

REFINEMENT OF DYNAMIC EQUILIBRIUM

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ABSTRACT. In this paper we propose a refinement of dynamic equilibria based on small random deviations from the perfect foresight equilibrium in a class of one step forward dynamic models. Specifically, when the backward perfect foresight (bpf) map is a unimodal function exhibiting cycles or complex dynamics, we define a small random deviation from the perfect foresight equilibrium as a sequence of random variables generated from small stochastic errors on the perfect prevision of the future state variable value. First, we show that the stochastic process generated in that way is stationary provided that the support of the perturbation is small enough. Second, when the bpf map exhibits ergodic chaos, we show that the stationary measures converge to the Bowen-Ruelle-Sinai invariant measure of the bpf map as the size of the perturbations approaches zero. Third, if the bpf map has an attracting cycle, then the stationary measure is close to a convex linear combination of Dirac measures supported on that cycle. Therefore, depending on the parameter value which defines the bpf map, small random deviations are close to combination of Dirac measures supported on a determinate cycle or to the stationary measure of the chaotic sunspot equilibrium defined by Araujo and Maldonado (2000). Neither indeterminate cycles nor other kind of sunspot equilibrium is found as the limiting behavior of the small random deviations. Finally, we provide two examples - the classical overlapping generations model with fiat money and the Shapley-Shubik market game - to illustrate the refinement of the dynamic equilibria in those models given by the small random deviations from the perfect foresight equilibrium.

Keywords: Equilibrium selection; Sunspot equilibrium; Stochastic dynamical system; Stochastic stability.

JEL classification: C61, D84, E32

1. INTRODUCTION

The Rational Expectations Hypothesis (REH) requires not only individuals maximizers of their objective functions, but also consistency between the perceived randomness of future variables and the actual distribution of them. An even stronger concept is that of perfect foresight equilibrium. It proposes that agents are able to have a perfect prevision of the exact value of the future state variable.

The strong version of the REH given by the perfect foresight assumption allows to find important equilibrium paths as indeterminate steady states or cycles and chaotic paths [10, 13, 19, 25]. From those equilibria, it is possible to construct stochastic equilibria based on extrinsics, the so called “sunspot equilibria” [1, 5, 15]. Although the perfect foresight assumption gives us a clear idea of several types of equilibria that a system may have, the question of what would happen if such assumption is not completely satisfied remains open; for instance, how the state variable sequence behaves under small deviations from the perfect foresight equilibrium.

In this paper we provide a selection criterion of those dynamic equilibria in one-step forward looking economic models. Such selection criterion or *refinement* of dynamic equilibria is based on small deviations from the perfect foresight equilibrium. The stochastic process generated in that way is stationary and for a typical family of unimodal backward perfect foresight (bpf) maps distinct outcomes may occur, depending on the parameter values of the model. If the unperturbed bpf dynamics exhibits an attracting cycle, the stationary measure associated with the small deviations is close to an atomic measure with support on that attracting cycle. More interestingly, in the case where the bpf dynamics are chaotic and an ergodic and absolutely continuous (with respect to Lebesgue) invariant measure for it exists, then the small deviations give rise to a stochastic process with a stationary measure that is close to the stationary measure of the global chaotic sunspot equilibrium presented in Araujo and Maldonado [1]. Additionally, as the size of the deviations goes to zero, the stationary measures of the stochastic processes converge to invariant measures associated to the bpf map. As a consequence, we assert that both the atomic measures with support on attracting cycles and the stationary measure of the chaotic sunspot equilibrium are robust under that refinement criterion of dynamic equilibria given by the small random perturbations of the perfect foresight equilibrium. It is worth noting that an attracting cycle of the bpf map corresponds to a deterministic cycle in the forward perfect foresight dynamics. Therefore, neither indeterminate cycles (or steady states) nor other kind of sunspot equilibrium other than the global chaotic sunspot equilibrium are robust under this type of refinement.

We can interpret those random deviations as small expectations coordination failures (ECF) in the model. The ECF have been a concern in the recent literature of dynamic models with heterogeneous agents [2, 17, 27, 28, 45], and since most of macro models deal with representative agents, this can be a simple way to introduce that imperfection into those models. Thus, within the setup of bpf maps belonging to an one-dimensional parameter family of unimodal maps with negative Schwarzian derivative and non-flat critical point, small failures in coordination will produce stochastic processes with stationary measures close to a chaotic SE, in the case where the original unperturbed dynamics are chaotic on a finite union of intervals. When the deterministic dynamics exhibits an attracting cycle, the stationary measure of the perturbed system will remain close to an atomic measure with support on a periodic orbit.

The term *sunspot* was coined by Cass and Shell [14] who defined the concept of sunspot equilibrium in the context of general equilibrium to study the influence of the agents expectations on market outcomes. More recently, Lucas and Stokey [35] have argued that sunspots and contagion effects are sources of liquidity crises. They brought the argument of Cass and Shell [14] on expectation coordination to explain the bank runs and consequently, the financial crises of 2008. Applications of the concept of sunspot equilibrium are the modeling of such bank-runs [39], lotteries [41, 42] and behavioral economics [23].

The existence of sunspot equilibrium requires that agents not only coordinate expectations, but also they have to do it by using an extrinsic event. In spite of this seeming to be an extremely strong hypothesis, our result asserts that allowing for small stochastic deviations from the perfect foresight equilibrium, the resulting stochastic dynamics are almost the same.

In the literature we find other selection criteria of dynamic equilibria, most of them applied to linear or monotonic dynamic models of rational expectations. These selection criteria are important because typically those models exhibit multiplicity of equilibria, thus refinements of those equilibria are needed. The first criterion is the well-known local stability criterion ([9, 11]). This simply proposes that the unique equilibrium must be stable under small perturbations of the initial condition. The second criterion is the minimal state variable criterion ([36, 37]) which states that the equilibrium selection must be done using forecasting functions with a minimal set of state variables and with parameters that are continuous functions around key values of the structural parameters. A recently defined criterion is the expectational stability ([21, 22, 34]), where the parameters defining the equilibrium must be stable under updating rules in *meta-time*. That dynamics represents the continuous and instantaneous revision of beliefs regarding those parameters. Finally, Driskill [20] provided a new selection criterion of equilibria, the “finite-horizon” or “backward-induction” criterion, consisting in taking the finite horizon model associated to the original one and finding the limit of the finite-horizon equilibria as time goes to infinity. Our work complements the criteria already proposed for the case of truly non-linear dynamics governing the state variable evolution.

The dynamics of unimodal maps, a key topic in the modern dynamical system theory (see e.g. [18] for an overview), plays a central role in our analysis. Of particular importance in what follows are properties such as the existence and uniqueness of absolutely continuous invariant probability measures for this class of dynamical system. Benedicks and Carleson [7] provide conditions satisfied by a positive measure set of values of the parameter c of the quadratic family $Q_c(x) = x^2 + c$ and implying the conclusion of Jakobson’s theorem [29], namely, that such maps admit a unique invariant probability measure which is absolutely continuous with respect to Lebesgue. See, e.g. [12, 31, 38], for more results regarding the existence of an absolutely continuous invariant measure μ for a unimodal map. Yet another notion that will be of great relevance for the results herein, is the stability of dynamical systems under perturbations given by sequences of independent and identically distributed random variables, known as *stochastic stability* and initially introduced by Kolmogorov and Sinai [44]. Results in this direction have been obtained by Kifer [32, 33] for uniformly expanding maps, Axiom A attractors and geometric Lorenz attractors of flows, and Young [46] for the stochastic stability of Axiom A diffeomorphisms. In what concerns one-dimensional maps, Katok and Kifer [30] proved stochastic stability for the quadratic family in the Misiurewicz case. This result was later extended for sets of values of the parameter with positive Lebesgue measure by Benedicks and Young [8], with respect to the convergence induced by the weak*-topology, and by Baladi and Viana [6], with respect to the norm topology and for a wider class of unimodal maps. More recently, Ávila and Moreira [3, 4] proved that quadratic maps are stochastically stable for Lebesgue almost every parameter value, their results holding also for generic families of unimodal maps of the interval.

This paper is organized as follows. In Section 2, we delimitate the class of economic dynamic models that we are going to consider, define the concept of small deviations from the perfect foresight equilibrium and prove the main theorem. In Section 3 we illustrate our main result through two examples. The first one is the classical overlapping generations model with fiat money and the second one is the

Shapley-Shubik market game model. For a large range of values of the risk aversion parameter (for the first model) and of the market thickness parameter (for the second model), we find that the stationary measure describing the small deviations of the perfect foresight equilibrium is close to: *i*) the invariant measure of the global chaotic sunspot equilibrium whenever the backward perfect foresight map is chaotic on a finite union of intervals or *ii*) an atomic measure with support on a periodic orbit whenever the backward perfect foresight map possesses an attracting cycle.

2. THE FRAMEWORK AND THE MAIN RESULT

We will consider one-period forward looking models of the type

$$(2.1) \quad F(x_t) = E_t[G(\tilde{x}_{t+1})],$$

where F and G are differentiable functions defined in open subsets of \mathbb{R} , x_t is the value of the state variable of the model in period t , \tilde{x}_{t+1} is the random variable representing the possible values of the state variable of the model in period $t + 1$ and $E_t[\cdot]$ is the mathematical expectation operator conditioned to the information available up to time t . As usual, the interpretation of the model (2.1) is the following: if the agents believe that the probability distribution of the state variable in period $t + 1$ is given by that of the random variable \tilde{x}_{t+1} , then the current state variable value that equilibrates individual decisions and markets is x_t .

When F is an invertible function it is possible to define the backward perfect foresight map, as follows.

Definition 2.1. The *backward perfect foresight map* (*bpf map*) associated to the model (2.1) is a real valued map such that $F(\phi(x)) = G(x)$ for all x in the domain of ϕ .

Thus, if there exists F^{-1} , the bpf map is defined by $\phi(x) = F^{-1}(G(x))$. In general, $\phi(x)$ represents the current value of the state variable that equilibrates the markets when the expectation regarding the future value of the state variable is the Dirac measure δ_x . That map allows us to define perfect foresight paths for the model.

Definition 2.2. A *perfect foresight equilibrium path* from x_0 is a sequence $\{x_t\}_{t \geq 0}$ such that $x_t = \phi(x_{t+1})$ for all $t \geq 0$.

Therefore, in a perfect foresight equilibrium path, individuals have perfect foresight regarding the future value of the state variable given by x_{t+1} , and the corresponding current value is $x_t = \phi(x_{t+1})$. Our point in this definition is that the future state value may have some perturbation and this will affect the current state variable value. Thus, we introduce the following definition.

Definition 2.3. Let $\{\epsilon_t\}_{t \geq 1}$ be a sequence of independent and identically distributed random variables. The associated *small deviation from a perfect foresight equilibrium path* from x_0 is a sequence $\{\tilde{x}_t\}_{t \geq 0}$ such that $\tilde{x}_0 = x_0$ and $\tilde{x}_t = \phi(\tilde{x}_{t+1} + \epsilon_{t+1})$ for all $t \geq 0$.

The reasons to introduce the definition above are the following. First, we want to analyze the dynamics of the process governing the state variable under the possibility of small deviations from the perfect prevision hypothesis. Second, we want to find the limit behavior of that dynamics when the support of the deviations shrinks

to the singleton $\{0\}$. This is an important analysis that allows us to test the robustness of equilibrium paths under perfect foresight, thus it will provide a criterion for equilibrium selection when a multiplicity of equilibria is present. Third, as long as most of the macro models deal with a representative agent, this (small) random perturbation can be interpreted as representing expectations coordination failures in more general models with heterogeneous agents. Finally, we will see that the limiting dynamics is closely related to the concept of sunspot equilibrium.

The reading of the definition above must be done as follows. Given the state variable value for the next period, \tilde{x}_{t+1} , a random perturbation of this value is perceived by the agents, $\tilde{x}_{t+1} + \epsilon_{t+1}$; then the current state variable value that rationalizes that perception is $\tilde{x}_t = \phi(\tilde{x}_{t+1} + \epsilon_{t+1})$. According to this, the process $\{\tilde{x}_t\}_{t \geq 0}$ is a backward Markovian stochastic process.

Next, we enumerate the hypotheses to be used throughout this work. We state and discuss our assumptions regarding the bpf map before focusing our attention on the properties required from the random perturbations.

Hypothesis A: The backward perfect foresight map ϕ is a C^3 unimodal map defined on an interval of the form $[0, \alpha]$, $0 < \alpha \leq +\infty$, with a non-flat critical point x^* for which $\phi^2(x^*) > 0$, negative Schwarzian derivative¹ and such that $\phi(0) = 0$ and $\phi'(0) > 1$. Moreover, we will assume that ϕ satisfies the following additional conditions:

- (A1) *Slow recurrence:* $|\phi^k(x^*) - x^*| \geq e^{-\alpha k}$ for all $k \geq H_0$,
- (A2) *Expanding critical orbit:* $|D\phi^k(\phi(x^*))| \geq e^{\gamma k}$ for all $k \geq H_0$,
- (A3) *Visiting property:* ϕ is topologically mixing in the dynamical interval $X = [\phi^2(x^*), \phi(x^*)]$,

where $H_0 \geq 1$, $\gamma > 0$ and $0 < \alpha < \gamma/4$ are fixed constants.

Under Hypothesis A, the bpf map ϕ restricted to its dynamical interval $X \subset \mathbb{R}^+$, $\phi : X \rightarrow X$, has a Bowen-Ruelle-Sinai invariant probability measure, namely, there exists an absolutely continuous measure μ with support in X such that $\mu(\phi^{-1}(A)) = \mu(A)$ for any Borel set $A \subset X$ and there exists a subset $C \subset X$ with positive Lebesgue measure such that for all $x \in C$ the sequence of measures

$$\mu_N = \frac{1}{N} \sum_{k=0}^{N-1} \delta_{\phi^k(x)}$$

converges to μ in the weak topology. The unimodal shape of ϕ and the existence of an invariant and absolutely continuous measure associated to it are the conditions for the existence of a sunspot equilibrium, as shown by Araujo and Maldonado [1]. The invariant measure of the Markovian process proposed by them is μ . It is also worth remarking that such measure is ergodic, describing the typical asymptotic behavior of ϕ in the sense that $\mu_N \rightarrow \mu$ in the weak topology for Lebesgue almost every $x \in X$.

In order to have stability of the stochastic perturbations, we will assume the following hypothesis for the random perturbations.

Hypothesis B: The probability distribution of each term in the process $\{\epsilon_t\}_{t \geq 1}$ has a probability density function $\theta_\epsilon : \mathbb{R} \rightarrow \mathbb{R}_0^+$, such that

¹See the textbook [18] for a detailed treatment of one-dimensional dynamics.

- (i) $\text{supp}(\theta_\epsilon) \subset \Omega_\epsilon = [-\epsilon, \epsilon]$,
- (ii) $\int_{\Omega_\epsilon} \theta_\epsilon(y) dy = 1$,
- (iii) $M = \sup_{\epsilon > 0} (\epsilon \sup |\theta_\epsilon|) < \infty$,
- (iv) $J_\epsilon := \{t \mid \theta_\epsilon(t) > 0\}$ is an interval containing 0 and $\eta_\epsilon := \log(\theta_\epsilon|_{J_\epsilon})$ is a concave function.

A quick remark regarding the well-posedness of the random dynamical system obtained by perturbing the bpf map ϕ by a random noise satisfying (B) is in order at this point. When a stochastic perturbation is introduced, the orbits of such stochastic dynamical system may leave the dynamical interval X associated with the bpf map ϕ with positive probability. However, since $\phi^2(x^*) > 0$ there exists $\epsilon_0 > 0$ such that for every probability density function θ_ϵ satisfying hypothesis (B), where $\epsilon < \epsilon_0$, there exists a compact interval

$$X_\epsilon = [\phi(\phi(x^*) + \epsilon) - \epsilon, \phi(x^*) + \epsilon] \subset \mathbb{R}^+$$

such that $X \subset X_\epsilon$ and, moreover, X_ϵ is invariant under both the deterministic and stochastic dynamics associated to ϕ . Since we will study the limit where the maximum size of the random perturbations $\epsilon \rightarrow 0$, we can take the size of the random perturbation to be strictly smaller than $\epsilon_0 > 0$, thus yielding well defined random dynamics in the compact interval X_ϵ .

It is worth noting that the process $\tilde{x}_t = \phi(\tilde{x}_{t+1} + \epsilon_{t+1})$ can be written as $\tilde{z}_t = \phi(\tilde{z}_{t+1}) + \epsilon_t$ after performing the change of variable $z_t = x_t + \epsilon_t$. From now on, for the sake of simplicity, we will write $\tilde{x}_t = \phi(\tilde{x}_{t+1}) + \epsilon_t$. Thus, given the expectation of the future state variable value $\tilde{x}_{t+1} = x$, and a random perturbation of it with support $[-\epsilon, \epsilon]$, the probability that the current value of the state variable belongs to the Borel set $A \in \mathcal{B}(X_\epsilon)$ is given by:

$$(2.2) \quad P_\epsilon(x, A) = \int_{A - \{\phi(x)\}} \theta_\epsilon(t) dt = \int_A \theta_\epsilon(y - \phi(x)) dy$$

If we suppose that the small random perturbations represent failures in expectations coordination, (2.2) represents the probability of the current value of the state variable being in A given that the expectation of the value for the future state is x and there is expectations coordination failures. Thus, $P_\epsilon(x, A)$ can be seen as the backward conditional probability induced by the ECF process.

The Markov process with transition probability given by (2.2) has an absolutely continuous stationary measure, which we denote by ν^ϵ and satisfies the identity:

$$(2.3) \quad \nu^\epsilon(A) = \int P_\epsilon(x, A) d\nu^\epsilon(x).$$

Our next task is to prove that the stationary measures of the perturbed process (ν^ϵ) converge to the B-R-S measure μ as $\epsilon \rightarrow 0$. This will show the robustness of the stationary measure of Araujo and Maldonado [1] sunspot equilibrium with respect to small random perturbations. Let $\mathcal{A} = (A_i)_{i=1}^n$ be a partition of X . Define the map $P : X \rightarrow \mathcal{A}$ by $P(x) = A_i$ if and only if $x \in A_i$. Clearly, the map P is well defined for every $x \in X$. We define $\mu_N(P(x))$ to be given by

$$\mu_N(P(x)) = \frac{\#\{\phi^k(x) \in P(x) : k \in \{0, \dots, N-1\}\}}{N},$$

that is, the histogram defined by the deterministic N -history of the perfect foresight equilibrium path $\{x_t\}_{t \geq 0}$ and bin range defined by the partition \mathcal{A} .

Denote the Lebesgue measure by λ and the density of μ by ρ . Then, for μ -a.e. $x \in X$ we have that

$$\rho_N^{\mathcal{A}}(x) := \frac{\mu_N(P(x))}{\lambda(P(x))}$$

is an estimate of the density (Radon-Nikodym derivative) of the invariant measure μ and, as $N \rightarrow \infty$ and the size of the partition \mathcal{A} converges to zero, we get that

$$\rho_N^{\mathcal{A}}(x) \rightarrow \rho(x)$$

with respect to the L^1 norm.

Let us now consider the small deviation from a perfect foresight equilibrium path $\{\tilde{x}_t\}_{t \geq 0}$ associated to the stationary discrete-time process $\{\epsilon_t\}_{t \geq 1}$ described above, i.e. the terms of $\{\tilde{x}_t\}_{t \geq 0}$ are such that

$$(2.4) \quad \tilde{x}_t = \phi(\tilde{x}_{t+1}) + \epsilon_t .$$

As discussed above, for every sufficiently small $\epsilon > 0$, the stochastic process defined by (2.4) has a unique ergodic absolutely continuous stationary measure ν^ϵ with density ρ^ϵ (see [8]). Thus, an estimate of the measure ν^ϵ can be defined by the asymptotic distribution of the Birkhoff average of Dirac measures over the path with expectations coordination failures $\{\tilde{x}_t\}_{t \geq 0}$ with $\tilde{x}_0 = x$, i.e. the measure

$$(2.5) \quad \nu_N^\epsilon := \frac{1}{N} \sum_{k=0}^{N-1} \delta_{\tilde{x}_{-k}}$$

converges to ν^ϵ in the weak topology. Analogously to the deterministic case, for every $x \in X_\epsilon$ and every sufficiently small $\epsilon > 0$, $\nu_N^\epsilon(P(x))$ is well defined by:

$$\nu_N^\epsilon(P(x)) = \frac{\#\{(\phi + \epsilon_k) \circ \dots \circ (\phi + \epsilon_1)(x) \in P(x) : k \in \{0, \dots, N-1\}\}}{N} .$$

For ν^ϵ -a.e. $x \in X_\epsilon$, we can estimate the density ρ^ϵ of the measure ν^ϵ by:

$$\rho_N^{\epsilon, \mathcal{A}}(x) := \frac{\nu_N^\epsilon(P(x))}{\lambda(P(x))} .$$

Moreover, we note that $\rho_N^{\epsilon, \mathcal{A}}(x) \rightarrow \rho^\epsilon(x)$ with respect to the L^1 norm as $N \rightarrow \infty$ and the size of the partition converges to zero.

We will now state the stochastic stability of backward perfect foresight equilibria. It provides conditions under which the stationary measure associated to the small random perturbations of the perfect foresight path converges to the B-R-S measure μ as the size of the perturbations decreases to zero. For the family of backward perfect foresight maps under consideration here, this result follows as a consequence of Baladi and Viana Theorem regarding the strong stochastic stability of a certain family of unimodal maps [6].

Theorem 2.4. *Assume that the model (2.1) has its associated bpf map $\phi : X \rightarrow X$ satisfying hypothesis (A), and that the random perturbations $\{\epsilon_j\}_{j \geq 1}$ associated to a small deviation from a perfect foresight equilibrium path have density function satisfying (B). Then, the following statements hold:*

- i) *The approximate measure ν_N^ϵ associated to the small deviation from a perfect foresight equilibrium path converges to the measure μ of the deterministic dynamics ϕ in the weak*-topology, when $N \rightarrow \infty$ and $\epsilon \rightarrow 0$.*

- ii) *The approximate density $\rho_N^{\epsilon, A}$ associated to the small deviation from a perfect foresight equilibrium path converges to the density ρ of ϕ in the norm topology, when $N \rightarrow \infty$, $\epsilon \rightarrow 0$ and for every sequence of partitions $A = (A_i)_{i=1}^n$ such that $\max_i |A_i| \xrightarrow{n \rightarrow \infty} 0$.*

Proof. We prove item (i) first. Let $\epsilon > 0$ be small enough so that ν^ϵ is the unique invariant and ergodic probability measure associated to $P_\epsilon(\cdot, \cdot)$ (namely, satisfying (2.3)). Furthermore, recall that for $\epsilon > 0$ small enough, ν^ϵ is absolutely continuous with respect to the Lebesgue measure. By the definition of the measure ν_N^ϵ given in (2.5) and ergodicity of ν^ϵ , we have that $\nu_N^\epsilon \xrightarrow{N \rightarrow \infty} \nu^\epsilon$ in the weak*-topology. Under assumptions (A) and (B), Baladi and Viana Theorem [6] ensures that $\nu^\epsilon \rightarrow \mu$ in the weak*-topology. Therefore, combining the two statements above, we obtain that $\nu_N^\epsilon \rightarrow \mu$ in the weak*-topology when $N \rightarrow \infty$ and $\epsilon \rightarrow 0$.

The proof of item (ii) is similar. By ergodicity of ν^ϵ , we have that $\rho_N^{\epsilon, A} \rightarrow \rho^\epsilon$ in $L^1(dx)$ when $N \rightarrow \infty$ and the size of largest element of the partition goes to zero. The result then follows by noting that Baladi and Viana Theorem [6] also ensures that $\rho^\epsilon \rightarrow \rho$ in $L^1(dx)$ as $\epsilon \rightarrow 0$. \square

Theorem 2.4 has the following important consequences. Random deviations from perfect foresight maps are stable, provided that they are small. Notice that we are not analyzing the small deviation of each element of the path $\{x_t\}_{t \geq 0}$, where $x_t = \phi(x_{t+1})$, but rather the stability of the systematic perturbation of the system, namely, of the process $\tilde{x}_t = \phi(\tilde{x}_{t+1}) + \epsilon_t$.

Moreover, the stability of the process is around the invariant B-R-S measure of the backward perfect foresight map ϕ . Recalling Araujo and Maldonado [1] result, when the bpf map is a unimodal map with an absolutely continuous invariant measure, there exists a stationary sunspot equilibrium for the system (2.1) given by the Markovian process

$$(2.6) \quad Q(x, A) = \frac{d\mu \circ f}{d\mu}(x) \delta_{f(x)}(A) + \frac{d\mu \circ g}{d\mu}(x) \delta_{g(x)}(A),$$

where f and g are the local inverses of ϕ .

Remark: To define the sunspot equilibrium (2.6) Araujo and Maldonado [1] used the definition provided by [16]: A sunspot equilibrium for the system (2.1) is a $X_0 \subset X$ and a transition function $Q : X_0 \times \mathcal{B}(X_0) \rightarrow [0, 1]$ such that i) $F(x) = E_{Q(x, \cdot)}[G(\tilde{x})]$ for all $x \in X_0$ and ii) there exists $x_0 \in X_0$ such that $Q(x_0, \cdot)$ is truly stochastic. The sunspot equilibrium is stationary if there exists a stationary probability measure μ for the transition function Q . The transition function defined in (2.6) satisfies that definition and the ergodic and ϕ -invariant absolutely continuous measure μ is also invariant for that Q . It is clear that once a stationary sunspot equilibrium is found for (2.1) in that way, it can be defined the process $s_{t+1} = F(x_t) - G(\tilde{x}_{t+1})$, which explicitly represents (extrinsic) sunspot variable.

The stationary process of (2.6) is the measure μ . Thus we conclude that small deviations from the perfect foresight map converge to the stationary measure of the sunspot equilibrium of Araujo and Maldonado [1] when the size of the deviations converge to zero. This is a robustness result for that type of sunspot equilibria.

Finally, when the invariant measure of the bpf map is not an absolutely continuous measure and ϕ has an attracting cycle (conditions (A2) and (A3) do not

hold), the measure (2.3) is close to a measure which is a convex linear combination of Dirac masses with support on that attracting cycle. In this case we will say that ϕ is an *ordered map*. Notice that such attracting cycle is with respect to the bpf map ϕ , thus it is a locally determinate cycle for the model (2.1). Hence, we will have that small deviations from the perfect foresight map converge to that measure supported in that cycle. Therefore, and depending on the model parameter values, the stationary measure associated with small random perturbations of the perfect foresight equilibrium may converge to the stationary measure of the chaotic sunspot equilibrium (2.6) if the unperturbed dynamics are chaotic over a finite union of intervals, or it may converge to an atomic measure with support on an periodic cycle, if the deterministic dynamics have an attracting cycle.

In the following section we show two examples where the bpf map is a unimodal map and we track the parameter values where the convergence of the small random perturbations to a B-R-S measure or to a cycle is obtained.

3. TWO EXAMPLES

In this section we provide two examples where the convergence of the stationary measure associated with small perturbations of perfect foresight maps is illustrated. Depending on the parameter values of these models, as the maximal size of the perturbations decreases to zero, we have convergence of the stationary measure to the absolutely continuous invariant measure of the sunspot equilibrium (2.6), provided the perfect foresight map is chaotic on a finite union of intervals, and convergence to a convex linear combination of Dirac measures supported on a periodic orbit, provided the perfect foresight map has an attracting cycle (namely, a locally deterministic cycle). Taking that convergence as a selection criterion when multiplicity of equilibria is present, we will have that the only equilibria that are robust under small random perturbations of deterministic paths are either the chaotic sunspot equilibrium (2.6) or a deterministic cycle of the model. Neither indeterminate equilibria nor other types of sunspot equilibria remain as a limiting behavior of small random perturbations.

The first example is the classical overlapping generation model with fiat money and the second is the market game model of Shapley and Shubik.

3.1. An OLG model with fiat money. In this section we will consider a two-period overlapping generations (OLG) model like the one introduced in [26] and provide an application for our main results. An analogous model was analyzed by Azariadis and Guesnerie in [5] to prove the existence of cycles and sunspots with finite support. We will show that depending on the parameter values of the model and up to a set of measure zero in parameter space, small deviations from the perfect foresight equilibrium can be either close to the stationary measure of the sunspot equilibrium (2.6) or close to an atomic measure with support on an attracting cycle for the deterministic dynamics.

The economy is populated by a large number of young and old agents. The population sizes are the same and remain constant over time. We assume that there exists a representative agent with preferences given by a separable utility function

$$U(c_t, c_{t+1}) = \frac{c_1^{1-\alpha_1}}{1-\alpha_1} + \frac{c_2^{1-\alpha_2}}{1-\alpha_2} ,$$

where $\alpha_1 > 0$ and $\alpha_2 > 0$ are the coefficients of relative risk aversion of the agent, c_t, c_{t+1} denote the corresponding consumption plan. We suppose also that one unit of the good is produced with one unit of the unique productive factor (labor) and let l_1^* and l_2^* denote the agents labor endowments in the first and second periods of their lives, respectively. Finally, we assume that there is a risk-free asset (fiat money) that can be purchased by the agents providing a gross return $z_t = 1$, and the money supply is constant, i.e. $M_t = M_0$ for all $t \geq 0$.

We will now state the consumption-saving problem of the representative agent. Let p_t and p_{t+1} denote the prices of the unique good in the economy during the first and second periods of the individual's life. Note that while p_t is known by the individual during the first stage of her life, her knowledge of p_{t+1} consists of a probability distribution μ_{t+1} representing the likelihood of occurrence of particular values of p_{t+1} and reflecting the individual's beliefs about the state of economy during the second period of her life. The agent must choose a consumption plan (c_t, c_{t+1}) and the first period saving m_t as the solution of the following optimization problem

$$(3.1) \quad \max_{\{c_t, c_{t+1}, m_t\}} \frac{c_1^{1-\alpha_1}}{1-\alpha_1} + E_t \left[\frac{c_2^{1-\alpha_2}}{1-\alpha_2} \right]$$

subject to the budget constraints

$$\begin{aligned} p_t c_t + m_t &= p_t l_1^* \\ p_{t+1} c_{t+1} &= p_{t+1} l_2^* + m_t \end{aligned}$$

where $E_t[\cdot]$ is the mathematical expectation with respect to the probability measure μ_{t+1} . Working out the first order condition for an interior solution of (3.1) leads to

$$(3.2) \quad -\frac{1}{p_t} \left(l_1^* - \frac{m_t}{p_t} \right)^{-\alpha_1} + E_t \left[\frac{1}{p_{t+1}} \left(l_2^* + \frac{m_t}{p_{t+1}} \right)^{-\alpha_2} \right] = 0 .$$

The normalized monetary equilibrium condition is $M_t = m = 1$. After introducing the new variable $x_t = 1/p_t$, the first order condition (3.2) may be rewritten as

$$(3.3) \quad F(x_t) = E_t [G(x_{t+1})] ,$$

where F and G are auxiliary functions defined by

$$(3.4) \quad F(x) = x(l_1^* - x)^{-\alpha_1} \quad \text{and} \quad G(x) = x(l_2^* + x)^{-\alpha_2} ,$$

thus yielding an identity of form (2.1). defining the equilibrium dynamics for the state variable $x_t = 1/p_t$.

The map $F : [0, l_1^*] \rightarrow \mathbb{R}^+$ defined in (3.4) is strictly increasing, and thus invertible. Hence, we obtain that the corresponding backward perfect foresight map is of the form

$$(3.5) \quad \phi(x) = F^{-1}(G(x)) ,$$

with the associate bpf dynamical system being determined by

$$x_t = \phi(x_{t+1}) = F^{-1}(G(x_{t+1})) .$$

3.2. The Shapley-Shubik market game model. We will now consider the one commodity overlapping generations version of the Shapley-Shubik market game [43], presented by Goenka et. al. [24].

This is a model of a pure exchange economy with overlapping generations of agents and a single good being traded. Time is discrete and labeled as $t = 1, 2, \dots$. In each period $t > 0$ agents are born and live two periods. Thus, young and old individuals coexist in each period t , with n elder individuals alive at time $t = 1$. Individuals are assumed to have identical utilities for consumption and identical initial endowments in each period of life. Denoting by (c_t^t, c_t^{t+1}) the consumption plan of an individual born in period t , her utility is given by:

$$(3.6) \quad U(c_t^t, c_t^{t+1}) = u_1(c_t^t) + u_2(c_t^{t+1}).$$

We will suppose that U is a Von Newman-Morgenstern utility function; thus, in the presence of uncertainty, an expected value operator $E[\cdot]$ is set in the second term of (3.6). The initial endowments in each period of life are ω_1 and ω_2 .

The trade mechanism is as follows. There exists a fixed amount \bar{m} of fiat money in the economy. Each individual $i \in \{1, 2, \dots, n\}$ born at time t provides a (monetary) bid $b_{it} = (b_{it}^t, b_{it}^{t+1})$ to get monetary resources to buy goods, decides her savings m_t and also provides an offer of goods $q_{it} = (q_{it}^t, q_{it}^{t+1})$. Bids and offers operations are performed in trading posts and in case of presence of uncertainty the negotiations are conditioned to each state of the nature. The aggregate bids and offers of individuals born at time $s = t, t + 1$ are defined by

$$(3.7) \quad B_t^s = \sum_{i=1}^n b_{it}^s \quad \text{and} \quad Q_t^s = \sum_{i=1}^n q_{it}^s, \quad s = t, t + 1$$

and the bids and offers in the market are:

$$(3.8) \quad B_t = B_{t-1}^t + B_t^t \quad \text{and} \quad Q_t = Q_{t-1}^t + Q_t^t.$$

The terms of trade, that will be used as a ‘‘price’’ of the commodities in terms of fiat money, is defined by B_t/Q_t . Then, the lifetime budget constraints of each individual are given by

$$(3.9) \quad b_t^t + m_t = \frac{B_t}{Q_t} q_t^t \quad \text{and} \quad b_t^{t+1} = m_t + \frac{B_{t+1}}{Q_{t+1}} q_t^{t+1}.$$

If (3.9) is satisfied, then the lifetime consumption of each individual is:

$$(3.10) \quad x_t^t = \omega_1 - q_t^t + \frac{Q_t}{B_t} b_t^t \quad \text{and} \quad x_t^{t+1} = \omega_2 - q_t^{t+1} + \frac{Q_{t+1}}{B_{t+1}} b_t^{t+1}.$$

To solve the individual problem in terms of the demand of money, we have to express the terms in (3.10) as functions of m_t . To do this, let us express the aggregate bids (offers) in t and $t + 1$ in terms of the individual bid (offer) and the other agents aggregate bids (\hat{B}_t, \hat{B}_{t+1}) (offers (\hat{Q}_t, \hat{Q}_{t+1})) in t and $t + 1$ respectively:

$$(3.11) \quad B_t = b_t^t + \hat{B}_t \quad \text{and} \quad Q_t = q_t^t + \hat{Q}_t.$$

Analogously, we have that

$$(3.12) \quad B_{t+1} = b_t^{t+1} + \tilde{B}_{t+1} \quad \text{and} \quad Q_{t+1} = q_t^{t+1} + \tilde{Q}_{t+1}.$$

Substituting (3.11) into the first constraint in (3.9), we obtain the identity

$$(\hat{Q}_t + q_t^t)(b_t^t + m_t) = (\hat{B}_t + b_t^t)q_t^t,$$

which, rearranging terms, yields

$$(3.13) \quad b_t + m_t = \left[\frac{\hat{B}_t - m_t}{\hat{Q}_t} \right] q_t^t .$$

Similarly, substituting (3.12) into the second constraint in (3.9) results in:

$$(3.14) \quad b_t^{t+1} = m_t + \left[\frac{\tilde{B}_{t+1} + m_t}{\tilde{Q}_{t+1}} \right] q_t^{t+1} .$$

Substituting (3.9) into (3.13) and (3.14), we obtain the terms of trade in each period, given by

$$(3.15) \quad \frac{B_t}{Q_t} = \frac{\hat{B}_t - m_t}{\hat{Q}_t} \quad \text{and} \quad \frac{B_{t+1}}{Q_{t+1}} = \frac{\tilde{B}_{t+1} + m_t}{\tilde{Q}_{t+1}} .$$

Finally, putting together (3.9), (3.10), (3.13) and (3.15) (as well as (3.9), (3.10), (3.14) and (3.15)), we obtain the individuals lifetime consumptions in terms of the money demand and the strategies of all other individuals:

$$(3.16) \quad x_t^t(m_t) = \omega_1 - \frac{\hat{Q}_t}{\hat{B}_t - m_t} m_t \quad \text{and} \quad x_t^{t+1}(m_t) = \omega_2 + \frac{\tilde{Q}_{t+1}}{\tilde{B}_{t+1} + m_t} m_t .$$

If in period t , agents do not know neither the value of the aggregate bid \tilde{B}_{t+1} nor the value of the aggregate offer \tilde{Q}_{t+1} for the next period, then they will use beliefs regarding those variables (μ_{t+1}). Thus a mathematical expectation operator must have to be included for the utility of the second period of life. Therefore, the individual problem is to find the demand of money such that:

$$(3.17) \quad \max_{m_t} u_1(x_t^t(m_t)) + E_t [u_2(x_t^{t+1}(m_t))] ,$$

where $E_t[\cdot]$ is the mathematical expectation with respect to the probability measure μ_{t+1} . The First Order Condition for interior solutions of the optimization problem above is:

$$(3.18) \quad \left(\frac{\hat{Q}_t}{\hat{B}_t - m_t} + \frac{\hat{Q}_t m_t}{(\hat{B}_t - m_t)^2} \right) u_1' \left(\omega_1 - \frac{\hat{Q}_t m_t}{\hat{B}_t - m_t} \right) = E_t \left[\left(\frac{\tilde{Q}_{t+1}}{\tilde{B}_{t+1} + m_t} - \frac{\tilde{Q}_{t+1} m_t}{(\tilde{B}_{t+1} + m_t)^2} \right) u_2' \left(\omega_2 + \frac{\tilde{Q}_{t+1} m_t}{\tilde{B}_{t+1} + m_t} \right) \right] .$$

It should be recalled that in this kind of models there is indeterminacy in either the offers or the bids (Peck et.al. [40]). Thus, we will suppose that the sequence of offers (q_t^t, q_t^{t+1}) is given and the amount of fiat money \bar{m} is fixed. In this context we give the following definition of equilibrium.

Definition 3.1. Given the exogenous offers (q_t^t, q_t^{t+1}) and the amount of fiat money \bar{m} , a *monetary Nash equilibrium* is a sequence of bids (b_t^t, b_t^{t+1}) that makes $m_t = \bar{m}$ the solution of (3.18).

To obtain the dynamics of the bids, let us notice that from (3.15) one can get

$$\frac{\tilde{Q}_t}{\tilde{B}_t + m_{t-1}} = \frac{\hat{Q}_t}{\hat{B}_t - m_t} ,$$

which, rearranging terms, yields

$$\tilde{B}_t = \frac{\tilde{Q}_t}{\hat{Q}_t}(\hat{B}_t - m_t) - m_{t-1} .$$

Substituting the identity immediately above into the right-hand side of (3.18) results in:

$$(3.19) \quad \left(\frac{\hat{Q}_t}{\hat{B}_t - m_t} + \frac{\hat{Q}_t m_t}{(\hat{B}_t - m_t)^2} \right) u'_1 \left(\omega_1 - \frac{\hat{Q}_t m_t}{\hat{B}_t - m_t} \right) = \\ E_t \left[\frac{\hat{Q}_{t+1}}{(\hat{B}_{t+1} - m_{t+1})^2} \left(\hat{B}_{t+1} - m_{t+1} - \frac{\hat{Q}_{t+1} m_t}{\hat{Q}_{t+1}} \right) u'_2 \left(\omega_2 + \frac{\hat{Q}_{t+1} m_t}{\hat{B}_{t+1} - m_{t+1}} \right) \right] .$$

Suppose now that the sequence of exogenous offers is $q_t^t = q_1$ and $q_t^{t+1} = q_2$ for all $t \geq 1$ and $\bar{m} = m$. Then, for all $t \geq 1$, one must have that

$$\hat{Q}_t = \hat{Q} = (n-1)q_1 + nq_2 \quad \text{and} \quad \tilde{Q}_t = \tilde{Q} = nq_1 + (n-1)q_2 .$$

Substituting the two identities above into (3.19) yields the dynamics in the aggregate bids:

$$(3.20) \quad \left(\frac{\hat{Q}}{\hat{B}_t - m} + \frac{\hat{Q}m}{(\hat{B}_t - m)^2} \right) u'_1 \left(\omega_1 - \frac{\hat{Q}m}{\hat{B}_t - m} \right) = \\ E_t \left[\frac{\hat{Q}}{(\hat{B}_{t+1} - m)^2} \left(\hat{B}_{t+1} - \frac{m(\hat{Q} + \tilde{Q})}{\tilde{Q}} \right) u'_2 \left(\omega_2 + \frac{\hat{Q}m}{\hat{B}_{t+1} - m} \right) \right] ,$$

in this case, the mathematical expectation operator is with respect to the probability distribution of the next period aggregate bids \hat{B}_{t+1} . Finally, we introduce the variable

$$x_t = \frac{\hat{Q}m}{\hat{B}_t - m}$$

and rewrite the equilibrium dynamics condition in (3.20) as

$$\frac{(x_t^2 + \hat{Q}x_t)}{\hat{Q}} u'_1(\omega_1 - x_t) = E_t \left[\frac{(\tilde{Q}x_{t+1} - x_{t+1}^2)}{\tilde{Q}} u'_2(\omega_2 + x_{t+1}) \right] ,$$

where, without loss of generality, the induced probability distribution on x_{t+1} is also denoted by μ_{t+1} .

Let us now consider, as done in the example of Section 3.1, the case of constant relative risk aversion utility functions:

$$(3.21) \quad u_i(c) = \frac{c^{1-\alpha_i}}{1-\alpha_i} , \quad \alpha_i \neq 1 ,$$

or the limit case of logarithmic utility functions (when $\alpha_i \rightarrow 1$):

$$(3.22) \quad u_i(c) = \ln(c) .$$

Under these functional specifications, we are able to define

$$F(x) := \frac{(x^2 + \hat{Q}x)}{\hat{Q}} (\omega_1 - x)^{-\alpha_1}$$

and

$$G(x) := \frac{(\tilde{Q}x - x^2)}{\tilde{Q}}(\omega_2 + x)^{-\alpha_2} ,$$

yielding, in the general stochastic case, the dynamical system

$$(3.23) \quad F(x_t) = E_t [G(x_{t+1})] ,$$

which is again of the form (2.1).

Noticing that the function F is strictly increasing in $[0, \omega_1]$ and the function G is a unimodal function on the interval $[0, \tilde{Q}]$, we obtain that the backward perfect foresight map

$$\phi(x) = F^{-1}(G(x))$$

is again a unimodal map. Moreover, as with (3.5), this map determines a dynamical system through the recursive relation.

$$x_t = \phi(x_{t+1}) = F^{-1}(G(x_{t+1})) .$$

Due to the striking similarities between the the bpf maps (and corresponding dynamical systems) of the models in this section and the previous one, we move on to perform a combined analysis for these two cases.

3.3. Numerical simulations. In this last subsection we provide the outcomes of some numerical simulations of the two models discussed above. Our aim is to illustrate the convergence of the stationary measures associated with the small stochastic deviations from the perfect foresight equilibrium as the maximal size of the stochastic deviations tends to zero. Depending on the models' parameter values, the limiting behavior of the stationary measure may be an absolutely continuous and ergodic invariant measure, in the case of deterministic dynamics which are chaotic in a finite union of intervals, and a linear convex combination of Dirac measures with support on an attracting cycle for the deterministic dynamics, for the case of regular deterministic bpf dynamics.

To keep things unified, let $\phi_\lambda(x)$ denote the bpf in any of the models described, where λ is one of the parameters defining the function, taking values in a certain set Λ , as we describe below:

- a) For the OLG model with fiat money of Section 3.1, we fix every parameter (α_1 is fixed in such a way that $\alpha_1 \in (0, 1]$), except for $\lambda = \alpha_2$, which is allowed to vary in the set

$$\Lambda = \{\lambda \in [2, +\infty) : (l_1^*)^{\alpha_1} > (l_2^*)^\lambda\} .$$

Then, for every $\lambda \in \Lambda$, we have that ϕ_λ is a C^3 unimodal map with $\phi(0) = 0$, $\phi'(0) > 1$, a (positive) nondegenerate critical point $\bar{x}(\lambda)$ and negative Schwarzian derivative² $S\phi(x) < 0$.

- b) For the market game model of Section 3.2, we fix every parameter value associated with the relative risk aversion coefficients and initial endowments in such a way that $\omega_1^{\alpha_1} > \omega_2^{\alpha_2}$. The parameter $\lambda = \tilde{Q}$ is allowed to vary in the set

$$\Lambda = (0, +\infty) .$$

²The Schwarzian derivative is defined by $S\phi = \frac{\phi'''(x)}{\phi'(x)} - \frac{3}{2} \left(\frac{\phi''(x)}{\phi'(x)} \right)^2$. A negative Schwarzian derivative for a unimodal map guarantees that ϕ is topologically mixing in the interval where the dynamics takes place

Under these conditions, for every $\lambda \in \Lambda$, ϕ_λ is a C^3 unimodal map with $\phi(0) = 0$, $\phi'(0) > 1$, a (positive) nondegenerate critical point $\bar{x}(\lambda)$ and $S\phi(x) < 0$.

Henceforth, we will focus our attention on those two one-parameter families ϕ_λ , $\lambda \in \Lambda$, of bpf maps of the examples above (parameterized by the relative risk aversion coefficient $\lambda = \alpha_2$ for the first example and the thickness parameter \tilde{Q} for the second).

Before proceeding with our discussion, we need to introduce an auxiliary concept.

Definition 3.2. We say that the one-parameter family ϕ_λ , $\lambda \in \Lambda$, has a *Misiurewicz parameter* $\lambda_* \in \Lambda$ with *generic unfolding* if the following conditions hold

- a) the map $(x, \lambda) \rightarrow (\phi_\lambda(x), D_x\phi_\lambda(x), D_x^2\phi_\lambda(x))$ is C^1 ;
- b) $\lambda_* \in \Lambda$ is such that ϕ_{λ_*} is a Misiurewicz map, i.e. ϕ_{λ_*} has no periodic attractors and the forward critical orbit does not accumulate on its critical point;
- c) the following transversality condition holds:

$$\lim_{n \rightarrow +\infty} \frac{D_\lambda \phi_{\lambda_*}^n(\bar{x}(\lambda_*))}{D_x \phi_{\lambda_*}^{n-1}(\phi_{\lambda_*}(\bar{x}(\lambda_*)))} \neq 0.$$

Note that condition a) only depends on the choice of a sufficiently regular parametrization for such family of bpf maps. In what concerns conditions b) and c), we remark that in the case where ϕ_{λ_*} is a post-critically finite Misiurewicz map, i.e. ϕ_{λ_*} has no periodic attractors and some iterate N of the critical point $\bar{x}(\lambda_*)$ reaches a repelling periodic point $P(\lambda_*)$, then condition c) is equivalent to the transversality of the curves $\lambda \mapsto \phi_\lambda^N(\bar{x}(\lambda))$ and $\lambda \mapsto P(\lambda)$.

The next result provides a set of conditions under which there exists a large set of bpf maps for which the strong stochastic stability of Theorem 2.4 holds. Nevertheless, we should remark that there may be additional parameter values yielding such strong stochastic stability.

Theorem 3.3. *Let ϕ_λ , $\lambda \in \Lambda \subset \mathbb{R}$, be a one-parameter family of bpf maps defined for one of the models above (the OLG model with fiat money of Section 3.1 or the market game model of Section 3.2).*

- (i) *If ϕ_λ has a Misiurewicz parameter $\lambda_* \in \Lambda$ with generic unfolding, then there exists a positive measure set $A \subset \Lambda$ having λ_* as a density point such that for every $\lambda \in A$ there exists an invariant measure μ_λ which is an absolutely continuous B-R-S measure.*
- (ii) *Moreover, if the dynamical system determined by ϕ_λ is perturbed by a random process $\{\epsilon_t\}_{t \geq 1}$ satisfying Hypothesis (B), then the results of Theorem 2.4 are also valid.*

Proof. For the families of utility functions with constant relative risk aversion introduced in Sections 3.1 and 3.2 and the choice of parameters discussed in items a) and b) above, we have that for every $\lambda \in \Lambda$ the following hold:

- i) ϕ_λ is a C^3 unimodal map;
- ii) ϕ_λ has a (positive) nondegenerate critical point $\bar{x}(\lambda)$;
- iii) ϕ_λ has a repelling fixed point at zero.

For one-parameter families of maps satisfying the conditions i), ii) and iii) above and having a Misiurewicz parameter $\lambda_* \in \Lambda$ with generic unfolding, there exists a

positive measure set A in the space of parameters with λ^* as a density point and such that conditions (A1) and (A2) from hypothesis A hold for every $\lambda \in A$ (see [18] for further details). As a consequence, for every $\lambda \in A$, we have that

- 1) ϕ_λ admits an absolutely continuous invariant measure μ_λ , with a L^p density for any $p < 2$;
- 2) μ_λ is a SBR measure;
- 3) ϕ_λ has positive Lyapunov exponent almost everywhere.

Hence, the strong stochastic stability of ϕ_λ follows from Baladi and Viana Theorem in [6] by observing that condition (A3) holds for unimodal maps with negative Schwarzian derivative and an absolutely continuous invariant measure. \square

As the theorems above assert, the set of parameter values where the stochastic stability is valid has positive Lebesgue measure. In that set, the limit of the stationary measure of the small random perturbations of ϕ_λ when the size of the perturbations goes to zero, is either the stationary measure of the chaotic sunspot equilibrium (2.6) or a convex linear combination of Dirac measures supported on a determinate (locally unique) cycle of the model. Notice that repelling cycles (namely, indeterminate cycles of the model) are not found as the limiting behavior of those perturbations. Since there exists a close relationship between indeterminate cycles (including steady states) and sunspots equilibria around them ([5], [9], [15]), those sunspot equilibria are also excluded as limiting behavior of the small random perturbations. Therefore, in those kind of models and in a set of parameter values with positive Lebesgue measure, the selection criterion of dynamic equilibria based on the small random perturbations of perfect foresight paths refines the multiplicity of equilibria only to two classes of equilibria, namely, determinate cycles or sunspot equilibria of the type (2.6).

We will now illustrate the large abundance of strong stochastic stable bpf maps for the OLG model of Section 3.1. We numerically determine values of parameters $(\alpha_1, \alpha_2, l_1^*, l_2^*)$ under which the map ϕ_λ is a post-critically finite Misiurewicz map, i.e. ϕ_λ has no periodic attractors and the critical orbit is pre-periodic to a repelling periodic orbit. Note that these are a subset of the set of Misiurewicz maps. To proceed with our numerical experiments, we fix the parameters $l_1^* = 3.51$ and $l_2^* = 0.55$ and work on the two parameter space of relative risk aversion coefficients $(\alpha_1, \alpha_2) \in (0, 1) \times (2, +\infty)$. The results described below are robust with respect to changes in the values of l_1^* and l_2^* . We then numerically compute any intersections between the first N iterates of the critical point and the periodic points up to some finite period M , excluding all the non-transverse intersections and all the intersections with attracting periodic points. Checking the stability of the periodic points is relevant because in the case where the critical point is pre-periodic to a repelling periodic point, then there are no stable or neutral cycles, since for unimodal maps with negative Schwarzian derivative, these would attract the critical orbit. In Figure 1 it is possible to notice the different dynamical behaviors of the bpf map as the risk aversion parameter α_2 increases from 2 to 7.5. For small values of α_2 , there exists a unique attracting fixed point of ϕ . Such attracting fixed point corresponds to a determinate steady state. As α_2 increases, attracting periodic points of higher periods are generated by period-doubling bifurcations. Each attracting periodic point corresponds to a determinate cycle of the model. All those ϕ_λ are ordered bpf maps and lead to invariant measures supported on

convex linear combination of Dirac measures supported on the determinate cycle. For large enough values of α_2 , Misiurewicz maps can be found. See Figure 2 for the distribution of Misiurewicz maps in parameter space $(\alpha_2, \alpha_1) \in (2, 7.5) \times (0.01, 0.29)$. As noted above, the values of parameters under which such maps occur are density points of positive measure sets where stability of small random perturbations of ϕ_λ can be found. Such parameters are associated with invariant B-R-S measures.

To leave clear the distinction between the behaviors associated with the two possibilities of convergence discussed above, for the parameter values used above and fixing additionally $\alpha_1 = 0.41$ and $\alpha_2 \in \{5.0; 6.5\}$, we plot in Figures 3 and 4 histograms associated with 10^6 iterations of paths generated by the bpf map ϕ for two different set of parameters values and varying sizes of the random perturbations.

The simulations discussed above can be criticized by the use of an excessively high relative risk aversion parameter. However, for the market game model of Section 3.2 with the constant relative risk aversion utility functions (3.21), we present the analogous analysis using more conservative values for that parameter. Specifically, fixing the parameter values $\alpha_1 = 0.5$, $\alpha_2 = 2$, $\omega_1 = 2$, $\omega_2 = 0.39$, $\hat{Q} = 0.1$ and varying $\tilde{Q} \in (0.0, 0.135)$ we can observe in Figure 5 the existence of parameters values determining bpf maps with B-R-S measures for values of \tilde{Q} below 0.098, approximately. For those parameter values we obtain the convergence of the stationary measures of small random perturbations to the stationary measure of the sunspot equilibria (2.6). For \tilde{Q} greater than 0.098, attracting cycles of the bpf map arise and therefore we obtain the convergence of the stationary measures of small random perturbations to convex linear combinations of Dirac measures supported on the corresponding deterministic cycle. Figures 6 and 7 contain histograms associated with varying maximal sizes for the random perturbations for the parameter values listed above and, respectively, $\tilde{Q} = 0.09$ and $\tilde{Q} = 0.12$, enabling us to compare once again the chaotic dynamics case against the regular dynamics case associated with an attracting cycle. For the case of unitary relative risk aversion (logarithmic utilities) and values of parameters $\omega_1 = 5.4$, $\omega_2 = 0.5$, $\hat{Q} = 0.2$ and varying $\tilde{Q} \in (0.0, 0.034)$, we observe in Figure 8 that stochastic stability with respect to an absolutely continuous invariant B-R-S measure holds for a wide range of values of \tilde{Q} below 0.255, approximately. The histograms in Figures 9 and 10 are obtained for varying maximal sizes of the random perturbations fixing, additionally, $\tilde{Q} = 0.23$ and $\tilde{Q} = 0.30$.

4. CONCLUSIONS

In this paper we propose a new selection criterion of dynamic equilibria in non-linear one-period forward looking economic models. The criterion is based on the limiting behavior of the stationary process generated by small random perturbations of the deterministic version of the model. Such small random perturbations are interpreted as small deviations from the perfect foresight dynamics that may arise from coordination failures in expectations, for example.

We apply that selection criterion to models with backward perfect foresight maps which are unimodal maps with negative Schwarzian derivative. For that class of models we prove that there exists a set of parameter values with positive Lebesgue measure where the only equilibria that are robust to this refinement are either deterministic cycles (including deterministic steady states) or the sunspot equilibrium

presented in [1]. Neither indeterminate cycles nor other types of sunspot equilibria remain as equilibrium selection according to that criterion.

To illustrate the large set of parameter values where those results are valid, we applied numerical simulations to two models in economic dynamics: the OLG model with fiat money and the Shapley-Shubick market game model. For both models we derived the equations defining the intertemporal equilibrium dynamics and showed the set of parameter values exhibiting convergence to a Bowen-Ruelle-Sinai measure, as well as parameter values exhibiting convergence to an atomic measure with support on an attracting cycle. Therefore, the proposed analysis aims at assessing the robustness of those equilibria, namely the global sunspot equilibrium of Araujo and Maldonado [1] or an attracting cycle, with respect to small stochastic deviations from the perfect foresight equilibrium.

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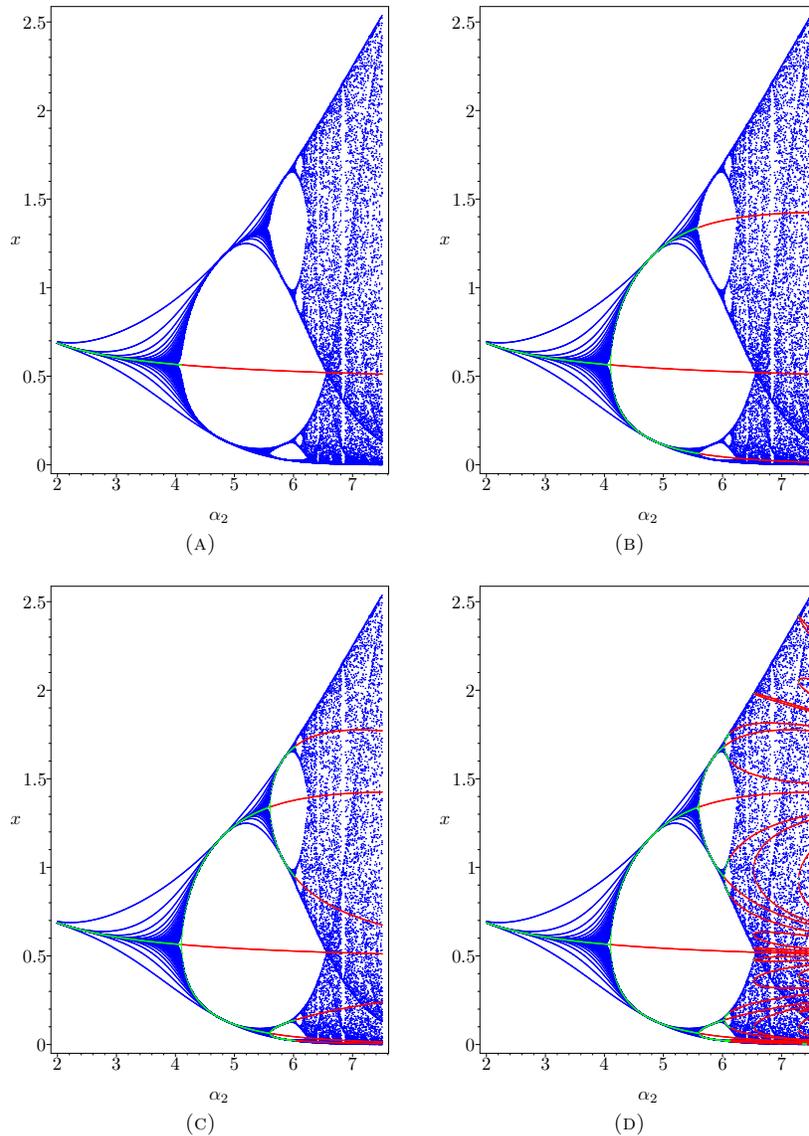


FIGURE 1. Plot in the (α_2, x) plane of the first 100 iterates of the critical point (in blue) and periodic points (up to period 8) of the bpf map ϕ of Section 3.1. We plot the stable periodic points in green and the unstable ones in red. Figure 1a contains only the fixed point of ϕ , Figure 1b contains the fixed point and the period 2 orbit, Figure 1c contains periodic points of periods 1, 2 and 4, and finally, Figure 1d contains all periodic points whose period divides 8. The remaining parameters are fixed and equal to $l_1^* = 3.51$, $l_2^* = 0.55$ and $\alpha_1 = 0.41$.

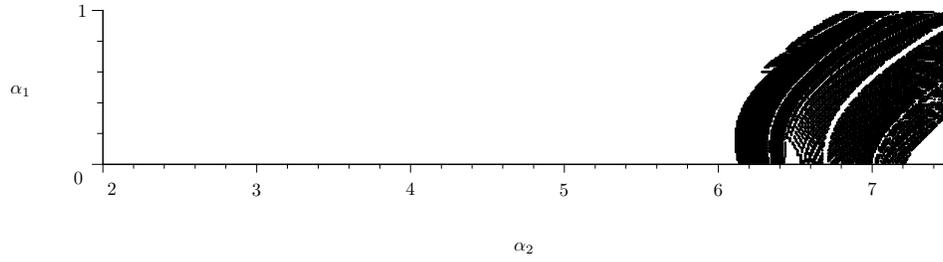


FIGURE 2. The distribution of Misiurewicz parameters for the family of bpf maps with $(\alpha_2, \alpha_1) \in (2, 7.5) \times (0.01, 0.99)$ for fixed $l_1^* = 3.51$ and $l_2^* = 0.55$. These are obtained by considering intersections of the first 100 iterates of the critical point with unstable periodic points of periods 1, 2, 4 and 8.

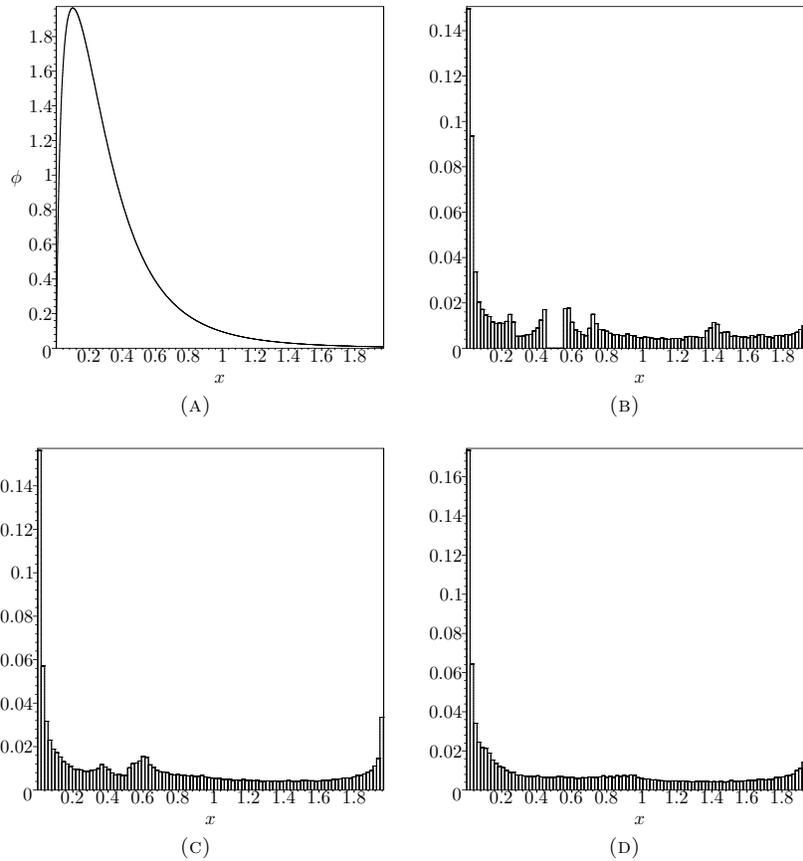


FIGURE 3. The bpf ϕ and the approximate densities associated with its stationary measure for the overlapping generations model of Section 3.1 and values of parameters $l_1^* = 3.51$, $l_2^* = 0.55$, $\alpha_1 = 0.41$ and $\alpha_2 = 6.5$ corresponding to deterministic bpf dynamics that are chaotic on a finite union of intervals. Fig. (3b) does not contain any randomness, while the maximum size of the random perturbation is $\epsilon = 0.001$ in Fig. (3c) and $\epsilon = 0.00804199$ in Fig. (3d).

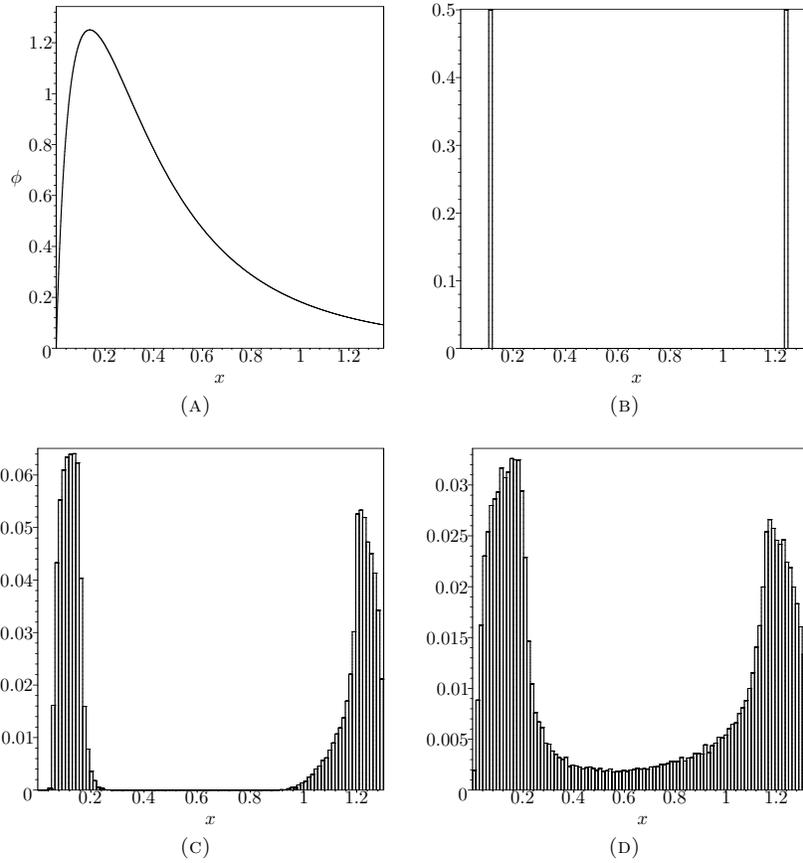


FIGURE 4. The bpf ϕ and the approximate densities associated with its stationary measure for the overlapping generations model of Section 3.1 and values of parameters $l_1^* = 3.51$, $l_2^* = 0.55$, $\alpha_1 = 0.41$ and $\alpha_2 = 5.0$ corresponding to deterministic bpf dynamics exhibiting an attracting cycle. Fig. (4b) does not contain any randomness, while the maximum size of the random perturbation is $\epsilon = 0.05$ in Fig. (4c) and $\epsilon = 0.09160628$ in Fig. (4d).

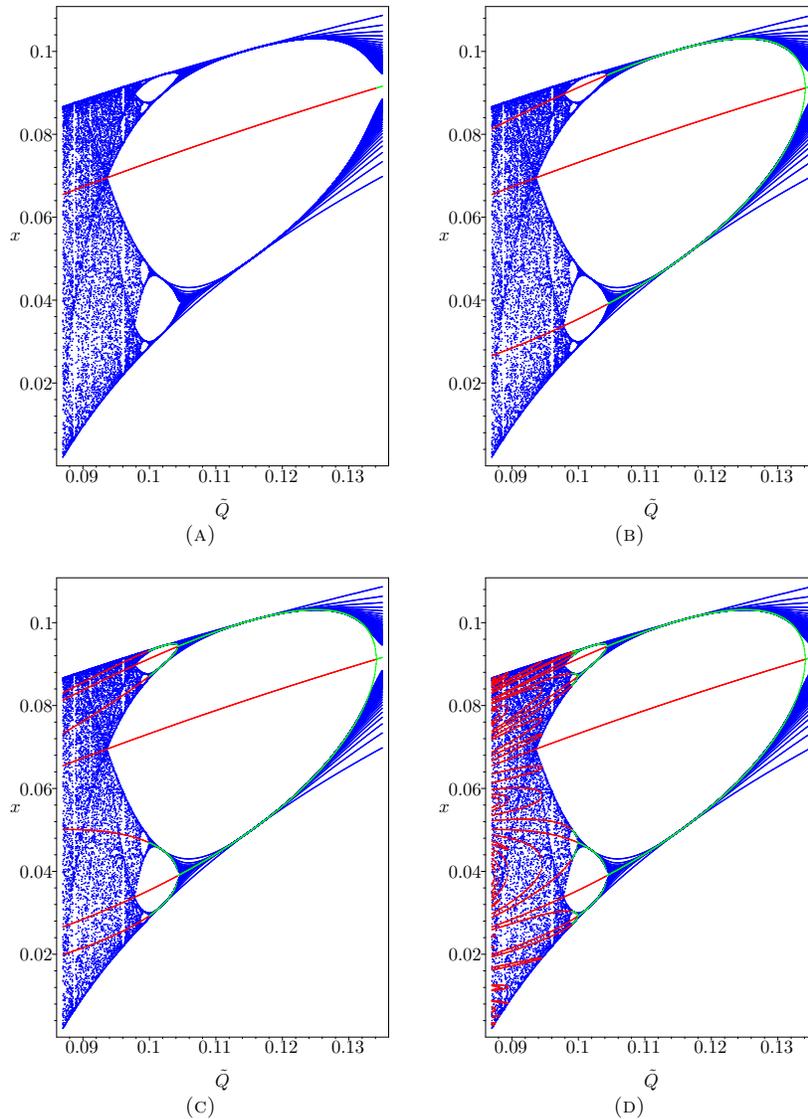


FIGURE 5. Plot in the (\tilde{Q}, x) plane of the first 100 iterates of the critical point (in blue) and periodic points (up to period 8) of the bpf map ϕ of Section 3.2 for the constant relative risk aversion utility functions (3.21). We plot the stable periodic points in green and the unstable ones in red. Figure 5a contains only the fixed point of ϕ , Figure 5b contains the fixed point and the period 2 orbit, Figure 5c contains periodic points of periods 1, 2 and 4, and finally, Figure 5d contains all periodic points whose period divides 8. The remaining parameters are fixed and equal to $\alpha_1 = 0.5$, $\alpha_2 = 2$, $\omega_1 = 2$, $\omega_2 = 0.39$ and $\hat{Q} = 0.1$.

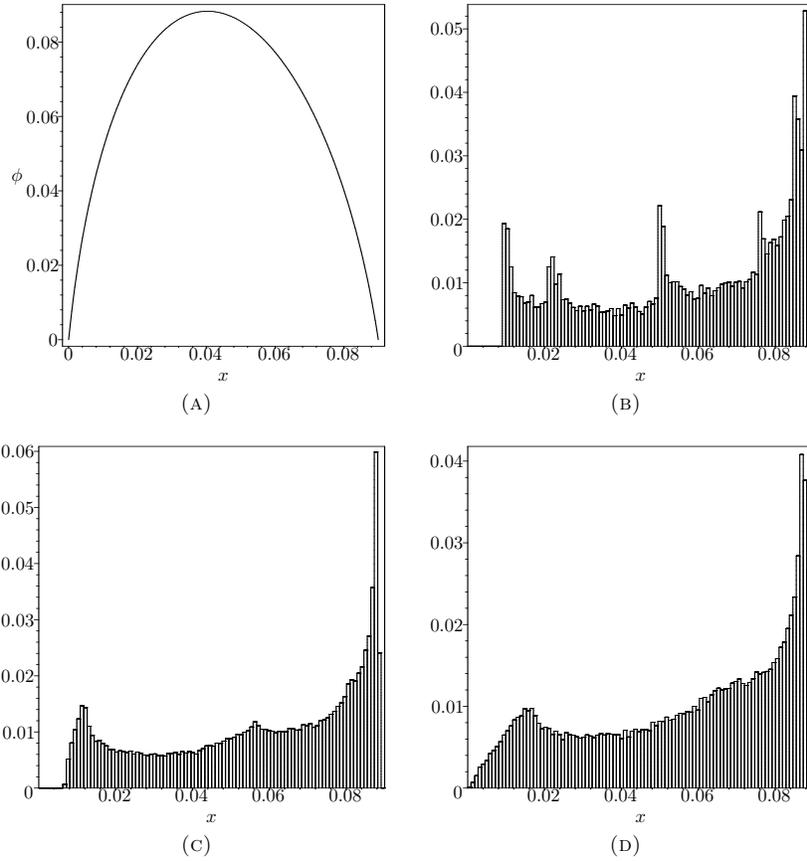


FIGURE 6. The bpf ϕ and the approximate densities associated with its stationary measure for the market game example of Section 3.2 with the constant relative risk aversion utility functions (3.21) and values of parameters $\alpha_1 = 0.5$, $\alpha_2 = 2$, $\omega_1 = 2$, $\omega_2 = 0.39$, $\hat{Q} = 0.1$ and $\bar{Q} = 0.09$ corresponding to deterministic bpf dynamics that are chaotic on a finite union of intervals. Fig. (6b) does not contain any randomness, while the maximum size of the random perturbation is $\epsilon = 0.0005$ in Fig. (6c) and $\epsilon = 0.00148461345$ in Fig. (6d).

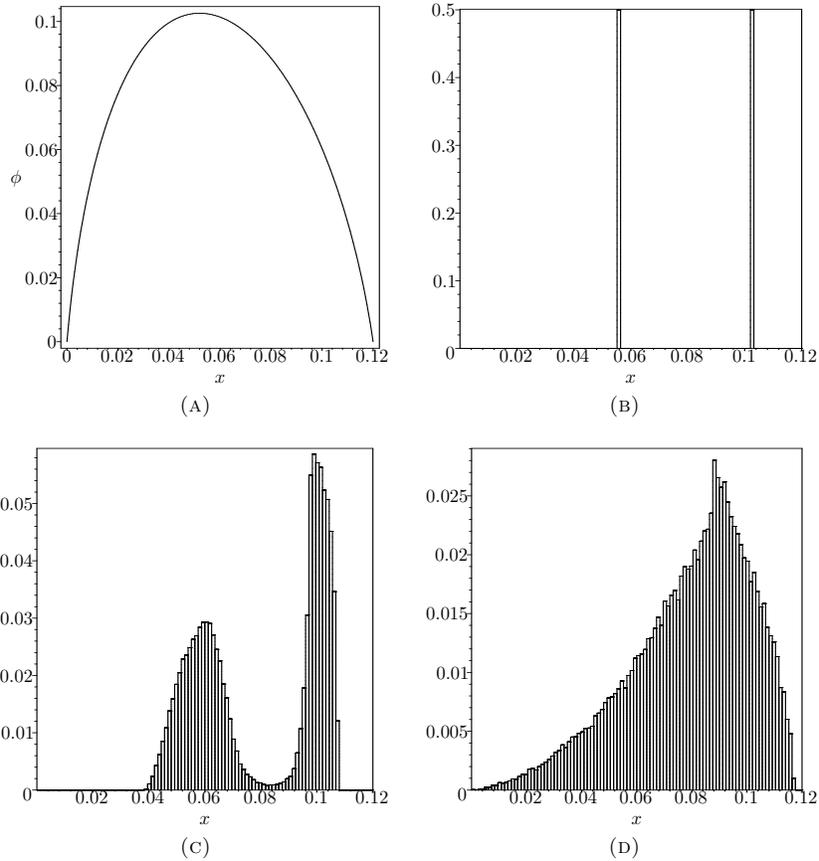


FIGURE 7. The bpf ϕ and the approximate densities associated with its stationary measure for the market game example of Section 3.2 with the constant relative risk aversion utility functions (3.21) and values of parameters $\alpha_1 = 0.5$, $\alpha_2 = 2$, $\omega_1 = 2$, $\omega_2 = 0.39$, $\hat{Q} = 0.1$ and $\bar{Q} = 0.12$ corresponding to deterministic bpf dynamics exhibiting an attracting cycle. Fig. (7b) does not contain any randomness, while the maximum size of the random perturbation is $\epsilon = 0.005$ in Fig. (7c) and $\epsilon = 0.014360401$ in Fig. (7d).

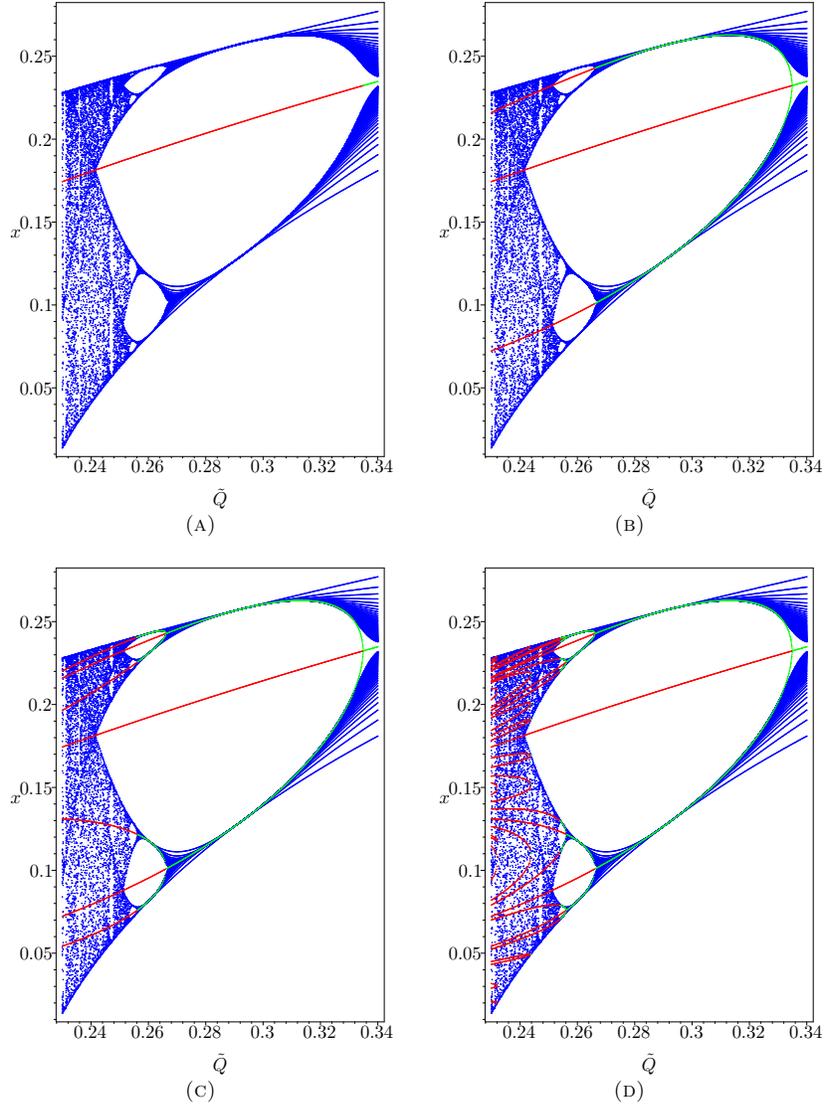


FIGURE 8. Plot in the (\tilde{Q}, x) plane of the first 100 iterates of the critical point (in blue) and periodic points (up to period 8) of the bpf map ϕ of Section 3.2 for the logarithmic utility function (3.22). We plot the stable periodic points in green and the unstable ones in red. Figure 8a contains only the fixed point of ϕ , Figure 8b contains the fixed point and the period 2 orbit, Figure 8c contains periodic points of periods 1, 2 and 4, and finally, Figure 8d contains all periodic points whose period divides 8. The remaining parameters are fixed and equal to $\omega_1 = 5.4$, $\omega_2 = 0.5$ and $\hat{Q} = 0.2$.

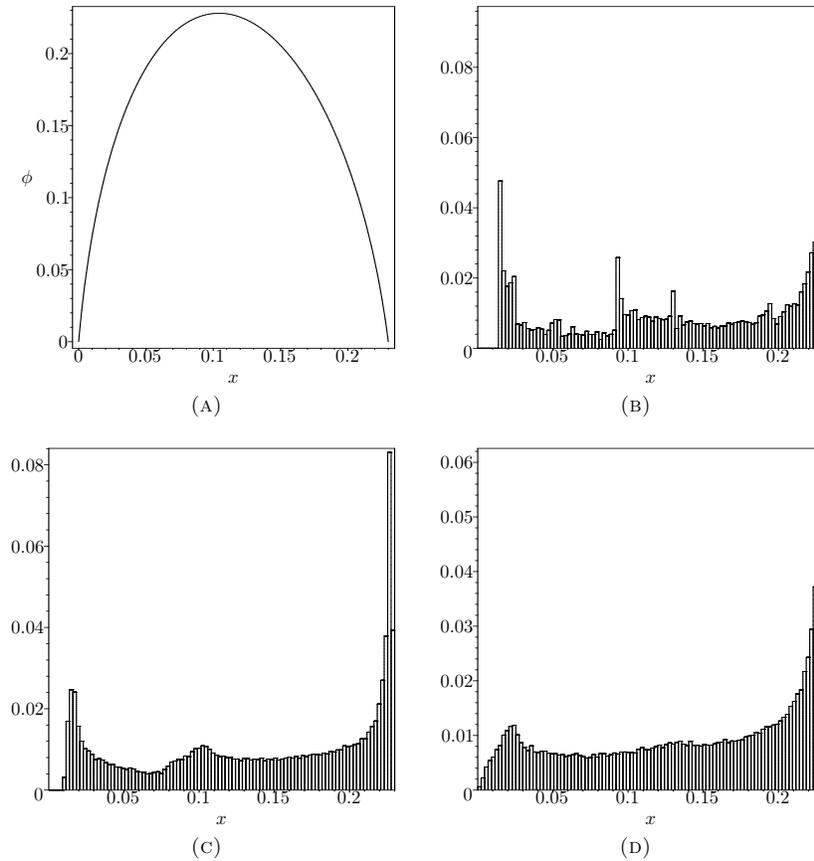


FIGURE 9. The bpf ϕ and the approximate densities associated with its stationary measure for the market game example of Section 3.2 with the logarithmic utility function (3.22) and values of parameters $\omega_1 = 5.4$, $\omega_2 = 0.5$, $\hat{Q} = 0.2$ and $\tilde{Q} = 0.23$ corresponding to deterministic bpf dynamics that are chaotic on a finite union of intervals. Fig. (9b) does not contain any randomness, while the maximum size of the random perturbation is $\epsilon = 0.0005$ in Fig. (9c) and $\epsilon = 0.0017744586$ in Fig. (9d).

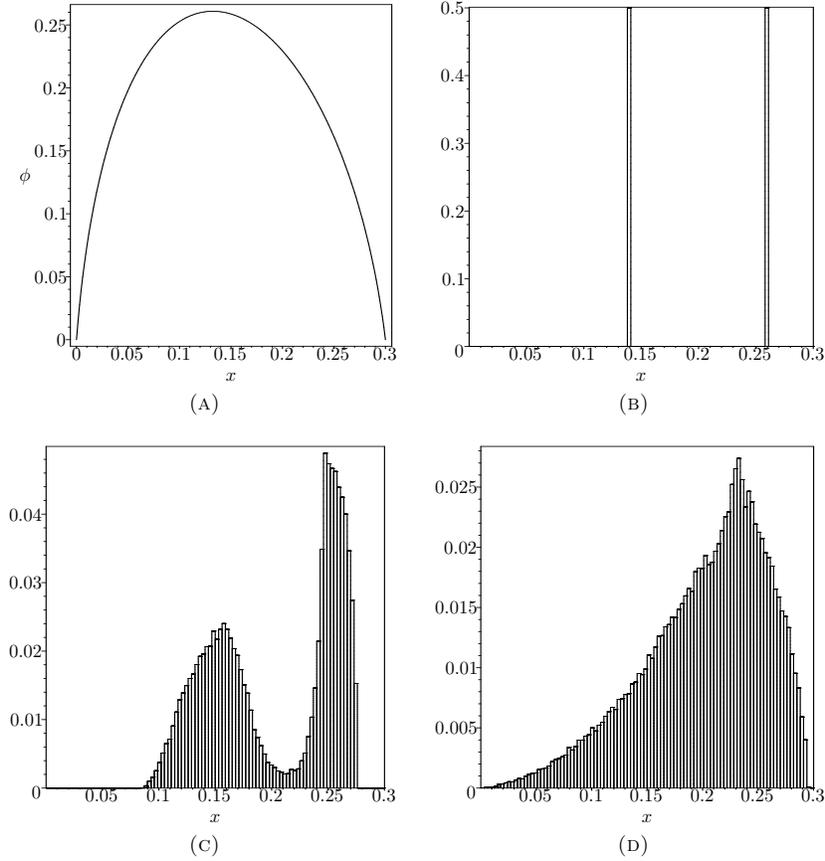


FIGURE 10. The bpf ϕ and the approximate densities associated with its stationary measure for the market game example of Section 3.2 with the logarithmic utility function (3.22) and values of parameters $\omega_1 = 5.4$, $\omega_2 = 0.5$, $\hat{Q} = 0.2$ and $\tilde{Q} = 0.30$ corresponding to deterministic bpf dynamics that are chaotic on a finite union of intervals. Fig. (10b) does not contain any randomness, while the maximum size of the random perturbation is $\epsilon = 0.015$ in Fig. (10c) and $\epsilon = 0.0332824589$ in Fig. (10d).