Empirical Evaluation of Overspecified Asset Pricing Models*

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Abstract

We study the effects of using an excessive number of pricing factors in empirical asset pricing, which may give rise to misleading statistical inferences. Unlike previous studies, which focus on the properties of the usual estimators and tests, we characterize the linear subspace of prices of risk compatible with the pricing restrictions of the model. We also propose tests to detect problematic cases, such as trivial SDFs unrelated to the cross-section of returns. Finally, we apply our methods to some of the most popular asset pricing models.

Keywords: CU-GMM, Factor pricing models, Underidentification tests, Set Estimation, Stochastic discount factor.

JEL: G12, G15, C12, C13.

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1 Introduction

The most popular empirically oriented asset pricing models effectively assume the existence of a common stochastic discount factor (SDF) that is linear in some risk factors, which discounts uncertain payoffs differently across different states of the world. Those factors can be either the excess returns on some traded securities, non-traded economy wide sources of uncertainty related to macroeconomic variables, or a combination of the two. The empirical success of such models at explaining the so called CAPM anomalies was initially limited, but researchers have progressively entertained a broader and broader set of factors, which has resulted in several success claims. Harvey, Liu and Zhu (2014) contains a comprehensive and up to date list of references, cataloguing 315(!) different factors.

However, several authors have warned that some of those factors, or more generally linear combinations of those factors, could be uncorrelated with the vector of excess returns that they are meant to price, which would render them redundant (see Burnside (2014), Gospodinov, Khan and Robotti (2014) and the references therein). Further, those papers forcefully argue that such redundancies can lead to misleading econometric conclusions.

In this context, the purpose of our paper is to study the estimation of risk prices and the testing of the cross-sectional restrictions imposed by overspecified linear factor pricing models in which the candidate SDF depends on too many factors. Our point of departure from the existing literature is that we do not focus exclusively on the properties of the usual estimators and tests. Instead, we use the econometric framework in Arellano, Hansen and Sentana (2012).[1] Specifically, we suppose that under the null hypothesis we may identify a linear subspace of risk prices compatible with the cross-sectional asset pricing restrictions. This set, which can be easily parameterized and efficiently estimated using a standard GMM approach, is of direct interest because it isolates the dimension along which identification of the original linear factor pricing model is problematic. The familiar J test from the work of Sargan (1958) and Hansen (1982) for overidentification of the augmented model now becomes a test for "underidentification" of the original model. If we can identify a linear subspace of risk prices without statistical rejection, then the original asset pricing model is not well identified and we refer to this phenomenon as underidentification. In contrast, a statistical rejection provides evidence that the prices of risk in the original model are indeed point identified, unless of course the familiar J test continues to reject its over-identifying restrictions.

We also follow Peñaranda and Sentana (2014) in using single-step procedures, such as the continuously updated GMM estimator (CU-GMM) of Hansen, Heaton and Yaron (1996), to ob-

[1] In this sense, our paper can be regarded as a substantial extension of Manresa (2009).

tain numerically identical test statistics and risk price estimates for SDF and regression methods, with uncentred or centred moments and symmetric or asymmetric normalizations. In addition, we propose simple tests that can diagnose economically unattractive but empirically relevant cases in which the expected value of the SDF is zero. For simplicity of exposition, we focus on excess returns, but we could easily extend our analysis to cover gross returns too, as in Peñaranda and Sentana (2014).

In our empirical applications we investigate the potential overspecification of several popular three-factor models with the cross-section of size and book-to-market sorted portfolios. We find that most of them are overspecified but the nature of the overspecification differs across models. We begin by using quarterly data to study the extensions of the Consumption CAPM by Yogo (2006) and Lettau and Ludvigson (2001). In this regard, we find that the admissible SDFs of the Yogo (2006) model lay on a two-dimensional subspace, which can be spanned by single-factor SDFs constructed from the two consumption factors. Nevertheless, we cannot reject the null hypothesis of zero SDF means, which effectively reflects lack of correlation between the consumption factors and the vector of excess returns. In turn, the admissible SDFs of the Lettau and Ludvigson (2001) model lay on a three-dimensional subspace, which means that any of its three factors can individually explain the cross-section of risk premia. In this case, though, we reject that the means of the SDFs in the admissible subspace are all zero.

We then use monthly data to study the extensions to the CAPM proposed by Fama and French (1993) and Jagannathan and Wang (1996). We find that the Jagannathan and Wang (1996) model is not identified because the admissible SDFs lay on a two-dimensional subspace. This subspace is spanned by two single-SDF factors constructed from the labor income and default spread factors, respectively. In contrast, we reject that there is any admissible SDF for the Fama and French (1993) model.

The rest of the paper is organized as follows. Section 2 provides the econometric framework for the empirical evaluation of overspecified asset pricing models. We then study empirically several asset pricing models in section 3. We will report the results of the simulation evidence in section 4. Finally, we summarize our conclusions and discuss some avenues for further research in section 5. A detailed description of the possible cases with models of one, two, and three factors are relegated to appendix A, while appendix B contains the Monte Carlo design.

2 Overspecified Asset Pricing Models

2.1 Stochastic discount factors and moment conditions

Let $\mathbf{r}$ be an $n \times 1$ vector of excess returns, whose means $E(\mathbf{r})$ we assume are not all equal to zero. Standard arguments such as lack of arbitrage opportunities or the first order conditions of a representative investor imply that

$$E(m\mathbf{r}) = \mathbf{0}$$

for some random variable m called SDF, which discounts uncertain payoffs in such a way that their expected discounted value equals their cost.

The standard approach in empirical finance is to model the SDF as an affine transformation of some $k < n$ observable risk factors $\mathbf{f}$, even though this ignores that m must be positive with probability 1 to avoid arbitrage opportunities, which would require non-linear specifications for m (see Hansen and Jagannathan (1991)). In particular, researchers typically express the pricing equation as

$$E\left[\left(a + \mathbf{b}'\mathbf{f}\right)\mathbf{r}\right] = \mathbf{0} \tag{1}$$

for some real numbers $(a, \mathbf{b})$, which we can refer to as the intercept and slopes of the affine SDF $a + \mathbf{b}'\mathbf{f}$.

When there is a solution different from the trivial one $(a, \mathbf{b}) = (0, \mathbf{0})$, we can at best identify directions in $(a, \mathbf{b})$ space, which leaves both the scale and sign of the SDF undetermined, unless we add an asset whose price is different from 0.[2] As forcefully argued by Hillier (1990) for single equation IV models, this suggests that we should concentrate our efforts in estimating the identified direction. However, empirical researchers often prefer to estimate points rather than directions, and for that reason they typically focus on some asymmetric scale normalization, such as $(1, \mathbf{b}/a)$. In this regard, note that $\boldsymbol{\delta} = -\mathbf{b}/a$ can be interpreted as prices of risk since we may rewrite (1) as $E(\mathbf{r}) = E\left(\mathbf{r}\mathbf{f}'\right)\boldsymbol{\delta}$.

Alternatively, we can express the pricing conditions (1) in terms of central moments. Specifically, we can add and subtract $\mathbf{b}'\boldsymbol{\mu}$ from $a + \mathbf{b}'\mathbf{f}$, define $c = a + \mathbf{b}'\boldsymbol{\mu}$ as the expected value of the affine SDF and re-write the pricing conditions as

$$E\left\{\begin{array}{c}\left[c + \mathbf{b}'\left(\mathbf{f} - \boldsymbol{\mu}\right)\right]\mathbf{r} \\ \mathbf{f} - \boldsymbol{\mu}\end{array}\right\} = \mathbf{0}. \tag{2}$$

The unknown parameters become $(c, \mathbf{b}, \boldsymbol{\mu})$ instead of $(a, \mathbf{b})$, but we have added k extra moments to estimate $\boldsymbol{\mu}$. Once again, we can only identify directions in $(c, \mathbf{b})$ space from (2) under the

[2] Peñaranda and Sentana (2014) show that the CU-GMM criterion function is numerically invariant to the addition of such an asset. For that reason, we will focus on excess returns in this paper.

null hypothesis of existence of some nontrivial solution to (2). In this regard, empirical work usually focuses on $(1, \mathbf{b}/c)$, where $\boldsymbol{\tau} = -\mathbf{b}/c$ can also be interpreted as prices of risk because (2) implies that $E(\mathbf{r}) = cov(\mathbf{r}, \mathbf{f})\,\boldsymbol{\tau}$.

We refer to those two variants as the uncentred and centred SDF versions since they rely on either $E(\mathbf{r}f)$ or $Cov(\mathbf{r}, f)$ in explaining the cross-section of risk premia. Peñaranda and Sentana (2014) show that the estimators and tests of both variants are numerically equivalent if one uses CU-GMM. In what follows, we will work with the uncentred SDF version, which does not require the estimation of $\boldsymbol{\mu}$, a non-trivial computational advantage.[3] In fact, we can still rely on the uncentred SDF moment conditions (1) to estimate c, which has the interpretation of the SDF mean, as long as we add the moment condition

$$E\left[a + \mathbf{b}'\mathbf{f} - c\right] = 0, \tag{3}$$

which is exactly identified for c given $(a, \mathbf{b})$. A non-trivial advantage of this approach is that (1) and (3) are linear in $(a, \mathbf{b}, c)$ while (2) is not linear in $(c, \mathbf{b}, \boldsymbol{\mu})$.

Our empirical applications will consider models where the elements of $\mathbf{f}$ are either nontraded or they are portfolios of $\mathbf{r}$. In those cases, the pricing conditions (1) and (3) contain all the relevant information to estimate and test the asset pricing model. Nevertheless, it would be very easy to extend our analysis to traded factors whose excess returns do not belong to the linear span of $\mathbf{r}$. In that case, we should add to (1) or (2) moment conditions such as

$$E\left[\left(a + \mathbf{b}'\mathbf{f}\right)\mathbf{f}\right] = \mathbf{0}$$

to complete the asset pricing information that we should consider.

2.2 Admissible SDFs sets

The pricing conditions (1) can be expressed in matrix notation as

$$\begin{pmatrix} E(\mathbf{r}) & E(\mathbf{r}\mathbf{f}') \end{pmatrix} \begin{pmatrix} a \\ \mathbf{b} \end{pmatrix} = \mathbf{M}\boldsymbol{\theta} = \mathbf{0}, \tag{4}$$

where $\mathbf{M}$ is an $n \times (k+1)$ matrix of data and $\boldsymbol{\theta}$ a $(k+1) \times 1$ parameter vector.

The highest possible rank of $\mathbf{M}$ is its number of columns $k+1$ since we have explicitly assumed that $k < n$. In that case, though, the asset pricing model cannot hold because the only value of $\boldsymbol{\theta}$ that satisfies (4) will be the trivial solution $a = 0$, $\mathbf{b} = \mathbf{0}$. On the other hand, if the rank of $\mathbf{M}$ is k then there is a one-dimensional subspace of $\boldsymbol{\theta}$'s that satisfy (4), in which case

[3] There is an alternative approach to test asset pricing models, which relies on the regression of $\mathbf{r}$ onto a constant and $\mathbf{f}$. The regression approach requires a higher number of parameters to estimate from a higher number of moments. Nevertheless, the results in Peñaranda and Sentana (2014) show that it provides numerically equivalent tests and prices of risk estimates.

the solution $\boldsymbol{\theta}$ is unique up to scale, as we have explained in the previous section. Therefore, $rank(\mathbf{M}) = k$ coincides with the usual identification condition required for standard GMM inference (see e.g. Hansen (1982) and Newey and McFadden (1994)).

Recently, though, Kan and Zhang (1999) and Burnside (2014) among others have forcefully argued that some empirical asset pricing models effectively rely on factors for which the matrix $Cov(\mathbf{r}, \mathbf{f})$ does not have full column rank. The best known example is a useless factor, which would yield a zero column in the matrix $Cov(\mathbf{r}, \mathbf{f})$. Alternatively, we may have included two proxies of a relevant pricing factor, so that their difference will be uncorrelated to the vector of excess returns. For instance, in a two-factor model with $Cov(\mathbf{r}, f_2) = 0$, the matrix $\mathbf{M}$ in (4) becomes

$$\begin{pmatrix} E(\mathbf{r}) & E(\mathbf{r}f_1) & E(\mathbf{r}f_2) \end{pmatrix} = \begin{pmatrix} E(\mathbf{r}) & E(\mathbf{r}f_1) & E(\mathbf{r})E(f_2) \end{pmatrix},$$

and hence trivially we find a vector $\boldsymbol{\theta}$ that satisfies (4). More formally, we find the one-dimensional subspace $(a, b_1, b_2) = b_2(-E(f_2), 0, 1)$ for any b_2. The matrix $\mathbf{M}$ will have rank 1 or 2 depending on $E(\mathbf{r}) = E(\mathbf{r}f_1)\delta_1$ or not, respectively. If the rank is 2, then we have an example where the rank of $Cov(\mathbf{r}, \mathbf{f})$ is $k-1$ but the rank of $\mathbf{M}$ is k. The model parameters remain econometric identified, so we can still use standard GMM inference. [4] But we lose identification when the rank of $\mathbf{M}$ is 1.

More generally, there will be rank failures in $Cov(\mathbf{r}, \mathbf{f})$ when we can find a valid asset pricing model with fewer factors. For example, assume that the true model is a (linearized) version of the CCPAM but the factor mimicking portfolio for consumption is perfectly correlated with the market portfolio. In this case both the CAPM and the CCAPM will hold in the sense that the market excess return and consumption growth can price a cross-section of excess returns in a single-factor model, say $E(\mathbf{r}) = E(\mathbf{r}f_1)\delta_1$ and $E(\mathbf{r}) = E(\mathbf{r}f_2)\delta_2$. Then an SDF that is linear in both factors, such as the one motivated by the (linearized) Epstein-Zin model, must have a matrix $Cov(\mathbf{r}, \mathbf{f})$ whose rank is one.[5] The matrix $\mathbf{M}$ in (4) becomes

$$\begin{pmatrix} E(\mathbf{r}) & E(\mathbf{r}f_1) & E(\mathbf{r}f_2) \end{pmatrix} = E(\mathbf{r})\begin{pmatrix} 1 & 1/\delta_1 & 1/\delta_2 \end{pmatrix},$$

and hence we find a two-dimensional subspace of vectors $\boldsymbol{\theta}$ that satisfy (4).

Therefore, it is of the upmost importance to develop statistical inference tools that can successfully deal with situations in which $rank(\mathbf{M}) < k$. Following Arellano, Hansen and Sentana

[4] If the rank is 2, then we have an example where the rank of $Cov(\mathbf{r}, \mathbf{f})$ is $k-1$ but the rank of $\mathbf{M}$ is k. The model parameters remain econometric identified, so we can still use standard GMM inference. Nevertheless, some asymmetric normalizations may be incompatible with these configurations (see section 4.4 in Peñaranda and Sentana (2014) for further details in the case of a single pricing factor).

[5] Note that in this case we have three equivalent mean-variance frontiers: The frontier constructed with the cross-section of excess returns, the frontier spanned by the market excess return and the frontier spanned by the consumption mimicking portfolio.

(2012), we begin by specifying the dimension of the subspace of solutions to (4), which we will denote as d, so that the rank of $\mathbf{M}$ will be $(k+1)-d$. Given that we maintain the hypothesis that $E(\mathbf{r}) \neq \mathbf{0},$ we could in principle consider any positive integer d up to a maximum value of k, or equivalently, ranks of $\mathbf{M}$ as low as 1.

As we have mentioned before, when $d=1$ we can rely on standard GMM to estimate a unique $\boldsymbol{\theta}$ (up to normalization) and use its associated J test to test the validity of the asset pricing restrictions. However, when $d \geq 2$, then we will have a multidimensional subspace of admissible SDFs even after fixing their scale. Nevertheless, we can efficiently estimate a basis of that subspace by replicating d times the moment conditions (4) as follows:

$$\left.\begin{array}{c} \mathbf{M}\boldsymbol{\theta}_1 = \mathbf{0}, \\ \mathbf{M}\boldsymbol{\theta}_2 = \mathbf{0}, \\ \vdots \\ \mathbf{M}\boldsymbol{\theta}_d = \mathbf{0}, \end{array}\right\} \tag{5}$$

and imposing enough normalizations on $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, ..., \boldsymbol{\theta}_d)$ to ensure identification of the basis. In this setting, the J test of the extended moment conditions provides a test of underidentification. As for the $c's$, which give the expected values of the basis SDF's, (5) can be supplemented with the moment conditions

$$\left.\begin{array}{c} \begin{pmatrix} 1 & E(\mathbf{f})' \end{pmatrix} \boldsymbol{\theta}_1 - c_1 = 0, \\ \begin{pmatrix} 1 & E(\mathbf{f})' \end{pmatrix} \boldsymbol{\theta}_2 - c_2 = 0, \\ \vdots \\ \begin{pmatrix} 1 & E(\mathbf{f})' \end{pmatrix} \boldsymbol{\theta}_d - c_d = 0, \end{array}\right\} \tag{6}$$

which are exactly identified for given values of $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, ..., \boldsymbol{\theta}_d)$.

As for the normalization, the usual approach when there is identification is to fix one entry of $\boldsymbol{\theta}$ to 1, as we saw in the previous section with $a + \mathbf{b}'\mathbf{f} = 1 - \boldsymbol{\delta}'\mathbf{f}$. For $d > 1$, we could make a $d \times d$ block of (a permutation of) the matrix $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, ..., \boldsymbol{\theta}_d)$ equal to the identity matrix of order d, as in Appendix A. We will follow an alternative approach in the presentation of our empirical results. We will keep the normalization $1 - \boldsymbol{\delta}'\mathbf{f}$ and we will impose zero entries in $\boldsymbol{\delta}$ to achieve identification of an SDF basis. The advantage of single-step methods such as CU-GMM is that inferences will be numerically invariant to the chosen normalization.

2.3 Testing restrictions on admissible SDF sets

As we have just seen, our inference framework allows us to estimate the set of SDFs that is compatible with the pricing conditions (1). But we can also use it to test if this subspace satisfies some relevant restrictions. For example, we may want to test if some factor, say f_1, does not appear in any admissible SDF. This test would be associated to the corresponding entry of $\mathbf{b}$ being zero in all the vectors $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, ..., \boldsymbol{\theta}_d)$. In some cases, though, such a test will not be interesting because it effectively amounts to $E(\mathbf{r}) = \mathbf{0}$. For instance, when $k = 2$ and $d = 2$ the joint null hypothesis that f_2 is not priced and the asset pricing model is true can only hold if $E(\mathbf{r}) = \mathbf{0}$ and the covariance of f_1 with $\mathbf{r}$ is zero. The situation is similar for $k = 3$ and $d = 3$. In contrast, such a degenerate implication would no longer be true if $d = 2$ but $k = 3$ instead.

An important test that we should always add to our analysis is that of zero means for all admissible SDFs, which are associated to the parameters $(c_1, c_2, ..., c_d)$. The reason is that if all these means are zero, then there will be no element in the admissible SDF set that explains the cross-section of expected returns from an meaningful economic perspective. More specifically, when $c_1 = c_2 = ... = c_d = 0$, all the vectors $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, ..., \boldsymbol{\theta}_d)$ compatible with the moment restrictions (5) are simply exploiting rank failures in $Cov(\mathbf{r}, \mathbf{f})$, and hence they will be unrelated to the vector $E(\mathbf{r})$.

To clarify this point, note that for a particular SDF $E(\mathbf{r})\,a + E\left(\mathbf{r}\mathbf{f}'\right)\mathbf{b} = E(\mathbf{r})\,c + Cov(\mathbf{r}, \mathbf{f})\,\mathbf{b}$, and hence $c = 0$ and the pricing errors are zero if and only if $Cov(\mathbf{r}, \mathbf{f})$ does not have full column rank. As we saw in the previous section for a two-factor example, when the two factors are such that $E(\mathbf{r}) = E(\mathbf{r}f_1)\,\delta_1$ and $E(\mathbf{r}) = E(\mathbf{r}f_2)\,\delta_2$, the matrix $\mathbf{M}$ in (4) has rank 1

$$\left(\begin{array}{ccc} E(\mathbf{r}) & E(\mathbf{r}f_1) & E(\mathbf{r}f_2) \end{array} \right) = E(\mathbf{r}) \left(\begin{array}{ccc} 1 & 1/\delta_1 & 1/\delta_2 \end{array} \right),$$

and hence we find a two-dimensional subspace of vectors $\boldsymbol{\theta}$ that satisfy (4). In this context, we still would like to distinguish an underidentified model with $1/\delta_1 = E(f_1)$ and $1/\delta_2 = E(f_2)$ from another underidentified model where $1/\delta_1 \neq E(f_1)$ and/or $1/\delta_2 \neq E(f_2)$. In the former model, all SDFs have zero mean and simply exploit a zero correlation with $\mathbf{r}$, the vector $E(\mathbf{r})$ does not belong to the span of $Cov(\mathbf{r}, \mathbf{f}) = \mathbf{0}$. In the latter model there are some SDFs that price $\mathbf{r}$ in a nontrivial way because $E(\mathbf{r})$ belongs to the span of $Cov(\mathbf{r}, \mathbf{f}) \neq \mathbf{0}$.

We can test any of the aforementioned constraints on the set of admissible SDFs by means of distance metric tests, which compare the J statistics computed with and without the constraints imposed on the estimation of the basis.

2.4 Comparison to the previous literature

Burnside (2014) and Gospodinov, Kan and Robotti (2014) apply the rank tests proposed by Cragg and Donald (1997) and Kleibergen and Paap (2006) to $Cov(\mathbf{r}, \mathbf{f})$ and $E\left(\mathbf{r}\mathbf{f}'\right)$, which are the matrices of centred and uncentred second moments between returns and pricing factors, respectively, in order to detect the existence of trivial SDFs that exploit the rank failure in those matrices to account for the cross-section of risk premia in $E(\mathbf{r})$. However, those tests cannot conclude if there are also nontrivial SDFs that can properly explain $E(\mathbf{r})$. In contrast, our econometric framework allows us to estimate a basis of the linear subspace of admissible SDFs, which we can then use to test whether or not all of them are trivial.

The other main difference with Burnside (2014) and Gospodinov, Kan and Robotti (2014) is that they focus their analysis on the implications of those rank failures for standard GMM procedures that assume point identification. In contrast, we develop a GMM framework that works under set identification.

3 Empirical Application

3.1 Set up

As we mentioned in the introduction, in this section we look at the empirical performance of some popular three-factor models. Therefore, the candidate SDFs can systematically be expressed as

$$m = 1 - \delta_1 f_1 - \delta_2 f_2 - \delta_3 f_3.$$

In order to test for one, two and three-dimensional linear subsets of valid SDFs, we will use the moment conditions (5) with $d = 1$, 2 and 3, respectively. In all cases, we augment those moment conditions with the exactly identified moment conditions (6) to estimate the associated SDF means.

We will estimate the different models using single-step GMM methods. The drawback is that we have to rely on numerical optimization to maximize the non-linear CU-GMM criterion even when the influence function are linear in the model parameters. For that reason, we compute the CU-GMM criterion by means of the auxiliary OLS regressions described in appendix B of Peñaranda and Sentana (2012). Given that single-step methods are invariant to different parametrizations of the SDF, we will use the uncentred version because it is the most parsimonious.

CU-GMM estimates are numerically invariant to normalizations, but we still must choose one to present the empirical results. We will chose the one that it is arguably easiest to interpret. In the case of $d = 2$, in particular, we will present the results for the simple normalization $(\delta_1, \delta_2, 0)$ and $(\delta_1, 0, \delta_3)$ in order to point identify a basis of the space of admissible SDFs. This normalization ensures linear independence of the estimated SDFs by imposing exclusion restrictions on the basis SDFs. Given that the models we study are extensions of models that only use the first factor, it seems natural to keep f_1 in both basis vectors. On the other hand, we will present the results for the simple normalization $(\delta_1, 0, 0)$, $(0, \delta_2, 0)$ and $(0, 0, \delta_3)$ in the case of $d = 3$. This normalization imposes that each factor can separately explain risk premia.

3.2 Quarterly data

In this section, we study two alternative three-factor extensions of the CCAPM over the sample period 1949 to 2012 using pricing factors borrowed from Burnside (2014). We evaluate both models against the Fama-French cross-section of size and book-to-market sorted portfolios, which can be obtained from Ken French's Data Library (see his web page, as well as Fama and French (1993) for further details). Following Burnside (2014), we compound monthly returns into quarterly returns and divide nominal excess returns by one plus the inflation rate. We present our empirical results for a cross-section of 6 portfolios to keep the number of moment conditions in a reasonable proportion to the sample size for the cases $d = 2$ and $d = 3$. Nevertheless, we have also repeated the computations for the Fama-French cross-section of 25 portfolios, obtaining qualitatively similar results.

As is well known, Yogo (2006) model extends the CCAPM by assuming recursive preferences over a consumption bundle of nondurable and durable goods. Therefore, the linearized version of his model implies that the SDF is linear in three factors: the market return, and the consumption growth of nondurables and durables.[6] In practice, we identify the log-growth rate of US real per capita consumption of nondurables and services and durables with f_2 and f_3, respectively. Finally, we associate the return on wealth, which is proxied by the real return on the value-weighted U.S. stock market, with f_1.

Table 1 shows the results of our overspecification analysis of the Yogo (2006) model. This table displays estimates of the SDF parameters with standard errors in square brackets [], J tests with degrees of freedom in braces {} and p-values in parenthesis (). The first, second and third blocks of columns report the results for sets of SDFs of dimension 1, 2 and 3, respectively. We complement the J tests with significance tests for the SDF parameters. In particular, we report the p-value of the distance metric test of the null hypothesis of a zero parameter in parenthesis to the right of the point estimates.

(Table 1)

The results for the one-dimensional set agree with the results in Yogo (2006), who finds that (i) the J test of two-step GMM does not reject his model for our sorted portfolios and (ii) durable consumption provides the only price of risk with a significant t-ratio. In this regard, the usual overidentification test reported in the first column of Table 1 does not reject the null hypothesis that there exists a SDF affine in the three factors that can price the cross-section of securities (p-value=89.2%). However, the validity of the asymptotic distribution of this J test

[6] Note that although the market return is a traded factor, we do not add its pricing condition to (1) because it can effectively be generated as a portfolio of the cross-section of excess returns that we want to price.

crucially depends on the model parameters being point identified. In fact, one should interpret as a warning sign the fact that none of the distance metric tests associated to the individual estimated parameters are statistically significant.

For that reason, we also report the overidentification test for $d = 2$. As explained before, this test assesses whether there is a linear subspace of dimension 2 of admissible SDFs that can price the cross section of risk premia. Since the p-value of this J test is 92%, we take the results as evidence that the original Yogo (2006) model is likely to be overspecified. Importantly, note that the first SDF specification presented in the two-dimensional set results coincides with the linearized version of the Epstein and Zin (1989) model In contrast, the overidentification test for $d = 3$ is rejected, which implies that the Epstein-Zin model is identified.

To infer the nature of the overspecification, we provide distance metric tests that each of the parameters are zero. The distance metric tests of zero prices of risk show that the market is not a significant pricing factor, while the consumption factors are. Hence, we find evidence of the two consumption factors being able to price this Fama-French cross-section by means of separate single-factor consumption growth models. However, the distance metric test of the entire set of admissible SDFs having zero means does not reject. This suggests that the seeming pricing ability of these two factors simply exploits their lack of correlation with $\mathbf{r}$. In other words, the vector of risk premia does not seem to lie in the span of the covariance matrix of the excess returns and the factors.

Our results are in line with Burnside (2014), who finds that the matrix $Cov(\mathbf{r}, \mathbf{f})$ associated to the Yogo (2006) model seems to have rank 1 only. His evidence implies that there are SDFs that trivially price the cross-section. Our results confirm that those trivial SDFs seem to be the only admissible SDFs that Yogo's model can generate.

The second SDF specification that we study was proposed by Lettau and Ludvigson (2001). Their model is an alternative extension of the linearized CCAPM, where f_1 represents now the consumption growth of nondurables, which is augmented by the lagged consumption-aggregate wealth ratio (*cay*) and the product between this variable and consumption growth as pricing factors. Table 2 shows the results of our analysis of the potential overspecification of this model[7]

(Table 2)

As in the previous table, the first block of columns shows the results of the standard J test, which does not reject. This finding is compatible with the results in Lettau and Ludvigson

[7]The sample period for this model starts in 1952 due to the previous unavailability of the *cay* variable in Burnside (2014) data. We also standardize *cay* to make it conmensurate with the consumption growth factor.

(2001), who do not reject this model for a similar cross-section on the basis of two different non-optimal versions of the GMM J tests. However, the results for the original model must be interpreted with considerable care because the overidentification test for $d = 2$ does not reject. The main difference with the Yogo (2006) model is that the overidentification test for $d = 3$ does not reject either. This strong lack of identification is probably behind the extremely high estimates and standard errors in the original model.

Therefore, we infer that each of the three factors can price this cross-section as single-factor SDFs.[8] We also report the distance metric test of zero means in the set of admissible SDFs. This null is rejected, which implies that risk premia are spanned by the covariance matrix of the excess returns with the Lettau-Ludvigson factors.

It is also of some interest to compare our results with those in Burnside (2014), who finds that the matrix $Cov(\mathbf{r}, \mathbf{f})$ associated to this model has rank 1 or at most 2. While his evidence implies that there are SDFs that trivially explain the cross-section of risk premia, our results indicate that the Lettau and Ludvigson (2001) model can also generate other non-trivial SDFs.

3.3 Monthly data

In this section, we study two alternative three-factor extensions of the CAPM over the period February 1959 to December 2012 using pricing factors borrowed from Gospodinov, Kan and Robotti (2014). As in the previous section, we also evaluate the models against the Fama-French cross-section of size and book-to-market sorted portfolios. The main difference is that we work with nominal excess returns, as it is customary in the empirical literature at the monthly frequency. Once again, we will present our empirical results for the Fama-French cross-section of 6 portfolios, but we have also repeated the computations for their cross-section of 25 portfolios, obtaining qualitatively similar.

We begin by studying the Jagannathan and Wang (1996) model. This model emphasizes that (i) the wealth portfolio should be used in place of the market portfolio and (ii) the conditional CAPM should be the relevant model instead of its unconditional counterpart. They captured the first point by including labor income growth (a proxy for the return on human capital) as a second factor, and the second point by identifying f_3 with the lagged default spread, which is a state variable related to interest rates. On the other hand, they set f_1 to the market excess return, as in the classic CAPM.

(Table 3)

[8] In the case of $d = 3$, we do not test if one of the prices of risk is zero because, as mentioned in Section 2.3, such a test is not interesting when $k = 3$ and $d = 3$. In this context, that test effectively amounts to $E(\mathbf{r}) = \mathbf{0}$.

As can be seen from the one-dimensional set results reported in Table 3, the usual overidentification tests does not reject the null hypothesis that there exists a SDF affine in the three factors that can price the cross-section of securities (p-value=25.9%). A researcher focusing on a one-dimensional set of admissible SDFs would also conclude that the labor income is insignificant, unlike the default spread. In turn, the market factor has a negative price of risk, which is significant with a distance metric test, even though it has a low t-ratio. If she had added the estimation of the SDF mean in her computations then she might be concerned about the actual pricing ability of these factors because the estimated c does not seem significantly different from 0.

Using a cross-section of portfolios sorted by size and market beta and a non-optimal version of GMM, Jagannathan and Wang (1996) concluded that their model is not rejected. They also found that the market factor has a negative but nonsignificant price of risk. Similarly, Gospodinov, Kan and Robotti (2014) do not reject the asset pricing restrictions on the Fama-French size and value sorted portfolios using a CU version of the J test. But they detect a failure in the rank condition for GMM identification. As we have repeatedly mentioned in previous cases, the validity of the asymptotic distribution of the J test crucially depends on the model parameters being point identified.

Once again, we cannot reject that the admissible SDFs generated by the Jagannathan and Wang (1996) model lie on a two-dimensional subspace. Moreover, it seems that the market factor does not affect this subspace, as was the case for the Yogo (2006) model in Table 1. However, unlike the Yogo model, we reject the null hypothesis that all the SDFs in the admissible subspace have zero mean. However, a SDF constructed from the default spread only does not have a mean significantly different from zero. In the case of the labor income factor, in contrast, the Wald test of a zero mean has a p-value of 3.4%. Nevertheless, the estimated mean is negative, which makes it difficult its interpretation as an economically meaningful SDF. Obviously, we could multiply this SDF by -1, but then we would have a negative price of risk for labor income, which is also difficult to rationalize.

Finally, we study the potential overspecification of the popular Fama and French (1993) model, in which the factors are excess returns on the market, a size factor portfolio and a value factor. As is well known, the second factor, usually denoted by SMB, is long/short in small/large capitalization stocks, while the third factor, usually denoted by HML, is long/short in high/low book-to market ones.

(Table 4)

The model is clearly rejected for the three dimensions that can be considered. In fact, Fama and French (1993) also rejected their original model by looking at the regression intercepts, although they defended their model because it still provides an improvement with respect to the CAPM and its rejection may be due to the high precision of the estimated pricing errors. Similarly, Gospodinov, Kan and Robotti (2014) also reject the Fama and French (1993) model using a CU version of the J test. They also conclude that the rank condition of GMM identification is satisfied. Therefore, it seems that our underidentification tests do not lack of power.

4 Monte Carlo (to be completed)

The design of our Monte Carlo experiment is described in Appendix B.

5 Conclusions

We study the testing of linear factor pricing models and the estimation of risk prices in potentially overspecified contexts in which we can only estimate the set of risk prices compatible with the pricing conditions. We use single-step GMM procedures, such as continuously updated GMM, to obtain identical test statistics and risk price estimates for SDF and regression methods, with uncentred or centred moments and symmetric or asymmetric normalizations. We also develop simple tests that can detect problematic cases such as the presence of trivial SDFs unrelated to the cross-section of returns.

In our empirical applications we investigate the potential overspecification of several popular three-factor models with the cross-section of size and book-to-market sorted portfolios. We find that most of them are overspecified but the nature of the overspecification differs across models. We begin by using quarterly data to study the extensions of the Consumption CAPM by Yogo (2006) and Lettau-Ludvigson (2001). In this regard, we find that the admissible SDFs of the Yogo (2006) model lay on a two-dimensional subspace, which can be spanned by single-factor SDFs constructed from the two consumption factors. Nevertheless, we cannot reject the null hypothesis of zero SDF means, which effectively reflects lack of correlation between the consumption factors and the vector of excess returns. In turn, the admissible SDFs of the Lettau-Ludvigson (2001) model lay on a three-dimensional subspace, which means that any of its three factors can individually explain the cross-section of risk premia. In this case, though, we reject that the means of the SDFs in the admissible subspace are all zero.

We then use monthly data to study the extensions to the CAPM proposed by Fama and French (1993) and Jagannathan and Wang (1996). We find that the Jagannathan and Wang (1996) model is not identified because the admissible SDFs lay on a two-dimensional subspace spanned by two single-SDF factors constructed from the labor income and default spread factors, respectively. In contrast, we reject that there is any admissible SDF for the Fama-French model.

There are some interesting avenues for further research. We have focused our econometric methods on optimal GMM, but there are alternative nonoptimal GMM variants like the Hansen and Jagannathan (1997) distance. We plan to extend this measure to the evaluation of overspecifed models.

References

Almeida, C. and R. Garcia (2012): "Assessing misspecified asset pricing models with empirical likelihood estimators", *Journal of Econometrics* 170, 519–537.

Arellano, M., L.P. Hansen and E. Sentana (2012): "Underidentification?", *Journal of Econometrics* 170, 256–280.

Burnside, C. (2014): "Identification and inference in linear stochastic discount factor models", mimeo, Duke.

Cragg, J.G. and S.G. Donald (1997): "Inferring the rank of a matrix", *Journal of Econometrics* 76, 223–250.

Epstein, L.G. and S.E. Zin (1989): "Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework", *Journal of Political Economy* 57, 937–969.

Fama, E.F. and K.R. French (1993): "Common risk factors in the returns on stock and bonds", *Journal of Financial Economics* 33, 3–56.

Gospodinov, N., R. Kan, and C. Robotti (2014): "Spurious fit in unidentified asset-pricing models", mimeo, Federal Reserve Bank of Atlanta.

Hansen, L.P. (1982): "Large sample properties of generalized method of moments estimators", *Econometrica* 50, 1029–1054.

Hansen, L.P., J. Heaton and A. Yaron (1996): "Finite sample properties of some alternative GMM estimators", *Journal of Business and Economic Statistics* 14, 262–280.

Hansen, L.P. and R. Jagannathan (1991): "Implications of security market data for models of dynamic economies", *Journal of Political Economy* 99, 225–262.

Hansen, L.P. and R. Jagannathan (1997): "Assessing specification errors in stochastic discount factor models", *Journal of Finance* 52, 557–590.

Harvey, C. R., Y. Liu, and H. Zhu (2014): ". . . and the cross-section of expected returns", mimeo, Fuqua Business School, Duke University.

Hillier, G.H. (1990): "On the normalization of structural equations: properties of direct estimators", *Econometrica* 58, 1181–1194.

Householder, A.S. (1964): *The Theory of Matrices in Numerical Analysis*, Blaisdell Publishing Co., London.

Jagannathan, R. and Z. Wang (1996): "The conditional CAPM and the cross-section of expected returns", *Journal of Finance* 51, 3–53.

Kan, R. and C. Robotti (2008): "Specification tests of asset pricing models using excess returns", *Journal of Empirical Finance* 15, 816–38.

Kan, R. and C. Zhang (1999): "GMM tests of stochastic discount factor models with useless factors", *Journal of Financial Economics* 54, 103–127.

Kleibergen, F. and R. Paap (2006): "Generalized reduced rank tests using the singular value decomposition", *Journal of Econometrics* 133, 97–126.

Lettau, M., and S. Ludvigson (2001): "Resurrecting the (C)CAPM: a cross-sectional test when risk premia are time-varying", *Journal of Political Economy* 109, 1238–1287.

Lewellen J., S. Nagel and J. Shanken (2010): "A skeptical appraisal of asset-pricing tests, *Journal of Financial Economics* 96, 175–194.

Manresa, E. (2009): "Application of rank tests to linear factor asset pricing models", unpublished Master thesis, CEMFI.

Newey, W.K. and D.L. McFadden (1994): "Large sample estimation and hypothesis testing", in Engle, R.F. and D.L. McFadden, *Handbook of Econometrics* vol. IV, 2111–2245, Elsevier.

Newey, W.K. and R.J. Smith (2004): "Higher order properties of GMM and generalized empirical likelihood estimators", *Econometrica* 72, 219–255.

Peñaranda, F. and E. Sentana (2012): "Spanning tests in portfolio and stochastic discount factor mean-variance frontiers: a unifying approach", *Journal of Econometrics* 170, 303–324.

Peñaranda, F. and E. Sentana (2014): "A unifying approach to the empirical evaluation of asset pricing models", forthcoming in *Review of Economics and Statistics*.

Sargan, J.D. (1958): "The estimation of economic relationships using instrumental variables", *Econometrica* 26, 393–415.

Yogo, M. (2006): "A consumption-based explanation of expected stock returns", *Journal of Finance* 61, 539–580.

Appendices

A Possible cases with one, two and three factors

Our empirical applications consider models where the elements of $\mathbf{f}$ are either nontraded or they are portfolios of $\mathbf{r}$. In those cases, the pricing conditions (1) or, equivalently, the matrix $\mathbf{M}$ in (4) contain all the relevant information to estimate and test the asset pricing model.

We describe all the possible cases for models with one, two or three factors under the only maintained assumption that $E(\mathbf{r}) \neq \mathbf{0}$. This assumption implies that we only need to study cases where the rank of $\mathbf{M}$ is one or higher. We also describe some normalizations for each case that we can use to implement GMM.

A.1 One factor

We cannot have an underidentified single-factor model because the valid SDFs are unique up to scale:

- Identification ($d = 1$): The rank of $\mathbf{M}$ is one.
  - $E(\mathbf{r})$ is not in the span of $Cov(\mathbf{r}, f)$: The rank of $Cov(\mathbf{r}, \mathbf{f})$ is zero.
  - $E(\mathbf{r})$ is in the span of $Cov(\mathbf{r}, f)$: The rank of $Cov(\mathbf{r}, \mathbf{f})$ is one.
- Lack of a valid SDF: The rank of $\mathbf{M}$ is one, while the rank of $Cov(\mathbf{r}, \mathbf{f})$ is one.

In a single-factor model, the parameter vector that we estimate is

$$\boldsymbol{\theta} = \begin{pmatrix} a \\ b \end{pmatrix}$$

subject to a normalization. For instance, if we normalize one entry to one then we have two possible normalizations

$$\begin{pmatrix} 1 \\ b \end{pmatrix}, \begin{pmatrix} a \\ 1 \end{pmatrix}.$$

A.2 Two factors

The valid SDFs may belong to a two-dimensional subspace and hence we may find underidentified two-factor models:

- Underidentification with $d = 2$: The rank of $\mathbf{M}$ is one.
  - $E(\mathbf{r})$ is not in the span of $Cov(\mathbf{r}, \mathbf{f})$: The rank of $Cov(\mathbf{r}, \mathbf{f})$ is zero.

- $E(\mathbf{r})$ is in the span of $Cov(\mathbf{r},\mathbf{f})$: The rank of $Cov(\mathbf{r},\mathbf{f})$ one.

- Identification ($d=1$): The rank of $\mathbf{M}$ is two.
  - $E(\mathbf{r})$ is not in the span of $Cov(\mathbf{r},\mathbf{f})$: The rank of $Cov(\mathbf{r},\mathbf{f})$ is one.
  - $E(\mathbf{r})$ is in the span of $Cov(\mathbf{r},\mathbf{f})$: The rank of $Cov(\mathbf{r},\mathbf{f})$ is two.
- Lack of a valid SDF: The rank of $\mathbf{M}$ is three, while the rank of $Cov(\mathbf{r},\mathbf{f})$ is two.

In a two-factor model, when $d=1$, the parameter vector that we estimate is

$$\boldsymbol{\theta} = \begin{pmatrix} a \\ b_1 \\ b_2 \end{pmatrix}$$

subject to a normalization. If we want to normalize one entry to one then we have three possible normalizations

$$\begin{pmatrix} 1 \\ b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} a \\ 1 \\ b_2 \end{pmatrix}, \begin{pmatrix} a \\ b_1 \\ 1 \end{pmatrix}.$$

If we analyze the case $d=2$ instead, then we estimate two parameter vectors

$$(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = \begin{pmatrix} a_1 & a_2 \\ b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

subject to normalizations. If we use the identity matrix to normalize these vectors then we have three possible normalizations

$$\begin{pmatrix} a_1 & a_2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ b_{11} & b_{12} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ b_{21} & b_{22} \end{pmatrix}.$$

A.3 Three factors

The valid SDFs may belong to a three-dimensional subspace:

- Underidentification with $d=3$: The rank of $\mathbf{M}$ is one.
  - $E(\mathbf{r})$ is not in the span of $Cov(\mathbf{r},\mathbf{f})$: The rank of $Cov(\mathbf{r},\mathbf{f})$ is zero.
  - $E(\mathbf{r})$ is in the span of $Cov(\mathbf{r},\mathbf{f})$: The rank of $Cov(\mathbf{r},\mathbf{f})$ is one.

- Underidentification with $d = 2$: The rank of $\mathbf{M}$ is two.
  - $E(\mathbf{r})$ is not in the span of $Cov(\mathbf{r}, \mathbf{f})$: The rank of $Cov(\mathbf{r}, \mathbf{f})$ is one.
  - $E(\mathbf{r})$ is in the span of $Cov(\mathbf{r}, \mathbf{f})$: The rank of $Cov(\mathbf{r}, \mathbf{f})$ is two.
- Identification ($d = 1$): The rank of $\mathbf{M}$ is three.
  - $E(\mathbf{r})$ is not in the span of $Cov(\mathbf{r}, \mathbf{f})$: The rank of $Cov(\mathbf{r}, \mathbf{f})$ is two.
  - $E(\mathbf{r})$ is in the span of $Cov(\mathbf{r}, \mathbf{f})$: The rank of $Cov(\mathbf{r}, \mathbf{f})$ is three.
- Lack of a valid SDF: The rank of $\mathbf{M}$ is four, while the rank of $Cov(\mathbf{r}, \mathbf{f})$ is three.

In a three-factor model, when $d = 1$, the parameter vector that we estimate is

$$\boldsymbol{\theta} = \begin{pmatrix} a \\ b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

subject to a normalization. There are four possible normalizations if we fix one entry to one

$$\begin{pmatrix} 1 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix}, \begin{pmatrix} a \\ 1 \\ b_2 \\ b_3 \end{pmatrix}, \begin{pmatrix} a \\ b_1 \\ 1 \\ b_3 \end{pmatrix}, \begin{pmatrix} a \\ b_1 \\ b_2 \\ 1 \end{pmatrix}.$$

If we analyze the case $d = 2$, then we estimate two parameter vectors

$$(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = \begin{pmatrix} a_1 & a_2 \\ b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$$

subject to normalizations. There are six possible normalizations if we use the identity matrix

$$\begin{pmatrix} a_1 & a_2 \\ b_{11} & b_{12} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a_1 & a_2 \\ 1 & 0 \\ b_{21} & b_{22} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a_1 & a_2 \\ 1 & 0 \\ 0 & 1 \\ b_{31} & b_{32} \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ b_{11} & b_{12} \\ b_{21} & b_{22} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ b_{11} & b_{12} \\ 0 & 1 \\ b_{31} & b_{32} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}.$$

Finally, if we analyze the case $d = 3$, then we estimate three parameter vectors

$$(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3) = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

subject to normalizations. There are four possible normalizations if we use the identity matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ b_{31} & b_{32} & b_{33} \end{pmatrix}.$$

B Monte Carlo design

We describe the Monte Carlo experiment for a two-factor model, $k = 2$, in a similar spirit to the one-factor design in Peñaranda and Sentana (2014). An unrestricted Gaussian data generating process (DGP) for $(\mathbf{f}, \mathbf{r})$ is

$$\mathbf{f} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}),$$
$$\mathbf{r} = \boldsymbol{\mu}_r + \mathbf{B}_r(\mathbf{f} - \boldsymbol{\mu}) + \mathbf{u}_r, \qquad \mathbf{u}_r \sim N(\mathbf{0}, \boldsymbol{\Omega}_{rr}),$$

where the $n \times 2$ matrix $\mathbf{B}_r$ is defined by the two beta vectors

$$\mathbf{B}_r = \begin{pmatrix} \boldsymbol{\beta}_1 & \boldsymbol{\beta}_2 \end{pmatrix}.$$

However, given that we use the simulated data to test that an affine function of $\mathbf{f}$ is orthogonal to $\mathbf{r}$, the only thing that matters is the linear span of $\mathbf{r}$. As a result, we can substantially reduce the number of parameters characterizing the conditional DGP for $\mathbf{r}$ by means of the following steps:

1. a Cholesky transformation of $\mathbf{r}$ to get a residual variance $\boldsymbol{\Omega}_{rr}$ equal to the identity matrix,

2. a Householder transformation that makes the second to the last entries of the vector of risk premia $\boldsymbol{\mu}_r$ equal to zero (see Householder (1964)),

3. another Householder transformation that makes the third to the last entries of $\boldsymbol{\beta}_1$ equal to zero, and

4. a final third Householder transformation that makes the fourth to the last entries of $\boldsymbol{\beta}_2$ equal to zero.

We also construct the two factors such that their variance is the identity matrix. As a result, our simplified DGP for excess returns will be

$$\mathbf{r} = \mu_r \mathbf{e}_1 + (\beta_{11}\mathbf{e}_1 + \beta_{21}\mathbf{e}_2)(f_1 - \mu_1) + (\beta_{21}\mathbf{e}_1 + \beta_{22}\mathbf{e}_2 + \beta_{23}\mathbf{e}_3)(f_2 - \mu_2) + \mathbf{u}_r,$$
$$\mathbf{u}_r \sim N(\mathbf{0}, \mathbf{I}_n),$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are the first, second and third columns of the identity matrix, and

$$\mathbf{f} \sim N(\boldsymbol{\mu}, \mathbf{I}_2).$$

The two parameters in $\boldsymbol{\mu}$ can be directly calibrated from data on $\mathbf{f}$. In turn, the six parameters that define $\mathbf{r}$ can be calibrated as follows. We can define a Hansen-Jagannathan (HJ) distance for this two-factor model as the minimum with respect to $\boldsymbol{\phi}$ of the quadratic form

$$\boldsymbol{\phi}'\mathbb{M}'Var^{-1}(\mathbf{r})\,\mathbb{M}\boldsymbol{\phi},$$

where

$$\mathbb{M}\boldsymbol{\phi} = \begin{pmatrix} E(\mathbf{r}) & Cov(\mathbf{r},\mathbf{f}) \end{pmatrix} \begin{pmatrix} c \\ \mathbf{b} \end{pmatrix}.$$

Note that $\mathbb{M}\boldsymbol{\phi} = \mathbf{M}\boldsymbol{\theta}$ and the rank of $\mathbb{M}$ is equal to the rank of $\mathbf{M}$.

The 3×3 weighting matrix

$$\mathbb{M}'Var^{-1}(\mathbf{r})\,\mathbb{M} = \begin{pmatrix} E(\mathbf{r})'Var^{-1}(\mathbf{r})\,E(\mathbf{r}) & E(\mathbf{r})'Var^{-1}(\mathbf{r})\,Cov(\mathbf{r},\mathbf{f}) \\ \cdot & Cov(\mathbf{r},\mathbf{f})'Var^{-1}(\mathbf{r})\,Cov(\mathbf{r},\mathbf{f}) \end{pmatrix}$$

can be interpreted as the variance matrix of three important portfolios, one that yields the maximum Sharpe ratio

$$r_0 = \mathbf{r}'Var^{-1}(\mathbf{r})\,E(\mathbf{r}),$$

and the two factor mimicking portfolios

$$r_1 = \mathbf{r}'Var^{-1}(\mathbf{r})\,Cov(\mathbf{r}, f_1), \qquad r_2 = \mathbf{r}'Var^{-1}(\mathbf{r})\,Cov(\mathbf{r}, f_2).$$

Note that if we minimize the quadratic form subject to the symmetric normalization $\boldsymbol{\phi}'\boldsymbol{\phi} = 1$ then this HJ distance is equal to the minimum eigenvalue of this variance matrix.

The first entry of the weighting matrix is the variance of r_0 or, equivalently, the squared maximum Sharpe ratio. The other two diagonal entries are the variances of r_1 and r_2 or, equivalently, the R^2 of their respective regressions. Finally, the three different off-diagonal elements correspond to the covariances between these three portfolios, which we can pin down by their correlations. In this way, we have six parameters that are easy to interpret and calibrate, and from them we can pin down the six parameters that our DGP requires for $\mathbf{r}$.

Moreover, it is easy to impose all the scenarios that we are interested in. All the cases have the same $\boldsymbol{\mu}$ and they also share the same value of the maximum Sharpe ratio, which pins down μ_r given the rest of parameter values.

- Underidentification with $d = 2$ when $E(\mathbf{r})$ is not in the span of $Cov(\mathbf{r}, \mathbf{f})$: In this case the rank of $Cov(\mathbf{r}, \mathbf{f})$ is zero but the rank of $\mathbb{M}$ is one. This case corresponds to a matrix $\mathbb{M}' Var^{-1}(\mathbf{r}) \mathbb{M}$ where only the first entry is different from zero. The five betas in our DGP are equal to zero.

- Underidentification with $d = 2$ when $E(\mathbf{r})$ is in the span of $Cov(\mathbf{r}, \mathbf{f})$: In this case the rank of both $Cov(\mathbf{r}, \mathbf{f})$ and $\mathbb{M}$ is one. This case corresponds to a matrix $\mathbb{M}' Var^{-1}(\mathbf{r}) \mathbb{M}$ where only the first 2×2 block is different from zero if we focus on the extreme case of a useless f_2. Nevertheless, this first block must be singular. All the betas in our DGP are equal to zero except β_{11}. We chose its value to match a particular value of R^2 in the regression that defines r_1.

- Identification when $E(\mathbf{r})$ is not in the span of $Cov(\mathbf{r}, \mathbf{f})$: In this case the rank of $Cov(\mathbf{r}, \mathbf{f})$ is one but the rank of $\mathbb{M}$ is two. This case corresponds to a matrix $\mathbb{M}' Var^{-1}(\mathbf{r}) \mathbb{M}$ where only the first 2×2 block is different from zero if we focus once again on the extreme case of a useless f_2. This first block is nonsingular in this case. Both β_{11} and β_{12} are different from zero, while we keep the other three betas at zero. Now we add the information of a particular correlation between r_0 and r_1.

- Identification when $E(\mathbf{r})$ is in the span of $Cov(\mathbf{r}, \mathbf{f})$: In this case the rank of both $Cov(\mathbf{r}, \mathbf{f})$ and $\mathbb{M}$ is two. This case corresponds to a matrix $\mathbb{M}' Var^{-1}(\mathbf{r}) \mathbb{M}$ with a single zero eigenvalue again. The novelty is that this singularity is not due to a useless f_2, but to r_0 being spanned by (r_1, r_2). Our DGP satisfies this property by choosing $\beta_{23} = 0$ and a full rank matrix
$$\begin{pmatrix} \beta_{11} & \beta_{21} \\ \beta_{12} & \beta_{22} \end{pmatrix}.$$

Now we add the information of a given value of R^2 in the regression that defines r_2 and a given correlation between r_0 and r_2. The correlation between r_1 and r_2 is implicitly determined by the other two correlations since we must have a singularity.

- Lack of a valid SDF: In this case the rank of $Cov(\mathbf{r}, \mathbf{f})$ is two but the rank of $\mathbb{M}$ is three. This case corresponds to parameters such that the matrix $\mathbb{M}' Var^{-1}(\mathbf{r}) \mathbb{M}$ has full rank and hence the HJ distance cannot be zero. The DGP parameters satisfy $\beta_{23} \neq 0$ and full rank of the matrix

$$\begin{pmatrix} \beta_{11} & \beta_{21} \\ \beta_{12} & \beta_{22} \end{pmatrix}.$$

  Now we add the information of a particular correlation between r_1 and r_2, different from the implicit one in the previous case.

Table 1:
Empirical evaluation of the Yogo model

	One-dimensional Set		Two-dimensional Set			Three-dimensional Set		
Market	0.519	(0.539)	-1.680	0.660	(0.083)	4.108	0	0
	[0.858]		[0.980]	[0.487]				
Nondurables	10.994	(0.874)	203.034	0	(0.000)	0	153.895	0
	[61.418]		[29.206]					
Durables	93.286	(0.201)	0	98.354	(0.000)	0	0	114.793
	[32.678]			[10.290]				
Mean	0.023	(0.822)	0.085	0.024	(0.775)	0.873	0.303	-0.103
	[0.094]		[0.109]	[0.091]				
J test	0.619	(0.892)		3.211	(0.920)		48.991	(0.000)
	{3}			{8}			{15}	

Note: This table displays estimates of the SDF parameters with standard errors in brackets [], as well as the J tests with degrees of freedom in braces {} and p-values in parenthesis (). The first, second and third block of columns report the results for sets of SDFs of dimension 1, 2 and 3, respectively. We display the estimates for a particular normalization, but our results are numerically invariant to the chosen normalization because we use continuously updated GMM. The J tests are complemented with significance tests of some SDF parameters. In particular, the p-value of the distance metric test of the null hypothesis of a zero parameter is reported in parenthesis to the right of the estimate. Standard errors and distance metric tests are not reported when the p-value of the J test is lower than 0.01. The payoffs to price are 6 quarterly excess returns of size and book-to-market sorted portfolios from the first quarter of 1949 to the fourth quarter of 2012.

Table 2:
Empirical evaluation of the Lettau-Ludvigson model

	One-dimensional Set		Two-dimensional Set			Three-dimensional Set			
Nondurables	3671.817	(0.640)	170.676	148.126	(0.087)	169.881	0	0	
	[122499.85]		[21.850]	[58.884]		[17.576]			
Cay	-10.411	(0.745)	-0.059	0	(0.928)	0	3.303	0	
	[360.098]		[0.211]				[1.299]		
Cay*Nondur.	-11855.333	(0.098)	0	72.711	(0.012)	0	0	919.164	
	[412724.93]			[228.716]				[436.304]	
Mean	-13.364	(0.429)	0.192	0.286	(0.048)	0.185	1.067	1.006	(0.016)
	[474.018]		[0.090]	[0.238]		[0.074]	[0.208]	[0.266]	
J test	2.232	(0.526)		6.716	(0.568)		22.463	(0.096)	
	{3}			{8}			{15}		

Note: This table displays estimates of the SDF parameters with standard errors in brackets [], as well as the J tests with degrees of freedom in braces {} and p-values in parenthesis (). The first, second and third block of columns report the results for sets of SDFs of dimension 1, 2 and 3, respectively. We display the estimates for a particular normalization, but our results are numerically invariant to the chosen normalization because we use continuously updated GMM. The J tests are complemented with significance tests of some SDF parameters. In particular, the p-value of the distance metric test of the null hypothesis of a zero parameter is reported in parenthesis to the right of the estimate. Standard errors and distance metric tests are not reported when the p-value of the J test is lower than 0.01. The payoffs to price are 6 quarterly excess returns of size and book-to-market sorted portfolios from the second quarter of 1952 to the fourth quarter of 2012.

Table 3:
Empirical evaluation of the Jagannathan-Wang model

	One-dimensional Set		Two-dimensional Set			Three-dimensional Set		
Market	-1.095 [0.890]	(0.018)	0.477 [1.357]	-0.894 [0.406]	(0.090)	3.823	0	0
Labor	-61.841 [74.765]	(0.245)	337.748 [59.200]	0	(0.000)	0	347.271	0
Default	109.154 [25.288]	(0.038)	0	91.993 [5.640]	(0.000)	0	0	79.257
Mean	0.188 [0.125]	(0.053)	-0.558 [0.263]	0.076 [0.053]	(0.002)	0.971	-0.641	0.205
J test	4.027 {3}	(0.259)		14.382 {8}	(0.072)		84.627 {15}	(0.000)

Note: This table displays estimates of the SDF parameters with standard errors in brackets [], as well as the J tests with degrees of freedom in braces {} and p-values in parenthesis (). The first, second and third block of columns report the results for sets of SDFs of dimension 1, 2 and 3, respectively. We display the estimates for a particular normalization, but our results are numerically invariant to the chosen normalization because we use continuously updated GMM. The J tests are complemented with significance tests of some SDF parameters. In particular, the p-value of the distance metric test of the null hypothesis of a zero parameter is reported in parenthesis to the right of the estimate. Standard errors and distance metric tests are not reported when the p-value of the J test is lower than 0.01. The payoffs to price are 6 monthly excess returns of size and book-to-market sorted portfolios from February 1959 to December 2012.

Table 4:
Empirical evaluation of the Fama-French model

	One-dimensional Set		Two-dimensional Set			Three-dimensional Set		
Market	3.534		-1.651	4.404		2.367	0	0
Size	2.589		16.497	0		0	11.315	0
Value	7.922		0	9.784		0	0	-6.624
Mean	0.940		0.944	0.943		0.992	0.993	0.993
J test	36.630 {3}	(0.000)		106.978 {8}	(0.000)		234.783 {15}	(0.000)

Note: This table displays estimates of the SDF parameters with standard errors in brackets [], as well as the J tests with degrees of freedom in braces {} and p-values in parenthesis (). The first, second and third block of columns report the results for sets of SDFs of dimension 1, 2 and 3, respectively. We display the estimates for a particular normalization, but our results are numerically invariant to the chosen normalization because we use continuously updated GMM. The J tests are complemented with significance tests of some SDF parameters. In particular, the p-value of the distance metric test of the null hypothesis of a zero parameter is reported in parenthesis to the right of the estimate. Standard errors and distance metric tests are not reported when the p-value of the J test is lower than 0.01. The payoffs to price are 6 monthly excess returns of size and book-to-market sorted portfolios from February 1959 to December 2012.