

Duality with Time-Changed Lévy Processes*

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Abstract

In this paper we study the pricing problem of derivatives written in terms of a two dimensional time–changed Lévy processes. Then, we examine an existing relation between prices of put and call options, of both the European and the American type. This relation is called *put–call duality*. It includes as a particular case, the relation known as put–call symmetry. Necessary and sufficient conditions for put–call symmetry to hold are shown, in terms of the triplet of local characteristic of the Time–changed Lévy process. In this way we extend the results obtained in Fajardo and Mordecki (2004) to the case of time–changed Lévy processes.

Key Words: Lévy processes, Time Change, Symmetry.

JEL Classification: G12, G13

1 Introduction

Since Black and Scholes (1973) seminal paper, many researches have studied the true dynamics of the underlying asset returns process. This true return differs from this seminal model assumptions in three different aspects: asset prices jumps, so we do not observe normal returns, the volatility is

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stochastic and finally returns and volatility are correlated, frequently this correlation is negative, this feature is called *leverage effect*.

In the effort to capture these option features, many models have been suggested, we can mention the Merton (1976) model, where a compound Poisson process is introduced as the structure of the jumps and the stochastic volatility model of Heston (1993), where a mean reverting square-root process is used. Time-changed Lévy processes can be used to unify these approaches, allowing for more sophisticated jump and stochastic volatility structure models. Of course some of this work can be done with the affine diffusion models introduced by Duffie et al. (2000), but the use of compound Poisson process to model jumps limits these models.

As have been observed in the empirical literature, we have many small jumps in a finite time intervals, to deal with that feature more realistic jump structures have been suggested as for example the Inverse Gaussian (IG) model of Barndoff-Nielsen(1998), the Generalized Hyperbolic (GH) model of Eberlein et al (1998), the Variance-Gamma (VG) model of Madan et al. (1998) and the CGMY model of Carr et al. (2002).

Empirical works have shown that time-changed Lévy processes can capture the best features of the above models, namely: high jump activity and leverage effect (see Carr and Wu (2004)). The stochastic time change on Lévy process generates the stochastic volatility, we can understand the original clock as a calendar time and the new random clock as a business time, more activity in a business day generate a faster business clock, this randomness in business clock generates stochastic volatility. Finally, if we let the Lévy process to be correlated with the random clock, we can capture the correlation between returns and volatility.¹ For an analysis of the best specification option pricing model with time-changed Lévy processes see Huang and Wu (2004)

In the present paper we consider the problem of pricing European and American type derivatives written on a two dimensional time-changed Lévy processes, with a payoff function homogeneous of an arbitrary degree. In the second part of the paper we study an existing relation between prices of put

¹See Black (1976) and Bekaert and Wu (2000)

and call options, of both the European and the American type. This relation is called *put–call duality*. It includes as a particular case, the relation known as put–call symmetry. We suppose that the underlying stock in the market model is driven by a time-changed Lévy processes.

The paper is organized as follows: in Section 2 we introduce time-changed Lévy processes. In Section 3 we describe the market model and introduce the pricing problem, illustrating with some important examples of traded derivatives. In Section 4 we describe the *Dual Market Method*, a method which allows to reduce the two stock problem into a one stock problem. In Section 5 we study the put–call duality relation. In Section 6 we have the conclusions and finally an appendix.

2 Time-Changed Lévy processes

Let $X = (X^1, \dots, X^d)$ be a d -dimensional Lévy process respect to the complete filtration $\mathbf{F} = \{\mathcal{F}_t, t \geq 0\}$, this process is defined on the probability space (Ω, \mathbf{F}, P) , in other words X is a càdlàg process with independent and stationary increments.

We know by the Lévy-Khintchine formula that the characteristic function of X_t , $\phi_{X_t}(z) \equiv \mathbf{E}e^{zX_t} = \exp(t\psi(z))$ where the characteristic exponent ψ is given by:

$$\psi(z) = (a, z) + \frac{1}{2}(z, \Sigma z) + \int_{\mathbb{R}^d} \left(e^{(z, y)} - 1 - (z, y)\mathbf{1}_{\{|y| \leq 1\}} \right) \Pi(dy), \quad (1)$$

where $a = (a_1, \dots, a_d)$ is a vector in \mathbb{R}^d , Π is a positive measure defined on $\mathbb{R}^d \setminus \{0\}$ such that $\int_{\mathbb{R}^d} (|y|^2 \wedge 1) \Pi(dy)$ is finite, and $\Sigma = ((s_{ij}))$ is a symmetric nonnegative definite matrix, that can always be written as $\Sigma = A'A$ (where $'$ denotes transposition) for some matrix A .

Now let $t \mapsto \mathcal{T}_t$, $t \geq 0$, be an increasing càdlàg process, such that for each fixed t , \mathcal{T}_t is a stopping time with respect to \mathbf{F} . Furthermore, suppose \mathcal{T}_t is finite $P - a.s.$, $\forall t \geq 0$ and $\mathcal{T}_t \rightarrow \infty$ as $t \rightarrow \infty$. Then $\{\mathcal{T}_t\}$ defines a random change on time, we can also impose $E\mathcal{T}_t = t$.

Then, consider the process Y_t defined by:

$$Y_t \equiv X_{\mathcal{T}_t}, \quad t \geq 0,$$

using different triplet for X and different time changes \mathcal{T}_t , we can obtain a good candidate for the underlying asset return process. We know that if \mathcal{T}_t is another Lévy process we have that Y would be another Lévy process (see appendix). A more general situation is when \mathcal{T}_t is modelled by a non-decreasing semimartingale:

$$\mathcal{T}_t = b_t + \int_0^t \int_0^\infty y \mu(dy, ds)$$

where b is a drift and μ is the counting measure of jumps of the time change, as in Carr and Wu (2004) we take $\mu = 0$ and just take locally deterministic time changes, so we need to specify the local intensity ν :

$$\mathcal{T}_t = \int_0^t \nu(s_-) ds \tag{2}$$

where ν is the instantaneous activity rate, observe that ν must be non-negative. When X_t is the Brownian motion, ν is proportional to the instantaneous variance rate of the Brownian motion, when X_t is a pure jump Lévy process, ν is proportional to the Lévy intensity of jumps.

Now we can obtain the characteristic function of Y_t :

$$\phi_{Y_t}(z) = \mathbb{E}(e^{z'X_t}) = \mathbb{E} \left(\mathbb{E} \left(e^{z'X_u} / \mathcal{T}_t = u \right) \right)$$

If \mathcal{T}_t and X_t were independent, then:

$$\phi_{Y_t}(z) = \mathcal{L}_{\mathcal{T}_t}(\psi(z))$$

where $\mathcal{L}_{\mathcal{T}_t}$ is the Laplace transform of \mathcal{T}_t . So if the Laplace transform of \mathcal{T} and the characteristic exponent of X have closed forms, we can obtain a closed form for ϕ_{Y_t} . Using equation (2) we have:

$$\mathcal{L}_{\mathcal{T}_t}(\lambda) = \mathbb{E}(e^{-\lambda \int_0^t \nu(s_-) ds}) \tag{3}$$

From here we can understand $\lambda\nu$ as an instantaneous interest rate, then we can search in the bond pricing literature to obtain a closed form for ϕ_{Y_t} .

For example a symmetric Lévy process has ψ real and consider an independent time change, then Y_t has a symmetric distribution, that is ϕ_{Y_t} remains real and can be computed by (3).

If we introduce correlation between X and \mathcal{T} , we obtain an asymmetric distribution for Y_t , then ϕ_{Y_t} will be a complex number, we can treat this case with a complex change of measure introduced by Carr and Wu (2004) and compute a generalized Laplace transform.

3 Market Model and Problem

Consider a market model with three assets (S^1, S^2, S^3) given by

$$S_t^1 = e^{Y_t^1}, \quad S_t^2 = S_0^2 e^{Y_t^2}, \quad S_t^3 = S_0^3 e^{Y_t^3} \quad (4)$$

where (Y^1, Y^2, Y^3) is a three dimensional Lévy process, and for simplicity, and without loss of generality we take $S_0^1 = 1$. The first asset is the bond and is usually deterministic. Randomness in the bond $\{S_t^1\}_{t \geq 0}$ allows to consider more general situations, as for example the pricing problem of a derivative written in a foreign currency, referred as *Quanto option*.

Consider a function:

$$f: (0, \infty) \times (0, \infty) \rightarrow \mathbb{R}$$

homogenous of an arbitrary degree α ; i.e. for any $\lambda > 0$ and for all positive x, y

$$f(\lambda x, \lambda y) = \lambda^\alpha f(x, y).$$

In the above market a derivative contract with payoff given by

$$\Phi_t = f(S_t^2, S_t^3)$$

is introduced.

Assume that we are under a risk neutral martingale measure, that is to say, $\frac{S^k}{S^1}$ ($k = 2, 3$) are P -martingales, i.e. P is an equivalent martingale measure (EMM), we want to price the derivative contract just introduced. In the European case, the problem reduces to the computation of

$$E_T = E(S_0^2, S_0^3, T) = \mathbb{E} \left[e^{-Y_T^1} f(S_0^2 e^{Y_T^2}, S_0^3 e^{Y_T^3}) \right] \quad (5)$$

In the American case, if \mathcal{M}_T denotes the class of stopping times up to time T , i.e:

$$\mathcal{M}_T = \{\tau : 0 \leq \tau \leq T, \tau \text{ stopping time}\}$$

for the finite horizon case, putting $T = \infty$ for the perpetual case, the problem of pricing the American type derivative introduced consists in solving an optimal stopping problem, more precisely, in finding the value function A_T and an optimal stopping time τ^* in \mathcal{M}_T such that

$$\begin{aligned} A_T = A(S_0^2, S_0^3, T) &= \sup_{\tau \in \mathcal{M}_T} \mathbb{E} \left[e^{-Y_\tau^1} f(S_0^2 e^{Y_\tau^2}, S_0^3 e^{Y_\tau^3}) \right] \\ &= \mathbb{E} \left[e^{-Y_{\tau^*}^1} f(S_0^2 e^{Y_{\tau^*}^2}, S_0^3 e^{Y_{\tau^*}^3}) \right]. \end{aligned}$$

3.1 Examples of Bidimensional Derivatives

In what follows we introduce some relevant derivatives as particular cases of the problem described.

3.1.1 Option to Default. Consider the derivative which has the payoff

$$f(x, y) = \min\{x, y\}$$

if $Y^1 = rt$, then the value of the *Option to Default* a promise S_T^2 backed by a collateral guarantee S_T^3 , at the time T would be:

$$D = \mathbb{E} \left[e^{-rT} \min\{S_T^2, S_T^3\} \right]$$

3.1.2 Margrabe's Options. Consider the following cases:

- a) $f(x, y) = \max\{x, y\}$, called the *Maximum Option*,
- b) $f(x, y) = |x - y|$, the *Symmetric Option*,
- c) $f(x, y) = \min\{(x - y)^+, ky\}$, the *Option with Proportional Cap*.

3.1.3 Swap Options. Consider

$$f(x, y) = (x - y)^+,$$

obtaining the option to exchange one risky asset for another.

3.1.4 *Quanto Options.* Consider

$$f(x, y) = (x - ky)^+,$$

and take $S_t^2 = 1$, then

$$E_T = \mathbb{E}e^{Y_T^1}(S_T^1 - k)^+$$

where $e^{Y_T^1}$ is the spot exchange rate (foreign units/domestic units) and S_T^1 is the foreign stock in foreign currency. Then we have the price of an option to exchange one foreign currency for another.

3.1.5 *Equity-Linked Foreign Exchange Option (ELF-X Option).* Take

$$S = S^1 : \text{foreign stock in foreign currency}$$

and Q is the spot exchange rate. We use foreign market risk measure, then an ELF-X is an investment that combines a currency option with an equity forward. The owner has the option to buy S_t with domestic currency which can be converted from foreign currency using a previously stipulated strike exchange rate R (domestic currency/foreign currency).

The payoff is:

$$\Phi_t = S_t(1 - RQ_t)^+$$

Then take $S_t^2 = 1$ and $f(x, y) = (y - Rx)^+$.

3.1.5 *Vanilla Options.* Take

$$Y_t^1 = rt,$$

then in the call case we have

$$f(x, y) = (x - ky)^+$$

and

$$f(x, y) = (ky - x)^+$$

in the put case, with $S_t^3 = S_0^3 e^{Y_t}$ and $S_t^2 = 1$.

4 Dual Market method

The main idea to solve the posed problems is the following: make a change of measure through Girsanov's Theorem for Lévy processes, in order to reduce the original problems to a pricing problems for an auxiliary derivative written

on one Lévy driven stock in an auxiliary market with deterministic interest rate. This method was used in Shepp and Shiryaev (1994) and Kramkov and Mordecki (1994) with the purpose of pricing American perpetual options with path dependent payoffs. It is strongly related with the election of the *numéraire* (see Geman et al. (1995)). This auxiliary market will be called the *Dual Market*.

More precisely, observe that

$$e^{-Y_t^1} f(S_0^2 e^{Y_t^2}, S_0^3 e^{Y_t^3}) = e^{-Y_t^1 + \alpha Y_t^3} f(S_0^2 e^{Y_t^2 - Y_t^3}, S_0^3),$$

let $\rho = -\log \mathbf{E} e^{-Y_1^1 + \alpha Y_1^3}$, that we assume finite. The process

$$Z_t = e^{-Y_t^1 + \alpha Y_t^3 + \rho t} \tag{6}$$

is a density process (i.e. a positive martingale starting at $Z_0 = 1$) that allow us to introduce a new measure, the *dual martingale measure*, \tilde{P} by its restrictions to each \mathcal{F}_t by the formula

$$\frac{d\tilde{P}_t}{dP_t} = Z_t.$$

Denote now by $\tilde{Y}_t = Y_t^2 - Y_t^3$, and $S_t = S_0^2 e^{\tilde{Y}_t}$. Finally, let

$$F(x) = f(x, S_0^3).$$

With the introduced notations, under the change of measure we obtain

$$E_T = \tilde{\mathbf{E}} [e^{-\rho T} F(S_T)]$$

$$A_T = \sup_{\tau \in \mathcal{M}_T} \tilde{\mathbf{E}} [e^{-\rho \tau} F(S_\tau)]$$

The following step is to determine the law of the process Y under the auxiliary probability measure \tilde{P} . To this end we can use Girsanov's theorem for semimartingales (see appendix or Jacod and Shiryaev (1987) Ch.3 Theorem 3.24).

4.1 An application

Let $Y_t^1 = rt$ and (Y_t^2, Y_t^3) be a bidimensional time-changed Lévy Process. We show how to obtain a formula for the value of an option to exchange one

risky asset for another at the end of a determined period, as was considered by Margrabe (1978). Let S_T^2 and S_T^3 be two risky assets, a contract with payoff $(S_T^2 - S_T^3)^+$ can be priced using *The Dual Market Method*:

$$\begin{aligned} D &= \mathbf{E} [e^{-rT}(S_T^2 - S_T^3)^+] \\ &= \int_{\mathcal{A}} e^{-rT}(S_0^2 e^{Y_T^2} - S_0^3 e^{Y_T^3}) dP \end{aligned}$$

Assuming for simplicity $S_0^2 = S_0^3 = 1$, Then $\mathcal{A} = \{\omega \in \Omega : Y_T^2(\omega) > Y_T^3(\omega)\}$, we proceed to applied the method:

$$\begin{aligned} D &= \int_{\mathcal{A}} e^{-rT}(e^{Y_T^2} - e^{Y_T^3}) dP \\ &= \int_{\{S_T > 1\}} e^{-rT} e^{Y_T^3} (S_T - 1) dP \end{aligned}$$

where $S_T = e^{Y_T}$ and $Y = Y^2 - Y^3$. Now, to use the dual measure, observe that $\rho = -\log \mathbf{E} e^{-r+Y_1^3} = r - \log \mathbf{E} e^{Y_1^3}$, then:

$$d\tilde{P} = \frac{e^{Y_T^3}}{\mathbf{E} e^{Y_T^3}} dP$$

With all this:

$$\begin{aligned} D &= e^{-\rho T} \int_{\{S_T > 1\}} (S_T - 1) d\tilde{P} \\ D &= e^{-\rho T} \int_{\{S_T > 1\}} S_T d\tilde{P} - e^{-\rho T} \int_{\{S_T > 1\}} d\tilde{P} \end{aligned}$$

Now to reduce this expression we need to assume a distribution for Y and then apply Proposition 2(see appendix) to obtain the density of S_T under \tilde{P} .

5 Put-Call Duality and Symmetry

In this section we will obtain the put-call duality relationship. Consider a *Time-changed Lévy market* where $Y_t^1 = rt$, $Y_t^2 = 0$ and $Y_t^3 = Y_t$. In other

words we have a riskless asset that we denote by $B = \{B_t\}_{t \geq 0}$, with

$$B_t = e^{rt}, \quad r \geq 0,$$

where we take $B_0 = 1$ for simplicity, and a risky asset that we denote by $S = \{S_t\}_{t \geq 0}$,

$$S_t = S_0 e^{Y_t}, \quad S_0 = e^y > 0. \quad (7)$$

In this section we assume that the stock pays dividends, with constant rate $\delta \geq 0$, and as in section 3, we assume that the probability measure P is the chosen equivalent martingale measure. In other words, prices are computed as expectations with respect to P , and the discounted and reinvested process $\{e^{-(r-\delta)t} S_t\}$ is a P -martingale.

Let $\Psi = (B, C, \nu)$ be the characteristic triplet of Y . Then, the drift characteristic² B is completely determined by the other characteristics:

$$B_t = \int_0^t (r - \delta) ds - \frac{1}{2} \int_0^t c_s ds - \int_0^t \int_{\mathbb{R}} (e^x - 1 - x) \nu(ds, dx)$$

In the market model considered we introduce some derivative assets. More precisely, we consider call and put options, of both European and American types.

Let us assume that τ is a stopping time with respect to the given filtration \mathcal{F} , that is $\tau: \Omega \rightarrow [0, \infty]$ belongs to \mathcal{F}_t for all $t \geq 0$; and introduce the notation

$$\mathcal{C}(S_0, K, r, \delta, \tau, \Psi) = \mathbb{E} e^{-r\tau} (S_\tau - K)^+ \quad (8)$$

$$\mathcal{P}(S_0, K, r, \delta, \tau, \Psi) = \mathbb{E} e^{-r\tau} (K - S_\tau)^+ \quad (9)$$

If $\tau = T$, where T is a fixed constant time, then formulas (8) and (9) give the price of the European call and put options respectively.

5.1 Put–Call duality

The following proposition presents a relationship that we have called Put-Call duality.

²See appendix

Proposition 1. *Consider a Time-changed Lévy market with driving process Y with characteristic triplet $\Psi = (B, C, \nu)$. Then, for the expectations introduced in (8) and (9) we have*

$$\mathcal{C}(S_0, K, r, \delta, \tau, \Psi) = \mathcal{P}(K, S_0, \delta, r, \tau, \tilde{\Psi}), \quad (10)$$

where $\tilde{\Psi}(z) = (\tilde{B}, \tilde{C}, \tilde{\nu})$ is the characteristic triplet (of a certain additive process) that satisfies:

$$\begin{cases} \tilde{B}_t &= (\delta - r)t - \frac{1}{2} \int_0^t \sigma_s^2 ds - \int_0^t \int_{\mathbb{R}} (e^x - 1 - x \mathbf{1}_{\{|x| \leq 1\}}) \tilde{\nu}(ds, dx), \\ \tilde{C} &= C, \\ \tilde{\nu}(dy) &= e^{-y} \nu(-dy). \end{cases} \quad (11)$$

Proof. In this market the martingale $Z = \{Z_t\}_{t \geq 0}$ defined by (6) is given by

$$Z_t = e^{Y_t - (r - \delta)t} \quad (t \geq 0). \quad (12)$$

As we have done in the latter section we introduce the *dual martingale measure* \tilde{P} given by its restrictions \tilde{P}_t to \mathcal{F}_t by

$$\frac{d\tilde{P}_t}{dP_t} = Z_t,$$

where P_t is the restriction of P to \mathcal{F}_t . Now

$$\begin{aligned} \mathcal{C}(S_0, K, r, \delta, \tau, \Psi) &= \mathbb{E} e^{-r\tau} (S_0 e^{Y_\tau} - K)^+ \\ &= \mathbb{E} [\mathbb{E} Z_\tau e^{-\delta\tau} (S_0 - K e^{-X_{u_\tau}})^+ / \mathcal{T} = u] \\ &= \mathbb{E} [\tilde{\mathbb{E}} e^{-\delta\tau} (S_0 - K e^{\tilde{X}_{u_\tau}})^+ / \mathcal{T} = u] \\ &= \tilde{\mathbb{E}} e^{-\delta\tau} (S_0 - K e^{\tilde{Y}_\tau})^+ \end{aligned}$$

where $\tilde{\mathbb{E}}$ denotes expectation with respect to \tilde{P} , and the process $\tilde{Y} = \{\tilde{Y}_t\}_{t \geq 0}$ given by $\tilde{Y}_t = -Y_t$ ($t \geq 0$) is the *dual process* (see [3]). In order to conclude the proof, that is, in order to verify that

$$\tilde{\mathbb{E}} e^{-\delta\tau} (S_0 - K e^{\tilde{Y}_\tau})^+ = \mathcal{P}(K, S_0, \delta, r, \tau, \tilde{\Psi}),$$

we must verify that the dual process \tilde{Y} is an additive process with characteristic triplet defined by (11). To this end take $u = (-1, 0, 1)$ and $v = (0, 0 - 1)$ in Proposition (2) in appendix. This concludes the proof. \square

Our Proposition 1 is very similar to Proposition 1 in Schroder (1999). The main difference is that the particular structure of the underlying process (time-changed Lévy process are a particular case of the model considered in [23]) allows to completely characterize the distribution of the dual process \tilde{X} under the dual martingale measure \tilde{P} , and to give a simpler proof.

The proof of the proposition motivates us to introduce the following market model. Given a time-changed Lévy market with driving process characterized by Ψ consider a market model with two assets, a deterministic savings account $\tilde{B} = \{\tilde{B}_t\}_{t \geq 0}$, given by

$$\tilde{B}_t = e^{\delta t}, \quad r \geq 0,$$

and a stock $\tilde{S} = \{\tilde{S}_t\}_{t \geq 0}$, modelled by

$$\tilde{S}_t = K e^{\tilde{Y}_t}, \quad S_0 = e^x > 0,$$

where $\tilde{Y} = \{\tilde{Y}_t\}_{t \geq 0}$ is a semimartingale with local characteristics under \tilde{P} given by $\tilde{\Psi}$. This market is the *auxiliary market* in Detemple (2001), and we call it *dual market*; accordingly, we call *Put-Call duality* the relation (10). It must be noticed that Peskir and Shiryaev (2001) propose the same denomination for a different relation in [21]. Finally observe, that in the dual market (i.e. with respect to \tilde{P}), the process $\{e^{-(\delta-r)t} \tilde{S}_t\}$ is a martingale and relation (10) is the result known as put-call symmetry.

5.2 Symmetric markets

It is interesting to note, that in a market with no jumps the distribution (or laws) of the discounted (and reinvested) stocks in both the given and dual Lévy markets coincide. It is then natural to define a market to be *symmetric* when this relation hold, i.e. when

$$\mathcal{L}(e^{-(r-\delta)t+Y_t} | P) = \mathcal{L}(e^{-(\delta-r)t-Y_t} | \tilde{P}), \quad (13)$$

meaning equality in law. In view of (11), and to the fact that the characteristic triplet determines the law of a time-changed Lévy processes, when \mathcal{T} is a subordinator, we know that the time-changed Lévy processes is also a Lévy process, in that case Fajardo and Mordecki (2003) obtain that a necessary and sufficient condition for (13) to hold is

$$\nu(dy) = e^{-y} \nu(-dy). \quad (14)$$

This ensures $\tilde{\nu} = \nu$, and from this follows $b - (r - \delta) = \tilde{b} - (\delta - r)$, giving (13), as we always have $\tilde{C} = C$. Condition (14) answers a question raised by Carr and Chesney (1996), see [6].

6 Conclusions

In a Time-changed Lévy market where the Lévy processes and time change are independent we have shown how to price derivatives written in terms of two dimensional Time-changed Lévy processes. And also we have derived a put-call relation that we call *put-call duality*, different from the one obtained by Peskir and Shiryaev (2001), that allowed to obtain the put-call symmetry relation as a particular case.

An important extensions of the above results are of interest. In particular the extension to the case where the Lévy processe and the Time change are correlated, this will allow us to capture the leverage effect.

7 Appendix

7.1 Subordinators

Theorem 7.1. *Let $\{Z_t\}$ be a subordinator with Lévy measure ρ , drift β , and $\mathcal{L}(Z_1) = \lambda$. Let $\{X_t\}$ be a Lévy processes \mathbb{R}^d with generating triplet (A, ν, γ) and $\mathcal{L}(X_1) = \mu$. Assume that $\{X_t\}$ and $\{Z_t\}$ are independent. Define*

$$Y_t(\omega) = X_{Z_t(\omega)}(\omega).$$

Then $\{Y_t\}$ is a Lévy process on \mathbb{R}^d and

$$P(Y_t \in B) \int_{[0, \infty)} \mu^s(B) \lambda^t(ds), \quad B \in \mathcal{B}(\mathbb{R}^d)$$

$$E \left[e^{i \langle z, Y_t \rangle} \right] = e^{t \Psi(\log \hat{\mu}(z))}, \quad z \in \mathbb{R}^d$$

The generating triplet (A^*, ν^*, γ^*) of $\{Y_t\}$ is :

$$\begin{aligned} A^* &= \beta A, \\ \nu^*(B) &= \beta \nu(B) + \int_{(0, \infty)} \mu^s(B) \rho(ds), \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}), \\ \gamma^* &= \beta \gamma + \int_{(0, \infty)} \rho(ds) \int_{|x| \leq 1} x \mu^s(dx). \end{aligned}$$

Lemma 7.1. *Let $\{X_t\}$ be a Lévy process and $\{\tau_t\}_{t \leq T}$ be an independent increasing càdlàg process with stationary increments. Then $\{X_{\tau_t}\}$ has stationary increments.*

Lemma 7.2. *Let $\{X_t\}$ be a Lévy process such that , for any $t \geq 0$, $EX_t^2 < \infty$ and $EX_t = 0$. Let $\{\tau_t\}_{t \leq T}$ be and independent càdlàg process such that , for any $t \geq 0$, $E\tau_t < \infty$. Then, for any $t \geq 0$, $EX_{\tau_t}^2 < \infty$ and $EX_{\tau_t} = 0$. Moreover, the increments of X_{τ_t} over disjoint intervals are not correlated.*

7.2 Additive Processes

Let $Y = (Y^1, \dots, Y^d)$ be an additive process with finite variation, that is a semimartingale, the Law of Y is described by its characteristic function:

$$\mathbb{E} [e^{i \langle z, Y_t \rangle}] = e^{\Psi(z)}$$

where

$$\Psi(z) = \int_0^t \left[i \langle z, b_s \rangle - \frac{1}{2} \langle z, c_s z \rangle + \int_{\mathbb{R}^d} (e^{i \langle z, x \rangle} - 1 - i \langle z, x \rangle) \lambda_s(dx) \right] ds, \quad (15)$$

where $b_t \in \mathbb{R}^d$, c_t is a symmetric non negative definite $d \times d$ matrix and λ_t is a Lévy measure on \mathbb{R}^d , i.e. it satisfies $\lambda(\{0\}) = 0$ and $\int_{\mathbb{R}^d} \min\{1, |x|^2\} \lambda_t(dx) < \infty$, for all $t \leq T$. Under some technical conditions we know that the local characteristics of the semimartingale (see Jacod and Shirjaev (1987)) is given by:

$$B_t = \int_0^t b_s ds, \quad C_t = \int_0^t c_s ds, \quad \nu([0, t] \times A) = \int_0^t \int_A \lambda_s(dx) ds,$$

Where $A \in \mathbb{R}^d$, the triplet (B, C, ν) completely characterizes the distribution of Y and the dual process $\tilde{Y} = -Y$ has characteristic triplet $(-B, C, -\nu)$. Now a Girsanov type theorem for semimartingales

Proposition 2. Let Y be a d -dimensional additive process with finite variation with triplet (B, C, ν) under P , let u, v be vectors in \mathbb{R}^d and $v \in [-M, M]^d$. Moreover let $\tilde{P} \sim P$, with density

$$\frac{d\tilde{P}}{dP} = \frac{e^{\langle v, Y_T \rangle}}{\mathbb{E}[e^{\langle v, Y_T \rangle}]}.$$

Then the process $Y^* := \langle u, Y \rangle$ is a \tilde{P} - semimartingale with characteristic triplet (B^*, C^*, ν^*) with:

$$\begin{aligned} b_s^* &= \langle u, b_s \rangle + \frac{1}{2}(\langle u, c_s v \rangle + \langle v, c_s u \rangle) + \int_{\mathbb{R}^d} \langle u, x \rangle (e^{\langle v, x \rangle} - 1) \lambda_s(dx) \\ c_s^* &= \langle u, c_s u \rangle \\ \lambda_s^* &= \Lambda(\kappa_s) \end{aligned}$$

where Λ is a mapping $\lambda: \mathbb{R}^d \rightarrow \mathbb{R}$ such that $x \mapsto \Lambda(x) = \langle u, x \rangle$ and κ_s is a measure defined by;

$$\kappa(A) = \int_A e^{\langle v, x \rangle} \lambda_s(dx).$$

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