Optimal robust bilateral trade: burning money

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March 7, 2016

Abstract

We provide an example of a robust bilateral trading mechanism, which cannot be represented as a randomized posted price, contrary to the main theorem in [6], and which is not *ex post* dominated by any randomized posted price, contrary to the main theorem in [3]. In this trading mechanism there is a bid-ask spread in some trading events. This suggests that bid-ask spread is Pareto optimal in the absence of any external forces or frictions beyond the incentive and participation constraints due to the traders' private information.

1 Introduction

In recent work, [3] address the bargaining problem between a seller and a buyer with private information who bargain over an indivisible item. They ask what is the set of pricing mechanisms, which are incentive compatible, feasible, robust to the details of traders' information, and, while satisfying these requirements, as efficient as possible (optimal). A trading mechanism is robust if it satisfies *ex post* incentive compatibility, balanced budget and individual rationality, and it is optimal if it satisfies an appropriate Pareto criterion. They conclude that, "under the assumption that the two traders are risk neutral, we answer this question: such pricing mechanisms are equivalent to non-wasteful randomized posted prices." Their analysis is a follow-up on the ground-breaking research by [6], who write that "it is shown that posted-price mechanisms are essentially the only mechanisms such that each trader has a dominant strategy" and then prove that claim under some minor technical conditions (whence the word "essentially").

While the two above claims are valid and plausible, and the proofs mostly correct, both theorems requisite some qualification. For the two theorems to be true, one

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needs to assume that there is no bid-ask spread, that is, that the payment received by the seller equals the price paid by the buyer. Without that assumption both theorems are essentially wrong.[1]

A bid-ask spread (which may vary with traders' private information) diminishes the set of events where the probability of trade is positive, but may increase the probability of trade on that set. This immediately implies that such a trading mechanism cannot be characterized by a randomized posted price mechanism.[2] Intuitively, at the expense of lowering the utility of one of the traders in some events and excluding some types of that trader through the bid-ask spread (because trade is no longer individually rational for those types), the other trader's incentive constraints are in some events relaxed so that she obtains a higher utility in those events. Therefore, such a trading mechanism where bid-ask spread arises in some events essentially favors one of the traders at the expense of the other, relative to a trading mechanism where there is no bid-ask spread. This implies that such a robust trading mechanism is not Pareto dominated by a robust trading mechanism with no bid-ask spread.[3]

Besides the main purpose – to highlight the necessity of assuming one price in the aforementioned theorems – the present example also suggests a theoretical explanation for bid-ask spreads. Most existing explanations rest on arguments of competition between market makers who intermediate trade (see e.g., [5] and [4]). In contrast, the present example shows that in markets that are not perfectly competitive a bid-ask spread may in some instances be incentive efficient. That is, bid-ask spread may in a thin market, in the absence of any market maker or any other intermediation whatsoever, arise due to informational constraints alone. That point might in and of itself merit further investigation. In the next section we present the formal definitions, state the main results of [6] and [3], and then provide our example.

[1]The aforementioned authors indeed seem to unknowingly make the assumption in their analysis, which does not seem to be limited to these authors. Indeed, it has been a common wisdom that parametrizing the traders' expected utilities by the probability of trade and price, or price conditional on trade as in [3] is without loss of generality. However, that is essentially only true if one considers any possible region for the set of draws of traders' private parameters on which the probability of trade is positive. The reason is that together with participation and feasibility constraints (individual rationality and budget balance), this region determines whether there exists a bid-ask spread or not.

[2]This contrasts with a "naïve" bid-ask spread trading mechanism, which is simply a lottery over deterministic bid-ask spread mechanisms, i.e., a lottery over mechanisms of the form (p_1, p_2), where $p_1 \leq p_2$ and the traders trade if the seller agrees to the receipt of p_1 and the buyer agrees to the payment of p_2.

[3]A robust trading mechanism with no bid-ask spread cannot be Pareto dominated by robust trading mechanism with bid-ask spread so that a characterization by non-wasteful randomized posted prices provides a sufficient but not a necessary condition for optimal robust trading mechanisms. Given the non-separable structure of the trading mechanism with bid-ask spread presented here, such a necessary and sufficient conditions seems elusive.

2 Bid-ask spread

A seller, 1, and a buyer, 2, bargain over the price of an indivisible good, where v_1 is the seller's cost of producing the good, and v_2 is the buyer's value of the good. At the time of trade, each trader knows her own type (valuation) v_i, but doesn't know the type of the other trader. It is common knowledge that pairs of types $v = (v_1, v_2)$ are drawn from $(0,1)^2$ according to *some* joint probability distribution function F, with $support(F) = (0,1)^2$, and such that F is absolutely continuous with respect to the Lebesgue measure. The support of F is common knowledge, but F is not common knowledge, so that the details of F are not known to the traders, or any other entity, such as a social planner or a mechanism designer.[4]

In a standard environment with risk neutral traders, [6] and [2] assume that the price paid by the buyer is also the payment received by the seller, i.e., there is no bid-ask spread. In general, the price paid by the buyer may be different than the payment to the seller – budget balance, defined in a moment, imposes the restriction that the former is larger than the latter. Denote by $\ell \in \{0,1\}$ the allocation of the good, where $\ell = 1$ if the object is transferred to the buyer – there is trade, and $\ell = 0$ if there is no trade. When the allocation is ℓ, the receipt of the seller is $p_1 \in (0,1)$, the payment from the buyer is $p_2 \in (0,1)$, and the valuations are $(v_1, v_2) \in (0,1)^2$, the payoff to trader i is given by a utility function $u_i(\ell, p_i, v_i)$, where for the seller, $u_1(p_1, v_1) = p_1 - \ell v_1$, and for the buyer, $u_2(\ell, p_2, v_2) = \ell v_2 - p_2$.

A direct revelation mechanism μ is a mapping from traders' reports $\tilde{v}_i \in (0,1)$ of their valuations into outcomes. An outcome is given by a lottery $\mu[\tilde{v}]$ over the possible allocations of the good and prices, $\{0,1\} \times (0,1)^2$. That is, $\mu[\tilde{v}]$ is a lottery that the traders face *ex post*, after they made reports $(\tilde{v}_1, \tilde{v}_2)$ of their valuations to a computer, or a broker. Therefore, a mechanism μ is a mapping,

$$\mu : (0,1)^2 \to \Delta\left(\{0,1\} \times (0,1)^2\right).$$

At the time of trade, each trader is allowed to step back if she finds the terms of trade unfavorable and trade is not subsidized from an external source. Therefore, a trading mechanism must satisfy *ex post* individual rationality and *ex post* budget balance. Additionally, a robust trading mechanism must satisfy *ex post* incentive compatibility, that is, reporting valuations truthfully must be an *ex post* Nash equilibrium.[5]

[4] For example, traders may have different beliefs about F, and different beliefs about the beliefs of the other trader and so on, i.e., any type space is allowed, see [1]. The analysis here easily generalizes to the case where the support of F is given by $(\underline{v}, \overline{v}) \times (\underline{v}, \overline{v}), \underline{v} < \overline{v}$, as well as to the case when the support of F is closed; the assumption that the support of F is open is made mainly for notational convenience: when the price of the good is 0 that signifies no trade.

[5] In a separable environment, *ex post* incentive compatibility is equivalent to requiring that the trading mechanism is *interim* (or Bayesian) incentive compatible on any type space, and in

A mechanism μ satisfies *ex post* budget balance and individual rationality if,

$$\mathit{support}(\mu[v]) \subset \{(\ell, p_1, p_2) \mid v_1 \times \ell \leq p_1 \leq p_2 \leq v_2 \times \ell\}, \forall v \in (0,1)^2 .$$

Given a mechanism μ and reports $v = (v_1, v_2) \in (0,1)^2$, denote by $E_{\mu[v]}$ the expectation operator with respect to the probability measure $\mu[v]$. Denote by Ω the state space of realizations of lottery μ. By *ex post* individual rationality and budget balance, we can let $\Omega = [0,1)^2$. The state space Ω then represents all possible prices, where any prices s.t., $p_1 = 0$ signify no trade (note that if $p_2 = 0$, $p_1 = 0$ by budget balance). Throughout, j denotes the trader other than i.

A mechanism μ is *ex post* incentive compatible if,

$$E_{\mu[v_i, v_j]} u_i(\ell, p_i, v_i) \geq E_{\mu[\tilde{v}_i, v_j]} u_i(\ell, p_i, v_i), \forall \tilde{v}_i, \forall v_i, \forall v_j, i \in \{1, 2\} \tag{1}$$

Definition 1. A mechanism μ is a robust trading mechanism if it satisfies *ex post* budget balance, individual rationality, and incentive compatibility.

Consider a posted price $\bar{p}$. If the seller is willing to deliver the good to the buyer in exchange for the payment of $\bar{p}$, and the buyer is willing to pay $\bar{p}$ in exchange for the good, then trade is effected; otherwise the buyer does not obtain the good and no payment takes place. Formally,

$$\mu_{\bar{p}}[v] = \begin{cases} 1_{\{\ell=1, p=\bar{p}\}}, & \text{if } v_1 \leq \bar{p} \leq v_2, \\ 1_{\{\ell=0, p=0\}}, & \text{otherwise.} \end{cases}$$

More generally, a posted price can be randomized so that there is a predetermined lottery λ over prices in $[0,1)$, and a posted price is then randomly drawn according to λ. Such a randomized posted price is given by,

$$\mu_{\lambda}[v] = \begin{cases} \lambda(\bar{p}) 1_{\{\ell=1, p=\bar{p}\}}, & \text{if } v_1 \leq p \leq v_2, \\ \lambda(\bar{p}) 1_{\{\ell=0, p=0\}}, & \text{otherwise.} \end{cases}$$

Note that here, $p_1 = p_2$ in all events where trade occurs.

Denote by $U_i^{\mu}(v; \tilde{v}_i)$ the payoff to trader i in a mechanism μ when traders valuations are $v \in (0,1)^2$ but trader i reports $\tilde{v}_i$ (and trader j reports v_j truthfully),

$$U_i^{\mu}(v; \tilde{v}_i) = E_{\mu[\tilde{v}_i, v_j]} u_i(\ell, p_i, v_i)$$

Given a robust trading mechanism μ, denote by $U_i^{\mu}(v)$ the *ex post* payoff to trader i

particular, for any common prior distribution over payoff types F, see [1] and also [7]. The revelation principle holds (see, e.g., [8]) so that the restriction to direct revelation mechanisms is without loss of generality.

when valuations are $v \in (0,1)^2$,

$$U_i^\mu(v) = E_{\mu[v]} u_i(\ell, p_i, v_i).$$

Given two mechanisms μ, μ', we say that μ and μ' are payoff equivalent if,

$$U_i^\mu(v) = U_i^{\mu'}(v), \forall v \in (0,1)^2.$$

Lemma 1. Let a mechanism μ be *ex post* incentive compatible. Then $U_1^\mu(v)$ is strictly decreasing in v_1 whenever $U_1^\mu(v) > 0$, and $U_2^\mu(v)$ is strictly increasing in v_2 whenever $U_2^\mu(v) > 0$.

Proof. We provide the proof for the seller. Let $U_1^\mu(v_1, v_2) > 0$, for some $0 < v_1 < v_2$ and let $\tilde{v}_1 < v_1$. Then $\mu[v]$ assigns a positive probability to some feasible prices, and by strict monotonicity of u_1 in v_1, we have $U_1^\mu(v_1, v_2; \tilde{v}_1) > U_1^\mu(v_1, v_2; v_1)$. By *ex post* incentive compatibility,

$$U^\mu(\tilde{v}_1, v_2; \tilde{v}_1) \geq U^\mu(v_1, v_2; \tilde{v}_1) > U^\mu(v_1, v_2; v_1).$$

■

Lemma 2. Let a mechanism μ be *ex post* incentive compatible and let μ' be payoff-equivalent to μ. Then μ' is *ex post* incentive compatible.

Proof. Take a $v \in (0,1)^2$, $v_1 < v_2$, and by payoff equivalence of μ and μ',

$$E_{\mu[v]} p_1 - v_1 E_{\mu[v]} \ell = E_{\mu'[v]} p_1 - v_1 E_{\mu'[v]} \ell,$$
$$v_2 E_{\mu[v]} \ell - E_{\mu[v]} p_2 = v_2 E_{\mu'[v]} \ell - E_{\mu'[v]} p_2,$$

so that, $E_{\mu[v]} \ell = E_{\mu'[v]} \ell$. Now take $\tilde{v}_1 > v_1$, and by *ex post* incentive compatibility of μ,

$$U_1^\mu(v) - U_1^\mu(v; \tilde{v}_1) = U_1^\mu(v) - U_1^\mu(\tilde{v}_1, v_2) - (\tilde{v}_1 - v_1) E_{\mu[\tilde{v}_1, v_2]} \ell \geq 0.$$

Since $U_1^{\mu'}(v) = U_1^\mu(v)$, $U_1^{\mu'}(\tilde{v}_1, v_2) = U_1^\mu(\tilde{v}_1, v_2)$, and $E_{\mu[\tilde{v}_1, v_2]} \ell = E_{\mu'[\tilde{v}_1, v_2]} \ell$, *ex post* incentive compatibility of μ' for the seller follows. Similarly, we prove *ex post* incentive compatibility of μ' for the buyer. ■

Given a robust trading mechanism μ, let $\varphi(v) = E_{\mu[v]} \ell = \mu[v](\{\ell = 1\})$, and $\pi_i(v) = E_{\mu[v]} p_i$, so that $\varphi(v)$ is the probability that the good is allocated to the buyer, and $\pi_i(v)$ is the expected price faced by trader i. In the special case, when $\pi_1 \equiv \pi_2$ denote that one price by π. Note that by *ex post* budget balance and individual rationality, the prices can only be positive whenever the object is allocated, so that,

$$\pi_i(v) = \begin{cases} E_{\mu[v]}(p_i \mid \ell = 1), & \text{if } \mu(\{\ell = 1\}) > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

We can therefore write,

$$U_i^\mu(v) = \mu[v](\{\ell = 1\})u_i(1, \pi_i(v), v_i) = \varphi(v)u_i(1, \pi_i(v), v_i),$$

that is,

$$U_1^\mu(v) = \varphi(v)(\pi_1(v) - v_1), \text{ and, } U_2^\mu(v) = \varphi(v)(v_2 - \pi_2(v)), \forall v \in (0,1)^2.$$

The mechanism μ is payoff equivalent to (φ, π_1, π_2). Call (φ, π_1, π_2) the *probability-prices representation* of μ.

Note that one can also define a representation of the mechanism μ with a single price $\hat{\pi}$ and appropriately adjust the probability function $\hat{\varphi}$, so that these quantities are given as solutions to $U_i^\mu(v) = \hat{\varphi}(v)u_i(1, \hat{\pi}_i(v), v_i)$, $i = 1, 2$. This effectively ignores the information given by fine structure of the state space of events where traders trade and at what prices. In contrast, this information is not ignored when prices and the probability of trade are derived by formal integration over trading events in μ, as in (2) in the probability-prices representation. In general, by budget balance, $\pi_1(v) \leq \pi_2(v), \forall v$. By Lemma 2, since μ is *ex post* incentive compatible, (φ, π_1, π_2) is also *ex post* incentive compatible. Moreover, since μ satisfies *ex post* budget balance and individual rationality, (φ, π_1, π_2) also satisfies *ex post* budget balance and individual rationality, so that (φ, π_1, π_2) is a robust trading mechanism. Note that the functions φ, π_1, π_2 are measurable on $\Omega = [0,1)^2$ (recall that Ω is the state space of all possible prices for the lottery μ). Note also that by Lemma 1, $\varphi(.,.)$ is weakly decreasing in v_1 and weakly increasing in v_2, and $\pi_i(.,.)$ is weakly increasing in v_1 and in v_2. Of course, it can also be that the probability-prices representation results in $\pi_1(v) = \pi_2(v), \forall v$, e.g., in the randomized posted price described above. We call such a probability-prices representation *unitary probability-prices representation.* Or, one can implicitly assume that this is the case and hope that such an assumption is without loss of generality.

It is clear that a posted price is a robust trading mechanism. To the effect of the converse, Hagerty and Rogerson, prove the following claims (Corollaries 1-3 to Theorem 1 in [6]).

1. Suppose that μ is a robust trading mechanism and in its probability-price representation φ is twice-differentiable. Then μ is payoff equivalent to a randomized posted price.

2. Suppose that μ is a robust trading mechanism and in its probability-price representation φ maps onto $\{0, 1\}$. Then μ is payoff equivalent to a randomized posted price.

3. Suppose that μ is a robust trading mechanism and in its probability-price representation φ takes finitely many different values on a finite grid. Then μ

is payoff equivalent to a randomized posted price.

We will here not delve into the proofs 1-3 above but will provide the following key auxiliary result from Čopič and Ponsatí, [3], which can be used to prove these claims and which is also essential to our example below. Consider the seller and define,

$$\tilde{\varphi}^{v_2}(.) = \varphi(., v_2),$$
$$\tilde{\pi}_1^{v_2}(.) = \pi_1(., v_2).$$

One can define analogous functions for the buyer.

Lemma 3. The following claims hold for the seller (analogous claims can be proven for the buyer).

1. $\tilde{\pi}_1^{v_2}(.)$ is a simple function if and only if $\tilde{\varphi}^{v_2}(.)$ is a simple function.
2. If $\tilde{\pi}_1^{v_2}(.)$ and $\tilde{\varphi}^{v_2}(.)$ are simple functions, then there exists a probability distribution λ^{v_2} on $\{0\} \times (0, v_2]$, such that,

$$\tilde{\varphi}^{v_2}(v_1) = Pr_{\lambda^{v_2}}\left(\omega \in [v_1, v_2]\right),$$
$$\tilde{\pi}_1^{v_2}(v_1) = E_{\lambda^{v_2}}\left(\omega \mid \omega \in [v_1, v_2]\right).$$

Proof.
Step 1. $\tilde{\varphi}^{v_2}(.) \in \{0, \alpha\}$, for some $\alpha \in (0, 1]$, if and only if, $\tilde{\pi}_1^{v_2}(.) \in \{0, \bar{p}\}$, for some $\bar{p} \in (0, 1)$. Moreover, $\tilde{\varphi}^{v_2}(v_1) = \alpha, \tilde{\pi}_1^{v_2}(v_1) = \bar{p}$, if and only if, $v_1 \in (0, \bar{p}]$.
Proof. Suppose $\tilde{\pi}_1^{v_2}(.) \in \{0, \bar{p}\}$. If $\tilde{\pi}_1^{v_2}(v_1) = 0$, then by *ex post* individual rationality and budget balance $\tilde{\varphi}^{v_2}(v_1) = 0$. If $\tilde{\pi}_1^{v_2}(v_1) = \tilde{\pi}_1^{v_2}(\tilde{v}_1) = \bar{p}$, $v_1 > \tilde{v}_1$, then by *ex post* incentive compatibility, $\tilde{\varphi}^{v_2}(v_1) = \tilde{\varphi}^{v_2}(\tilde{v}_1)$, or v_1 would have incentives to misreport to $\tilde{v}_1$. By *ex post* individual rationality and budget balance $\tilde{\varphi}^{v_2}(v_1) = \alpha$, for some $\alpha \in (0, 1]$.

If $\tilde{\varphi}^{v_2}(v_1) = 0$, then by *ex post* individual rationality and budget balance (taking into account the individual rationality of trader 2), $\tilde{\pi}_1^{v_2}(v_1) = 0$.
Suppose $\tilde{\varphi}^{v_2}(v_1) = \alpha$. Then $\tilde{\pi}_1^{v_2}(v_1) \geq v_1$, by *ex post* individual rationality. If $\tilde{\pi}_1^{v_2}(v_1) = v_1$, then, by *ex post* incentive compatibility, $\tilde{\varphi}^{v_2}(\tilde{v}_1) = \alpha$ and $\tilde{\pi}_1^{v_2}(\tilde{v}_1) = v_1$, $\forall \tilde{v}_1 < v_1$, and $\tilde{\varphi}^{v_2}(\tilde{v}_1) = 0$ and $\tilde{\pi}_1^{v_2}(\tilde{v}_1, v_2) = 0$, $\forall \tilde{v}_1 > v_1$. If $\tilde{\pi}_1^{v_2}(v_1) > v_1$, then, by *ex post* incentive compatibility, $\tilde{\varphi}^{v_2}(\tilde{v}_1) = \alpha$ and $\tilde{\pi}_1^{v_2}(\tilde{v}_1) = \tilde{\pi}_1^{v_2}(v_1)$, $\forall \tilde{v}_1 \in (0, v_1]$.

Step 2. $\tilde{\varphi}^{v_2}(.) \in \{0, \alpha_1, ..., \alpha_K\}$, for some $\{\alpha_1, ..., \alpha_K\} \subset (0, 1], \alpha_k > \alpha_{k+1}, k < K$, if and only if, $\tilde{\pi}_1^{v_2}(.) \in \{0, p_1, ..., p_K\}$, for some $\{p_1, ..., p_K\} \subset (0, 1), p_k < p_{k+1}, k < K$. Moreover, $\tilde{\varphi}^{v_2}(v_1) = \alpha_{k+1}, \tilde{\pi}_1^{v_2}(v_1) = p_{k+1}$, if and only if, $v_1 \in (\bar{p}_k, \bar{p}_{k+1}]$, where,

$$p_k = \frac{\alpha_k - \alpha_{k+1}}{\alpha_k}\bar{p}_k + \frac{\alpha_{k+1}}{\alpha_k}p_{k+1}, \;\; k < K, \text{ and } p_K = \bar{p}_K. \tag{3}$$

Proof. The first part follows as in Step 1, i.e., by *ex post* individual rationality and budget balance $\tilde{\varphi}^{v_2}(.) > 0$, if and only if $\tilde{\pi}_1^{v_2}(.) > 0$, and by *ex post* incentive compatibility, $\tilde{\varphi}^{v_2}(.)$ is constant, if and only if, $\tilde{\pi}_1^{v_2}(.)$ is constant.
That $p_K = \bar{p}_K$ follows from *ex post* incentive compatibility for type $v_1 = \bar{p}_K$. Now take types $v_1 = \bar{p}_k, k < K$, and $\tilde{v}_1 = v_1 + \epsilon, \epsilon > 0$. By *ex post* incentive compatibility for v_1,

$$\alpha_k(p_k - \bar{p}_k) \geq \alpha_{k+1}(p_{k+1} - \bar{p}_k),$$

and by *ex post* incentive compatibility for $\tilde{v}_1$,

$$\alpha_{k+1}(p_{k+1} - \bar{p}_k - \epsilon) \geq \alpha_k(p_k - \bar{p}_k - \epsilon).$$

Let $\epsilon \to 0$, and combine the last two inequalities to obtain,

$$\alpha_k(p_k - \bar{p}_k) = \alpha_{k+1}(p_{k+1} - \bar{p}_k),$$

which yields (3). ■

Consider the example in Figure 1. below – the trading mechanism in the example is characterized by quantities, $\underline{\varphi}, \bar{\varphi}, \underline{p}, \bar{p}, \tilde{p}$, and in the NW corner, traders trade with probability 1. By Lemma 3, in the NW corner, from the incentive constraints of the buyer the price is given by, $p^* = \underline{\varphi}\underline{p} + (1 - \underline{\varphi})\bar{p}$, and from the incentive constraints of the seller, $p^* = \bar{\varphi}\tilde{p} + (1 - \bar{\varphi})\bar{p}$. Since $\tilde{p} < \bar{p}$ it follows that $\underline{\varphi} + \bar{\varphi} > 1$ so that this trading mechanism is not payoff equivalent to a lottery over posted prices. Therefore, this is a counter-example to claims 1-3, or effectively, to the main theorem in [6]. One can easily verify that this trading mechanism is also not equivalent to a lottery over personalized prices, that is a lottery over the objects of the form $(p_1, p_2), p_1 \leq p_2$, where each trader has a veto power should they find the price unfavorable.

Next, we turn to the question of optimality of trading mechanism. In [2], Čopič defines the following notion of efficiency suitable for the study of robustness. Given two robust trading mechanisms μ and μ', μ' *ex post* Pareto dominates μ if, $U_i^{\mu'}(v) \geq U_i^{\mu}(v)$, $\forall v \in (0,1)$, $i \in \{1,2\}$, and there is a trader $i \in \{1,2\}$ and a subset of types $O \subset (0,1)^2$, such that the Lebesgue measure of O is positive and,

$$U_i^{\mu'}(v) > U_i^{\mu}(v), \forall v \in O.$$

Based on this welfare criterion, Čopič and Ponsatí, [3], then give the following definition.

Definition 2. A robust trading mechanism μ is *optimal* if there does not exist a robust trading mechanism μ' which *ex post* Pareto dominates μ.

Accounting for the fact that in a randomized posted price some non-zero probability might be assigned to prices at which at least one of the traders would in no event be willing to trade, Čopič and Ponsatí, [3], define a randomized posted price to be *non wasteful* if $\lambda(\bar{p} = 0) = 0$. The main theorem in [3] is as follows.

Theorem 1. A mechanism is an optimal robust trading mechanism if and only if it is payoff equivalent to a non-wasteful randomized posted price.

The example from Figure 1 necessitates that in some events, there is a bid-ask spread. However, a bid-ask spread implies some inherent inefficiency given that some amount of transfer commodity must be disposed off. It is therefore intuitive and natural to think that optimal trading mechanisms must have a unitary probability-prices representation and such that they are non-wasteful. By Theorem 1 of Čopič and Ponsatí these mechanism are payoff equivalent to non-wasteful randomized prices.

Now consider again the example in Figure 1. If the only optimal trading mechanisms have a unitary probability-prices representation, then there must exist such a trading mechanism, which *ex post* Pareto dominates the mechanism in Figure 1. Call this dominating mechanism μ'. Then it must be that under μ', the price in the NW region remains constant (or either of the two traders would be worse off). Since the probability $\underline{\varphi}$ and the price $\underline{p}$ also cannot change (that region of trade is already pushed to the boundary of the individually-rational and budget balanced trades), this implies that $\bar{p}$ must also remain constant. Therefore, in μ', he NE region of trade (where in the mechanism in Figure 1 the probability of trade is $\bar{\varphi}$), the region of positive probability of trade can only be extended horizontally to the diagonal line, that is, to the point where $\pi_1(v) = \pi_2(v) = \bar{p}$. One can easily verify the following two facts:
1. Under such μ', the probability of trade in that region decreases from $\bar{\varphi}$ to $1-\underline{\varphi}$. 2. The expected utility of all types of sellers in that region increases under μ', relative to the mechanism in Figure 1.

In other words such μ' awards higher payoffs to the seller for all type draws in the NE region, while the payoffs to both traders remain the same in all other regions. However, notice that the price to the buyer did not change under μ' for all type draws in the NE region, while the probability of trade went down from $\bar{\varphi}$ to $1 - \underline{\varphi}$. Therefore, the buyer is worse off under μ' than she was under the original mechanism, for all the type draws in the NE region where the probability of trade was positive under the original trading mechanism. Therefore, there does not exist a trading mechanism μ' with a unitary probability-prices representation, which would *ex post* Pareto dominate the trading mechanism of Figure 1. Consequently, the trading mechanism in Figure 1 also provides a counter-example to Theorem 1 in [3]. It is therefore evident that there exist optimal robust trading mechanisms where in some trading events, there is a bid-ask spread (in the probability-prices representation).

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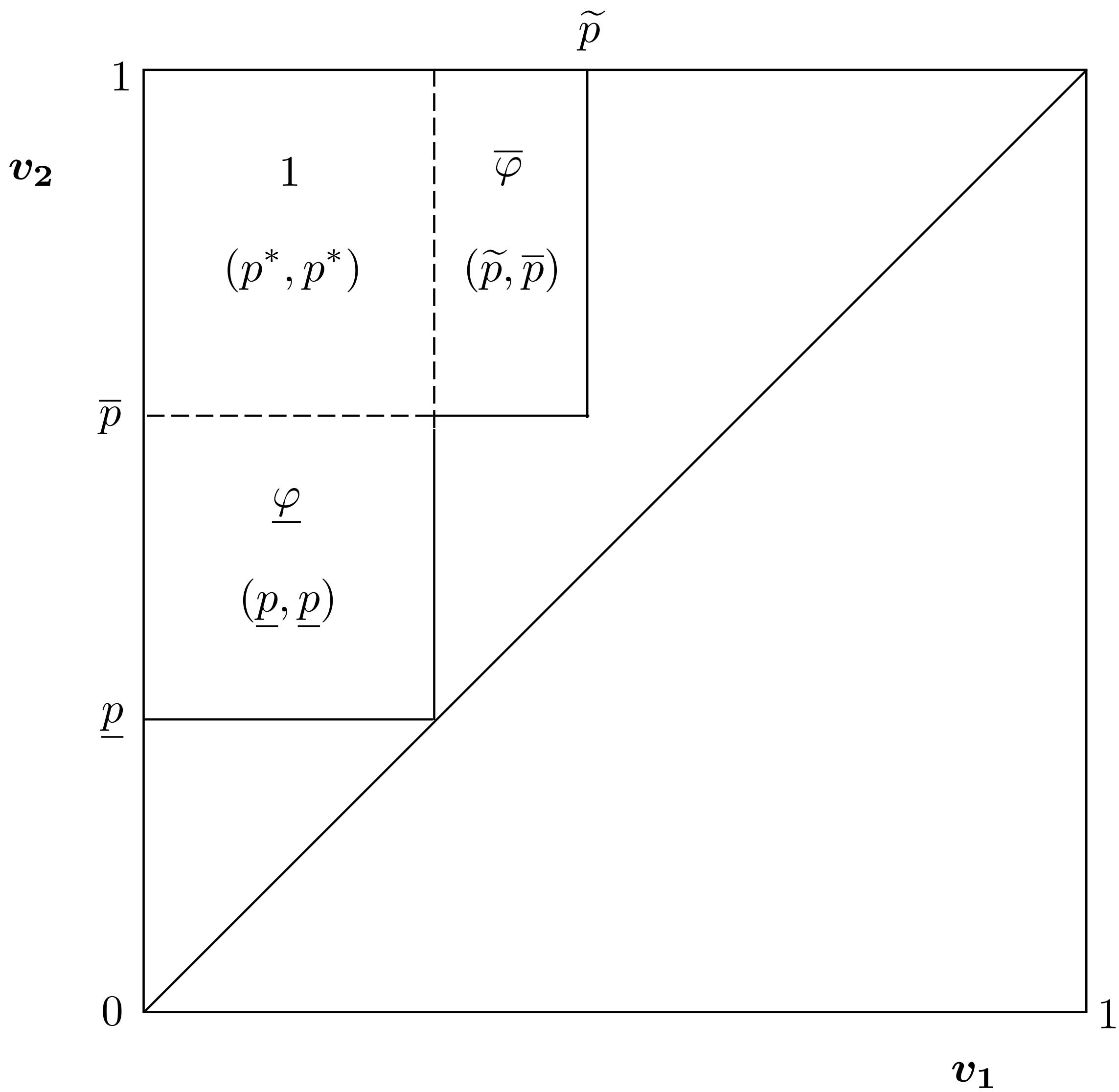
$\widetilde{p}$
1
v_2
1
(p^*, p^*)
$\overline{\varphi}$
$(\widetilde{p}, \overline{p})$
$\overline{p}$
$\underline{\varphi}$
$(\underline{p}, \underline{p})$
$\underline{p}$
0
1
v_1

Figure 1.