Actually, we can construct a decreasing sequence $(K_n, \bar{K}_n)_{n\in\mathbb{N}}$ where K_n is open, $\bar{K}_n$ is compact, $G(\alpha) \subseteq K_n \subseteq \bar{K}_n$ and $\cap_{n\in\mathbb{N}} \bar{K}_n = G(\alpha)$. It then follows that $\alpha'(G(\alpha)) = 1$. $\square$

The correspondence F has non-empty values. Indeed, since the function $a \mapsto V_\alpha(a)$ is continuous and A is compact, the demand set $G(\alpha)$ is always non-empty. If $\hat{a}$ is an element of $G(\alpha)$, then the Dirac measure on $\hat{a}$ belongs to $F(\alpha)$. Since by construction the correspondence F has convex values, we can apply Kakutani's Fixed-point Theorem to the correspondence F. $\square$

References

Braido, L. H. B., and V. F. Martins-da-Rocha (Forthcoming): "Output contingent securities and efficient investment by firms," International Economic Review.

[5] Notice that, for each n, we have $f(\omega, a_n) \leqslant y_{\mathrm{H}}$ and $m_{\alpha_n}(\omega) \leqslant u'_1(y_{\mathrm{L}})$.