

Monetary Dynamics in a General Equilibrium Version of the Baumol-Tobin Model

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Abstract

I study the welfare cost of inflation and the effect on prices after a permanent increase in the interest rate. In the steady state, the real money demand is homogeneous of degree one in income and its interest-rate elasticity is approximately equal to $-1/2$. Consumers are indifferent between an economy with 10% p.a. inflation and one with zero inflation if their income is 1% higher in the first economy. A permanent increase in the interest rate makes the price level to drop initially and inflation to adjust slowly to its steady state level.

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1. INTRODUCTION

This paper offers a monetary model with two applications: to measure the welfare cost of inflation, and to analyze the behavior of prices after an interest rate shock. The link between inflation and welfare is the deviation of resources from consumption to the management of money. Prices are flexible, but they do not adjust instantaneously to a change in the nominal interest rate.

As in Baumol (1952) and Tobin (1956), money is useful for transactions but agents incur a transfer cost whenever they exchange bonds for money. Moreover, money holdings do not receive interest. The economy has infinitely-lived consumers with different endowments and constant relative risk aversion utility. The utility function is in terms of goods only: neither money nor the transfer cost enter in the utility function.

The analysis offers two contributions. The first is to measure the welfare cost of inflation for different preference parameters with infinitely-lived consumers, consumption smoothing, and transfer cost in goods. The second is to describe the behavior of prices after a permanent increase in the nominal interest rate.

The welfare cost of inflation is defined as the income compensation required to leave consumers indifferent between an economy under positive interest rate and one with zero interest rate. According to the model, consumers are indifferent between an economy with 10% p.a. inflation and an economy with zero inflation if the income of each consumer is 1% higher in the first economy. This estimate agrees with the findings in Lucas (2000). The elasticity of intertemporal substitution has a small effect on the welfare cost of inflation. The model is calibrated using U.S. data from 1900 to 1997.

When the nominal interest rate is positive, agents use their resources to maintain the optimal level of money holdings. An increase in the interest rate decreases average

consumption and increases the variation of consumption. The first effect happens because agents exchange bonds for money more frequently. The second effect is a consequence of the concentration of consumption in the beginning of each holding period.

The optimal monetary policy is to decrease the money supply at the rate of time preferences and set the nominal interest rate to zero, as in Friedman (1969). When the nominal interest rate decreases towards zero, money demand converges to the present value of production discounted by the intertemporal discount rate.

We also study a 1% permanent increase in the nominal interest rate in an economy initially in the steady state with zero inflation. This shock produces an initial price drop and then a period of around 5 months with inflation under 1% p.a. After this period, inflation starts to vary between 5% and -3% p.a. in cycles of about 6 months. Eventually, inflation approaches its new steady state value of 1% p.a.

General equilibrium versions of the Baumol-Tobin model were also presented by Jovanovic (1982), Romer (1986), and Fusselman and Grossman (1989). In Jovanovic, consumption is assumed constant within holding periods. As a result, the optimal interval between transfers is finite even when the nominal interest rate is zero. Therefore, zero interest rate minimizes the welfare cost, but it does not reduce it to zero. In Romer, the economy is populated by consumers with finite life and zero intertemporal discount factor. Jovanovic and Romer do not calculate the welfare cost and they do not study a shock to the nominal interest rate. Fusselman and Grossman consider agents with logarithmic utility and transfer cost in utility terms. They have a different calibration and they do not discuss the welfare cost of inflation. With transfer cost in utility terms, the path of prices in their model does not converge to a new steady state after a nominal interest rate shock.

Romer (1987) studies the effect on money demand after a permanent change in the nominal interest rate using the framework in Romer (1986). But he keeps the real

interest fixed during the transition. Here, the real interest rate varies after the shock because the price level does not change instantaneously with the nominal interest rate.

The framework in this paper is adapted from the formulation in Grossman (1987). In this model, agents exchange bonds for money in fixed periods. I include the decision of the optimal interval between transfers¹. As in Alvarez, Atkeson and Edmond (2002), agents keep their resources in two separate accounts: one for the goods market and another one for the asset market.

The structure of the remaining of the paper is the following. Section 2 contains the description of the economy and the utility maximization problem. Section 3 has the analysis of the economy in the steady state. Section 4 contains the measurement of the welfare cost of inflation. Section 5 has the study of the effects of a change in the nominal interest rate. Section 6 concludes. All proofs are in the appendix.

2. THE MODEL

The model consists of a continuum of consumers who need to use money for goods transactions and who incur a cost to transfer resources from the asset market to the goods market. The transfer cost is paid in goods. It covers administrative, opportunity, and non-pecuniary costs involved in making a transfer, as motivated by Baumol (1952) and Tobin (1956). Time is continuous². The model is adapted from Grossman (1987)³.

¹The possibility of paying a proportional transaction cost between holding periods is removed. I use the same notation as in Grossman (1987) whenever possible.

²This is a simplifying assumption. It allows us to ignore integer constraints to find the optimal interval between transfers.

³In Grossman (1987), the interval between transfer is fixed and the derivation of the steady state is obtained supposing that agents equalize their marginal utility of wealth. The characterization of the steady state is done here explicitly as a function of the endowments.

Consumers are infinitely-lived and they discount the future at the rate $\rho > 0$. They choose how much to consume and the time of each transfer. Let $c(t)$ denote consumption at time t and $N_j \geq 0$ $j = 1, 2, \dots$ denote the interval between transfers. The time of each transfer is given by the summation of the intervals N_j , define thus $T_j \equiv \sum_{s=1}^j N_s$, $T_0 \equiv 0$. Each T_j denotes the time in which a transfer is made.

Consumers have preferences given by

$$U(c) = \sum_{j=0}^{\infty} \int_{T_j}^{T_{j+1}} e^{-\rho t} u(c(t)) dt. \quad (1)$$

The utility function is assumed to have the constant relative risk aversion form,

$$u(c) = \begin{cases} \frac{c^{1-\sigma}}{1-\sigma} & \text{for } \sigma \neq 1, \sigma > 0, \\ \log c & \text{for } \sigma = 1. \end{cases}$$

This family of preferences will generate an aggregate real money demand linear in income. Preferences are a function of goods only: neither money nor the transfer cost enter in the utility function⁴.

Each agent owns two accounts: a brokerage account and a bank account. The brokerage account contains all resources used in the asset market. The bank account contains the monetary resources used for goods purchases.

There is a single and nonstorable good. Each consumer produces Y units of the good in every period. Consumption goods are traded in the goods market. The price level at time t is given by $P(t)$ and inflation is given by $\pi(t)$. Taxes at each time t are denoted by $\tau(t)$. Taxes are lump sum and are levied in the brokerage account. At each time, production is sold in the goods market and the proceeds after taxes are deposited in the brokerage account. Although the good is sold in the goods market for money, agents cannot use this money to buy goods in the same period⁵.

⁴Since the model is deterministic, we will make reference to the elasticity of intertemporal substitution $1/\sigma$.

⁵It is interesting to view an agent as a family composed of two types of individuals, a worker and

In order to transfer resources from the brokerage account to the bank account, agents have to pay a fee in goods. The transfer cost is proportional to income, it is given by γY , $\gamma > 0$. A value $\gamma = 1$ means that the transfer cost is equal to one working day per transfer.

Government bonds are traded in the asset market. Let $Q(t)$ be the value at time zero of one dollar to be received at time t . The nominal interest rate is denoted by $r(t)$ and it is assumed positive in every period.

Let B_0 denote bond holdings held by each consumer at time zero and let W_0 denote total wealth initially available in the brokerage account. We have

$$W_0 = B_0 + \int_0^{\infty} Q(t) [P(t)Y - \tau(t)] dt. \quad (2)$$

All consumers have the same present value of production. Any difference in the balance of the brokerage accounts across consumers is given by the amount of bonds held.

At time zero, in addition to the deposits in the brokerage account, agents have M_0 in money holdings in the bank account.

What distinguishes consumers in this economy is the pair (M_0, W_0) . There is an initial given distribution F of M_0 and W_0 along with its density function f .

The importance of money is given by its transaction role: in order to buy goods, agents need to use money. Let $M(t, M_0, W_0)$ denote the quantity of money held at time t by consumer (M_0, W_0) . Whenever there is a purchase of goods, the amount of money necessary for the purchase is subtracted from the bank account. Therefore, the cash in advance constraint in this economy is

$$\dot{M}(t, M_0, W_0) = -P(t)c(t, M_0, W_0), \quad t \neq T_1, T_2, \dots \quad (3)$$

a shopper, as in Lucas (1990). The worker produces the consumption good Y in each period and deposits the proceeds in a brokerage account in the form of bonds. The shopper decides when to sell bonds for money and how to use the money to buy goods.

where $c(t, M_0, W_0)$ denotes consumption at time t of consumer (M_0, W_0) and $\dot{x}(t) \equiv \partial x(t) / \partial t$.

Notice that M_0 can be used promptly for good transactions. In order to use the balance in the brokerage account to buy goods, however, the agents need to pay γY , sell a certain amount of bonds and transfer the monetary proceeds to the goods market.

If it were costless to transfer resources from the asset market to the goods market, agents would hold only the quantity of money needed to buy goods for each particular time. The reason is that interest is accrued to bonds in every period whereas it is not accrued to money. After trading, consumers would have zero money holdings and they would transfer resources in every period.

A transfer cost induces agents to maintain resources sufficient for consumption in several periods. Just after the transfer, agents hold a relatively large quantity of money.

After the first transfer, at time T_1 , consumers will withdraw the exact amount of money necessary to consume until the next transfer. Agents use their money holdings until they exhaust them. When it happens, they make another transfer.

Agents have to decide consumption and money holdings for each time t , and when they will transfer resources between their accounts. This decision is done at time zero, given the path of the nominal interest rate and of the price level. Money holdings in the beginning of each holding period are denoted by $M^+(t)$.

The initial value of money holdings, M_0 , is exogenously given. It is not necessarily the amount of money required for consumption between 0 and T_1 . Agents are, therefore, allowed to transfer an amount $K \geq 0$ from the bank account to the brokerage account at time T_1 . The variable K denotes the quantity of money not used in $[0, T_1)$ and deposited in the brokerage account at $t = T_1$ ⁶.

⁶We have $K > 0$ if M_0 is higher than the value otherwise chosen by the agent. This is likely to

The individual maximization problem is to maximize (1), subject to (3) and

$$\sum_{j=1}^{\infty} Q(T_j) M^+(T_j) + \sum_{j=1}^{\infty} Q(T_j) P(T_j) \gamma Y \leq W_0 + Q(T_1) K, \quad (4)$$

$$\int_0^{T_1} P(t) c(t) + K \leq M_0 \quad (5)$$

plus the nonnegativity constraints for $c(t)$, $M(t)$, N_j , and K^7 .

The constraint (4) states that the present value of all money transfers is equal to the present value of the deposits in the brokerage account. The constraint (5) states that consumption until the first transfer must be financed from M_0 .

The initial amount of the money supply is given by M_0^S . The money supply at time t is given by $M^S(t)$ and its growth rate is given by $\alpha(t)$. The government sets taxes and issues money in order to finance its real purchases $g(t)$ and the initial quantity of bonds held by the public B_0^S .

The government budget constraint is given by

$$B_0^S + \int_0^{\infty} Q(t) P(t) g(t) dt = \int_0^{\infty} Q(t) \tau(t) dt + \int_0^{\infty} Q(t) P(t) \frac{\dot{M}^S(t)}{P(t)} dt. \quad (6)$$

If the government wants to inject money, it needs to exchange bonds for money with the consumers in the asset market. The present value of these open market operations is reflected by the term B_0^{S8} .

The market clearing conditions for money and bonds are given respectively by

$$\int M(t, M_0, W_0) dF(M_0, W_0) = M^S(t), \quad (7)$$

happen if M_0 is too high relative to W_0 or if inflation is high in the beginning periods. In principle we would need a variable K for each holding period, K_j . But we know that, with positive nominal interest rates, the quantity of money will be chosen to make $K_j = 0$ for $j \geq 2$.

⁷I remove the reference to (M_0, W_0) of these variables to simplify notation when it does not lead to ambiguity.

⁸Taxes $\tau(t)$ and government purchases $g(t)$ are described for completeness. These variables will be set to zero and bonds will be financed by seigniorage.

$$\int B_0(M_0, W_0) dF(M_0, W_0) = B_0^S, \quad (8)$$

for all t .

We have to take into account the transfer cost to write the market clearing condition for goods. Let

$$A(t, \delta) \equiv \{(M_0, W_0) : T_j(M_0, W_0) \in [t, t + \delta)\}$$

for a certain $j = 1, 2, \dots$ denote the set of consumers making a transfer during the interval $[t, t + \delta)$. The measure of $A(t, \delta)$ gives the number of transfers from time t to time $t + \delta$. The total amount of resources directed to financial transfers at time t is then given by

$$\gamma Y \lim_{\delta \rightarrow 0} \int_{A(t, \delta)} \frac{1}{\delta} dF(M_0, W_0). \quad (9)$$

The market clearing condition for goods is therefore given by

$$\int c(t, M_0, W_0) dF(M_0, W_0) + g(t) + \gamma Y \lim_{\delta \rightarrow 0} \int_{A(t, \delta)} \frac{1}{\delta} dF(M_0, W_0) = Y. \quad (10)$$

Equilibrium in this economy is defined as prices $P(t)$, $Q(t)$, demands $c(t, M_0, W_0)$, interval between transfers $N_j(M_0, W_0)$, government consumption $g(t)$ and taxes $\tau(t)$ such that (i) $c(t, M_0, W_0)$ solves the maximization problem of each individual (M_0, W_0) , (ii) the government budget constraint holds, and (iii) the market clearing conditions for goods, money and bonds hold.

It is interesting to rewrite the individual budget constraint for $t \geq T_1$ as

$$\sum_{j=1}^{\infty} Q(T_j) \int_{T_j}^{T_{j+1}} P(t) c(t) dt + \sum_{j=1}^{\infty} Q(T_j) P(T_j) \gamma Y \leq W_0 + Q(T_1) K.$$

For this, we use the fact that money holdings are exhausted in the end of each holding period.

The first order condition with respect to consumption in the interior of a holding period is given by

$$e^{-\rho t} u'(c(t)) = \lambda(M_0, W_0) Q(T_j) P(t), \quad T_j < t < T_{j+1}, \quad j = 1, 2, \dots, \quad (11)$$

for consumer (M_0, W_0) ⁹. $\lambda(M_0, W_0)$ is the Lagrange multipliers of the constraint (4). Notice that, since u is concave, if inflation at time t is greater than or equal to $-\rho$ then consumption within a holding period is decreasing. Consumers concentrate consumption in the beginning of a holding period to avoid losing resources for inflation.

Denote $c^+(t)$ and $c^-(t)$ as consumption respectively in the beginning and in the end of a holding period. The first order condition for $T_j, j = 2, 3, \dots$ is, after simplification,

$$\begin{aligned} & \gamma Y [r(T_j) - \pi(T_j)] + \frac{1}{1-\sigma} \left[c^-(T_j) \frac{Q(T_{j-1})}{Q(T_j)} - c^+(T_j) \right] \\ &= \left[c^-(T_j) \frac{Q(T_{j-1})}{Q(T_j)} - c^+(T_j) \right] - r(T_j) \int_{T_j}^{T_{j+1}} c(t) \frac{P(t)}{P(T_j)} dt. \end{aligned} \quad (12)$$

The left hand side of (12) is the marginal gain of increasing the interval N_j while its right hand side is the marginal loss. The marginal gain is given first by postponing the payment of the transfer cost at time T_j . The second term in the left hand appears because consumers smooth consumption within holding periods. The right hand side of (12) reflects the loss in interest foregone of the amount necessary to cover consumption during the holding period beginning at T_j ¹⁰.

The following proposition is useful for the computation of equilibrium.

Proposition 1. The optimal values $c(t, M_0, W_0; Y)$, $K(M_0, W_0; Y)$ and $M^+(t, M_0, W_0; Y)$ are homogeneous of degree one in (M_0, W_0, Y) and the optimal values of the intervals between transfers $N_j(M_0, W_0; Y)$ are homogeneous of degree zero in (M_0, W_0, Y) .

Proof. See appendix.

⁹See appendix for the full characterization of the first order conditions.

¹⁰It is interesting to consider equation (12) under the assumptions of the Baumol-Tobin model. In this case, the interest rate, the price level and consumption are constant. We would have

$$\gamma Y r = Y (e^{rN} - 1) - Y r N$$

and so $\gamma r \approx (rN)^2 / 2 \Rightarrow N \approx \sqrt{2\gamma/r}$.

For the proof, we use the linearity of the budget constraints and the fact that the transfer cost is proportional to income. Since all agents have the same income Y , we drop this identifier and index the individual solutions by (M_0, W_0) .

3. THE ECONOMY IN THE STEADY STATE

We now describe the economy in the steady state. In this case, the values of the nominal interest rate and of the inflation rate are constant. Moreover, the interval between transfers for all consumers are equal. We aggregate individual money holdings and find the aggregate money demand. We find the distribution of M_0 and W_0 such that the economy is in the steady state since $t = 0$. The steady state is unique.

The calibration of the transfer cost parameter γ is done so that the steady state money demand passes through the geometric mean of the nominal interest rates and of the money-income ratios for U.S. during the period 1900-1997. See appendix for the description of the data.

Suppose that the monetary authority sets the growth rate of money creation at a constant α . We look for an equilibrium in which individuals visit the asset market at constant intervals $N_2(M_0, W_0) = N_3(M_0, W_0) = \dots = N$. The values of $T_1(M_0, W_0)$ will be different across consumers and they will depend on (M_0, W_0) . The properties of the economy in the steady state allow us to call the equilibrium of this type a stationary equilibrium. We need first a formal definition.

Definition 1. A *stationary equilibrium* is given by prices r, π , demands $c(t, M_0, W_0)$, intervals between transfers $N_j(M_0, W_0)$, $j = 1, 2, \dots$, a distribution F of (M_0, W_0) , government consumption $g(t)$, and taxes $\tau(t)$ such that

- i. $r(t) = r, \pi(t) = \pi$;
- ii. $N_j(M_0, W_0) = N$ for all (M_0, W_0) and $j = 2, 3, \dots$ is such that consumers maximize utility given prices;
- iii. the distribution of $F(M_0, W_0)$ is such that $T_1(M_0, W_0)$ is uniformly distributed

along $[0, N)$;

- iv. the market clearing conditions for goods, money and bonds hold;
- v. the government budget constraint holds; and
- vi. agents have the same consumption pattern along their holding period. That is,

$$c(T_1(M_0, W_0) + t, M_0, W_0) = c(T_1(M'_0, W'_0) + t, M'_0, W'_0), \quad (13)$$

for all pairs $(M_0, W_0), (M'_0, W'_0)$ in the support of F and all $t \geq 0$.

Equation (13) states that consumption is the same for two different consumers if the time elapsed since the first transfer is the same for both consumers. The only difference across consumers in the stationary equilibrium is their position in the holding period. In particular, consumption just after a transfer is the same for all agents. Denote this value by c_0 . This is also the highest level of consumption within a holding period.

We now characterize the optimal behavior of a consumer given the individual initial endowments M_0 and W_0 . Set $g(t) = 0$ and $\tau(t) = 0$.

We first characterize the decision of $N_j(M_0, W_0)$, $j \geq 2$, and then proceed to the choice of $T_1(M_0, W_0)$. We write $N(r, \rho, \sigma, \gamma)$ to stress the dependence of the steady state interval between transfers to the parameters of the model.

Lemma 1. In the steady state, the level of consumption just after a transfer, c_0 , is

$$c_0(r, Y, \rho, \sigma, \gamma) = Y \left(1 - \frac{\gamma}{N(r, \rho, \sigma, \gamma)} \right) \frac{rN(r, \rho, \sigma, \gamma)/\sigma}{1 - e^{-rN(r, \rho, \sigma, \gamma)/\sigma}}, \quad (14)$$

given the interval between transfers $N(r, \rho, \sigma, \gamma)$.

Proof. See appendix.

The value of c_0 is obtained by the market clearing condition for goods and the consumption profile given by (11) in the steady state. Note that the density function in this case is given by $1/N$.

According to (14) individual consumption is homogeneous of degree one in income.

Consumers direct $\gamma Y/N$ of their resources to transfers. We need $NY > \gamma Y$ to have positive consumption: production during a holding period must be greater than the transfer cost paid to obtain money holdings for the same period.

Consider the case in which the elasticity of intertemporal substitution is high and preferences are close to linear ($\sigma \rightarrow 0$). In this case, the value of N is large and each agent consumes almost his present value of production in the beginning of the holding period. So c_0 is very large relative to Y and $c_0/Y \rightarrow +\infty$.

Proposition 2 gives the value of $N(r, \rho, \sigma, \gamma)$. Proposition 3 assures existence and uniqueness.

Proposition 2. The optimal interval between transfers, $N(r, \rho, \sigma, \gamma)$, is given implicitly by the positive root of the equation

$$-\frac{1 - e^{rN(1-1/\sigma)}}{1 - 1/\sigma} - r \frac{1 - e^{-\rho N} e^{rN(1-1/\sigma)}}{\rho - r(1 - 1/\sigma)} = \rho \gamma \frac{Y}{c_0(N)}, \quad (15)$$

for $\sigma \neq 1$, and

$$rN - \frac{r}{\rho} (1 - e^{-\rho N}) = \rho \gamma \frac{Y}{c_0(N)} \quad (16)$$

for $\sigma = 1$, where $c_0(N)$ is given by Lemma 1.

Proof. See appendix¹¹.

For the proof we use the first order conditions with respect to T_j when $r = \rho + \pi$ and impose $N_j = N$.

Proposition 3. A positive value of $N(r, \rho, \sigma, \gamma)$ exists and is unique for all $\sigma > 0$, $r > 0$, $\rho > 0$ and $\gamma > 0$.

Proof. See appendix.

With the implicit function theorem we obtain the following proposition about properties of the optimal interval between transfers.

¹¹Consumption just after each transfer is directly related to the marginal utility of wealth. Therefore, $\gamma Y/c_0$ is equivalent to the cost in utility terms. To confirm this interpretation, compare (16) to the value of N found in another version of this paper for $\sigma = 1$ and the transaction cost given in utility terms, γ_u : $rN - \frac{r}{\rho} (1 - e^{-\rho N}) = \rho \gamma_u$.

Proposition 4. The optimal value of the interval between transfers is such that (i) $\frac{\partial N}{\partial r} < 0$; (ii) $\frac{\partial N}{\partial \gamma} > 0$; (iii) $\frac{\partial N}{\partial \rho} > 0$; (iv) $N > \gamma$; (v) $\lim_{\gamma \rightarrow 0} N = 0$; (vi) $\lim_{r \rightarrow 0} rN = \varepsilon > 0$.

The interval between transfers is decreasing in the nominal interest rate and increasing in the transfer cost. It increases without bound as the interest rate decreases towards zero. The product rN , however, converges to a small positive constant when r goes to zero. We will see that this will make the real money demand bounded when r goes to zero.

The interval $N(r, \rho, \sigma, \gamma)$ goes to zero if the transfer cost goes to zero. We also obtained numerically that it decreases when the elasticity of intertemporal substitution decreases.

When r and ρ are close to zero, the terms rN and ρN are also close to zero¹². With a second-order Taylor expansion of the terms $e^{-\rho N}$ and $e^{-rN(1-\sigma)/\sigma}$ we obtain

$$N^2 - N\gamma \left(1 - \frac{1}{\sigma}\right) - \frac{2\gamma}{r} = 0.$$

With r small and σ bounded away from zero, the term $2\gamma/r$ is much larger than the term $(1 - 1/\sigma)$. Therefore, the value of N is approximated by

$$N \approx \sqrt{\frac{2\gamma}{r}}.$$

We have the square-root formula for the interval between transfers¹³. The elasticity of intertemporal substitution does not appear in this formula. It reflects the fact that

¹²When $\sigma = 1$, $\lim_{r \rightarrow 0} rN \approx \rho\gamma/(1 + \rho\gamma/2) = 1.5 \times 10^{-4}$ for the parameterization used here. See the proof of proposition 4 in the appendix for details.

¹³The approximation is better the higher is the value of σ . For an idea of the degree of the approximation, the true value of N is higher than the approximated value by 0.3%, 0.4% and 3.7% respectively for $\sigma = 1$, $\sigma = 0.5$, and $\sigma = 0.05$. The other parameters used in this example were $\gamma = 1.791$, $r = 4\%$ p.a. and $\rho = 3\%$ p.a. Note that as σ increases then consumption is less variable and we are closer to the assumptions of the Baumol-Tobin model. Jovanovic (1982) also obtains

the interval between transfers is more sensitive to the nominal interest rate and to the transfer cost.

By proposition 2, the steady state $N(r, \rho, \sigma, \gamma)$ does not depend on the initial conditions M_0 and W_0 , as we need for the stationary equilibrium. The time of the first visit to the asset market, T_1 , depends on these values. Proposition 5 gives the time of the first transfer.

Proposition 5. In the steady state, the first transfer from the brokerage account to the bank account, $T_1(r, \rho, \sigma, \gamma, M_0, W_0)$, is given implicitly by the solution of

$$rT_1 = \log \frac{\lambda}{\mu} + rN,$$

where $\lambda(r, \rho, \sigma, \gamma, W_0, T_1)$ and $\mu(r, \rho, \sigma, \gamma, M_0, T_1)$ are the Lagrange multipliers of (4) and (5) respectively.

Proof. See appendix.

We can now describe the behavior of consumption in the steady state within any holding period. Let $\hat{c}(x)$ denote consumption at the position $x \in [0, N(r, \rho, \sigma, \gamma))$. In the stationary equilibrium, consumption is periodic. That is, \hat{c} is such that

$$\hat{c}(x) = c[x + T_1 + (j - 1)N, M_0, W_0]$$

for all x in $[0, N)$ and for all (M_0, W_0) in the support of F , $j = 1, 2, \dots$. The function $\hat{c}(x)$ is given by

$$\hat{c}(x) = c_0 e^{-\frac{r}{\sigma}x}, \quad 0 \leq x < N(r, \rho, \sigma, \gamma),$$

where c_0 is given by Lemma 1 and Proposition 2.

The ratio of consumption in the beginning of a holding period to consumption in the end of a holding period, $e^{rN/\sigma}$, is approximately equal to $1 + \sqrt{2\gamma r}/\sigma$. If r is close to the square-root formula as an approximation of his model. Lucas (2000) obtains the square-root formula with the McCallum-Goodfriend framework, where time and real money balances interact via a transactions technology.

to zero or σ is high, then consumption is approximately constant. If the transfer cost γ is close to zero then the interval between transfers goes to zero and consumption is also approximately constant. Figure 1 shows the behavior of consumption within a holding period for an arbitrary consumer¹⁴.

Now we move to the money demand. In the steady state we have a uniform distribution of consumers along the interval $[0, N)$. Index consumers by their position in this interval, $n \in [0, N)$ ¹⁵. For $t > jN$, $j = 1, 2, \dots$, consumers with $n \in [0, t - jN)$ will be in their $(j + 1)$ th holding period, and consumers with $n \in [t - jN, N)$ will be in their j th holding period. Aggregate money demand is given by

$$M(t) = \frac{1}{N} \int_0^{t-jN} M(t, n) dn + \frac{1}{N} \int_{t-jN}^N M(t, n) dn,$$

where $M(t, n)$ denotes the individual money demand. Solving the integrals above yields the following proposition.

Proposition 6. In the steady state, the real money demand is given by

$$m(r, Y, \rho, \sigma, \gamma) = \frac{c_0(r, Y, \rho, \sigma, \gamma)}{\rho - r(1 - 1/\sigma)} \left(\frac{1 - e^{-rN/\sigma}}{Nr/\sigma} + e^{-rN/\sigma} \frac{1 - e^{(r-\rho)N}}{(r - \rho)N} \right), \quad (17)$$

where $c_0 = c_0(r, Y, \rho, \sigma, \gamma)$ is given by Lemma 1 and $N = N(r, \rho, \sigma, \gamma)$ is given by proposition 2.

Proof. See appendix¹⁶.

The real money demand is homogeneous of degree one in Y , as c_0 is homogeneous of degree one in Y . Thus, the elasticity of the money demand with respect to income is equal to one. Figures (2) and (3) show the graphs of the ratio of money demand to

¹⁴Notice that consumption can be lower than income in all points of time. The infimum of c_0 is given by $Y(1 - \gamma/N)$. This is the limit of c_0 when $\sigma \rightarrow +\infty$, for $r, \rho, \gamma > 0$. $c_0 > Y$ only if $\sigma < 1$. In particular, $c_0 = Y$ if $r = \rho$ and $\sigma = 1$.

¹⁵Later we will make this more precise by characterizing the values of M_0 and W_0 such that $N_1(M_0, W_0) = n$.

¹⁶We can also use the symmetry of the steady state and the cash in advance constraint to derive the money demand. See appendix for this alternative derivation.

GDP, $M/(PY)$, and the equilibrium value of the interval between transfers. Figures (2) and (3) also contain the data for U.S. during the period 1900 to 1997. Figure (4) shows the elasticities of real money demand and of the interval between transfers with respect to the nominal interest rate. Real money demand is decreasing in the interest rate¹⁷.

Consider the limit of the money demand in (17) when $r \rightarrow 0$. Since the value of the interval between transfers N increases without bound and the product rN converges to a positive constant, money demand converges to Y/ρ when $r \rightarrow 0$. Therefore, the optimal quantity of money is equal to the present value of production, discounted by the intertemporal discount rate¹⁸.

The model is calibrated following the procedure in Lucas (2000). The parameter ρ is set to 3% p.a. and, therefore, a nominal interest rate of 3% p.a. implies zero inflation. The value of γ is chosen so that the money-income ratio passes through the geometric mean of the data for $\sigma = 1$. This implies $\gamma = 1.791$. This value for γ means that consumers pay the equivalent of roughly 2 days of work to transfer resources from their brokerage account to the bank account. See also another possible calibration in the next section.

For the period 1900-1997, the nominal interest rate in U.S. varied from around 0.5% to 15% per year and velocity varied from around 2 to 8 per year. Lucas argues that, between a constant elasticity money demand and a constant semi-elasticity money

¹⁷Note that in models with fixed interval between transfers, real money demand is *increasing* in the interest rate.

¹⁸The behavior of the money demand is similar when r decreases towards zero and when $r = 0$. The model, however, is not well defined when $r = 0$ because the value of N increases towards infinity. In Jovanovic (1982), the fact that consumption is constant makes the transfer period bounded when r decreases towards zero. He then concludes that agents incur a welfare cost even with zero nominal interest rate. Here, $N \rightarrow +\infty$ as $r \rightarrow 0$ even though $m < \infty$. This happens because rN is bounded and this is the relevant measurement to calculate foregone interest costs.

demand, the function that best fits the data is a constant elasticity money demand with elasticity equal to $-1/2$.

According to the graphs in figures (2), (3) and (4) we have the following. (i) The money demand derived in propositions 2 and 6 is close to the Baumol-Tobin money demand. This is clear by figure (4) where the numerically-calculated elasticities are very close to $-1/2$. (ii) The elasticity of intertemporal substitution has a small effect on the money demand. Three values were used to illustrate this: 0.1, 1 and 10. Only when we increase the precision of the graph it is possible to distinguish the three curves. The difference is discernible for the elasticities with respect to the interest rate, but the differences are small. These calculations were also done for various values of ρ and the results were similar.

We calculated numerically the elasticities of real money demand and of the interval N with respect to the transfer cost γ . These values are also around 0.5 for the same elasticities of intertemporal substitution considered above.

We obtain the equilibrium value of the price level at time zero, P_0 , with the real money balances, m , and the initial supply of money, M_0^S . It is immediate to verify that in the steady state inflation is equal to the growth rate of money α and hence $r = \rho + \pi$. Given that the goods and money markets clear, the bonds market also clears by Walras' Law.

We now write the values of M_0 and W_0 that imply a uniform distribution of consumers along the interval $[0, N)$. Let $n \in [0, N)$ index consumers by the time of the first transfer. Define the functions $\tilde{M}_0(n)$ and $\tilde{W}_0(n)$ respectively as the initial amounts of deposits in the bank and brokerage accounts such that a consumer $(\tilde{M}_0(n), \tilde{W}_0(n))$ transfers resources at $t = n, n + N, n + 2N$ etc. After the first withdrawal, the interval between transfers is constant and equal to N .

$\tilde{M}_0(n)$ must be exactly enough to allow the consumer to consume at the steady state rate in the interval $[0, n)$. On the other hand $\tilde{W}_0(n)$ is equal to the present

value of all future transfers from $t = n$ and on, plus the present value of the total transfer cost. Proposition 7 gives the values of $\tilde{M}_0(n)$ and $\tilde{W}_0(n)$.

Proposition 7. Given the values of $N(r, \rho, \sigma, \gamma)$, $c_0(r, Y, \rho, \sigma, \gamma)$ and of the price level at $t = 0$, P_0 , the values of the initial money holdings in the bank account, $\tilde{M}_0(n)$, and of the initial wealth in the brokerage account, $\tilde{W}_0(n)$, such that the consumer chooses $T_1 = n$ are given by

$$\tilde{M}_0(n) = P_0 c_0 e^{\frac{r}{\sigma} n} e^{-\frac{r}{\sigma} N} \frac{1 - e^{-(\rho - r(1 - 1/\sigma))n}}{\rho - r(1 - 1/\sigma)}$$

and

$$\tilde{W}_0(n) = \frac{e^{-\rho n}}{1 - e^{-\rho N}} \left(P_0 c_0 \frac{1 - e^{-(\rho - r(1 - 1/\sigma))N}}{\rho - r(1 - 1/\sigma)} + P_0 \gamma Y \right),$$

for $n \in [0, N(r, \rho, \sigma, \gamma))$.

Proof. See appendix.

$\tilde{M}_0(n)$ is increasing in n while $\tilde{W}_0(n)$ is decreasing in n . A consumer with more initial money holdings and less initial bond holdings will make the first transfer later than a consumer with less money and more bonds.

4. WELFARE COST OF INFLATION

When the nominal interest rate is positive, agents deviate real resources from consumption to financial services to maintain the optimal level of money holdings.

The optimal monetary policy is to set the money growth rate at $-\rho$. In this case, the inflation rate is equal to $-\rho$ and the nominal interest rate is equal to zero. This is in accordance to Friedman (1969). But how much does a positive nominal interest rate cost to society?

The welfare cost of inflation¹⁹ is defined as the percentage income compensation required to leave consumers indifferent between $r > 0$ and $r = 0$. Let $U^T(r, Y)$ be the total welfare, derived from all consumers with equal weight, for an economy with

¹⁹Or, the “welfare cost of a positive nominal interest rate”.

income Y and nominal interest rate $r > 0$. The welfare cost $w(r)$ is defined as the solution to

$$U^T [r, (1 + w(r)) Y] = U^T (0, Y). \quad (18)$$

In the present model, consumption follows $c_0 e^{-rt/\sigma}$ within a holding period for each agent in the steady state. At each time t we have consumers along the interval $[0, N]$ in different positions of their holding period. To calculate total utility, we first sum total utility for each time t and then we sum this value from $t = 0$ to infinity. This yields

$$U^T (r, Y) = \frac{1}{\rho} \frac{1}{N} \int_0^N \frac{(c_0(r, Y, \rho, \sigma, \gamma; Y) e^{-rt/\sigma})^{1-\sigma}}{1-\sigma} dt, \quad (19)$$

where $c_0(r, Y, \rho, \sigma, \gamma; Y)$ is given by (14). Instead of comparing $r > 0$ to $r = 0$, it is useful to allow a comparison with $\bar{r} > 0$. Using (18) and (19) with $\bar{r} > 0$ we have the following proposition.

Proposition 8. The income compensation to leave consumers indifferent between $r > 0$ and $\bar{r} > 0$ is given by

$$1 + w(r) = \left[\left(1 - \frac{\gamma}{\bar{N}}\right) \frac{\bar{r}\bar{N}/\sigma}{1 - e^{-\bar{r}\bar{N}/\sigma}} \right] \left[\left(1 - \frac{\gamma}{N}\right) \frac{rN/\sigma}{1 - e^{-rN/\sigma}} \right]^{-1} \\ \times \left[\frac{rN}{\bar{r}\bar{N}} \frac{1 - e^{\bar{r}\bar{N}(1-1/\sigma)}}{1 - e^{rN(1-1/\sigma)}} \right]^{\frac{1}{1-\sigma}},$$

for $\sigma \neq 1$, and

$$1 + w(r) = \left[\left(1 - \frac{\gamma}{\bar{N}}\right) \frac{r\bar{N}}{1 - e^{-r\bar{N}}} \right] \left[\left(1 - \frac{\gamma}{N}\right) \frac{rN}{1 - e^{-rN}} \right]^{-1} \\ \times \exp \left(\frac{rN}{2} - \frac{\bar{r}\bar{N}}{2} \right),$$

for $\sigma = 1$.

Proof. See appendix.

Figure (5) shows the welfare cost for r between zero and 100% p.a. Bringing down the nominal interest rate from 15% p.a. to 0% p.a yields an increase in welfare

equivalent to a permanent increase in income of 2%. Figure (6) shows the welfare compensation in units that help its interpretation in terms of inflation. The welfare cost was set to zero for $r = 3\%$ p.a., the nominal interest rate compatible with zero inflation for the U.S. Bringing down inflation from 10% p.a. to 0% p.a. is equivalent to a permanent increase in income of 1%.

The variable m in the model was interpreted as $M1$ in the calibration above. With the transfer cost γ set to 1.791 working days per transfer we have velocity of around 4 per year and an interval between transfers of around six months when $r = 4\%$ p.a. This size for the interval may be viewed as excessive for $M1$ but it is not for broader definitions of money. As pointed out by Vissing-Jorgensen (2002) a large fraction of agents trade assets with higher yields very infrequently, less than once a year. Moreover, in the data summarized by Alvarez, Atkeson and Edmond (2002), the opportunity cost in terms of foregone interest is similar for currency, savings deposits and time deposits ($M2$ less retail money market mutual funds).

We then reinterpret m according to this broader definition of money and recalibrate the parameter γ . Alvarez, Atkeson and Edmond find that the opportunity cost of holding $M2$ less retail money market is about 200 basic points. Therefore, we set γ such that velocity equals 1.6 per year when the interest rate is 2% p.a., that is, the average nominal interest rate of 4% p.a. minus 2% p.a. in opportunity costs. We obtain $\gamma = 5.9$ working days per transfer. This implies an interval between transfers of around 470 days when the interest rate is 2% p.a. This lower frequency of transfers is compatible with this broader definition of money. Using this calibration, bringing down inflation from 10% p.a. to 0% p.a. is equivalent to a permanent increase in income of 1.75%.

We compare the welfare compensation implied by this model with the one calculated directly from the Baumol-Tobin money demand. As the interest foregone cancel out across consumers, the total cost for society is given only by the transfer cost, $\gamma Y/N$.

With $N = \sqrt{2\gamma/r}$, the amount of resources deviated from consumption is equal to $Y\sqrt{r\gamma/2}$. Therefore, total welfare in the Baumol-Tobin model under the nominal interest rate r is given by

$$U^{BT}(r, Y) = Y \left(1 - \sqrt{\frac{r\gamma}{2}} \right).$$

Using equation (18), the welfare cost of inflation using the Baumol-Tobin money demand is given by $w^{BT}(r) = \sqrt{\frac{r\gamma}{2}} (1 - \sqrt{\frac{r\gamma}{2}})^{-1}$. This value is very close to the welfare compensation for the model in this paper, as illustrated in figure (5)²⁰.

This calculation does not take into account the fact that $c \neq Y$. Use $N = \sqrt{\frac{2\gamma Y}{r c}}$ and $c/Y = 1 - \gamma/N$. Defining $x \equiv c/Y$ we write

$$x + \sqrt{\frac{r\gamma}{2}}\sqrt{x} - 1 = 0.$$

Solving this equation for x we calculate $w^{BT(c/Y)}(r)$ by $xY(1 + w^{BT(c/Y)}(r)) = Y$. This calculation is also very close to the welfare compensation for the present model, as it is illustrated in figure (5).

We saw that the money demand is virtually unaffected by the coefficient of risk aversion σ . The welfare compensation, however, could be more sensible to σ because consumption decreases at the rate r/σ within holding periods. Thus, a lower σ implies more variation on the slope of the consumption profile when r changes. We have that this effect is small. The welfare compensation does not depend on the coefficient of risk aversion for all practical purposes.

The Baumol-Tobin model does not take into account the change in consumption within a holding period. So the compensation would be too small in the Baumol-Tobin model. One way of thinking about a correction of the Baumol-Tobin compensation is to include a risk premium compensation.

²⁰When r is small, $w(r) \approx \sqrt{\frac{r\gamma}{2}}$. This approximation is equal to the area under the money demand curve.

The risk premium is given by

$$RP = \frac{1}{2} var(\varepsilon) \sigma,$$

where ε is the dispersion, in logs, and σ is the coefficient of relative risk aversion. In our case we have $c(x) = c_0 e^{-\frac{r}{\sigma}x}$ and consumers are distributed uniformly along the $[0, N)$ interval. So,

$$var = \frac{(\log c_0 - \log c_0 e^{-\frac{r}{\sigma}N})^2}{12} = \frac{(rN)^2}{12\sigma^2}.$$

The risk premium is then given by $RP = \frac{r\gamma}{12\sigma}$. Summing the transfer cost and the risk premium, the total compensation can be written as

$$\sqrt{\frac{r\gamma}{2}} + \frac{r\gamma}{12\sigma}.$$

When r is small, the term $\sqrt{r\gamma}$ is much larger than $r\gamma$ and therefore the coefficient of risk aversion affects little the welfare compensation.

Finally, note that the Baumol-Tobin demand implies an elasticity of $w(r)$ with respect to r close to 1/2. The elasticity of $w(r)$ with respect to r for the model presented here is also close to 1/2 for all coefficients of relative risk aversion.

5. A MONETARY SHOCK

The real interest rate is a function of real magnitudes. In the case of this paper, it is a function of the intertemporal discount ρ only. If this parameter is fixed then the real interest rate is also fixed. Thus, any change in the nominal interest rate should be accompanied by an equal change in the price level. The two effects interact to maintain the real interest rate fixed.

A higher nominal interest rate will make agents realize that they have more real money balances than they would like to have. As a consequence, each individual will try to spend more than with a lower interest rate. As total demand cannot be

higher than total output, and output is constant, the price level increases. The price change will curb any upward movement in demand. This explanation for the increase in prices can be found, for example, in Friedman (1969)²¹.

Therefore, eventually the price level will adapt to the new interest rate. Nothing indicates, however, how fast the price level will reach its new path. In the present model, the transfer cost makes agents economize the use of money. In order to avoid making a transfer too soon, agents will not increase their consumption as they would without the transfer cost. Consequently, prices do not change instantaneously with the change in the nominal interest rate.

This behavior can be seen in figure (7). It contains the price in logs after an increase in the nominal interest rate of one percentage point. The initial nominal interest rate is equal to the discount parameter $\rho = 3\%$ p.a. Hence, this change implies a 1% p.a. inflation rate in the new steady state from an economy with zero inflation.

According to the simulation results displayed in figure (7), the price level drops at the moment of the shock. Moreover, the inflation rate is initially lower than its steady state value of 1% p.a.

After about six months, however, two groups of consumers with different consumption patterns meet. The first group is composed of those who had substantial money holdings at time zero and have not made a transfer after the shock. The second group is formed by the consumers who have already made a transfer and are now doing their second transfer. When the two groups meet there is a fast increase in the price level, otherwise, demand would be much higher because the second group consumes at a faster rate. After the nominal interest rate shock, therefore, the model predicts an initial price drop, followed by relatively low inflation, and an inflation overshooting

²¹Friedman studies a permanent increase in the individual money holdings, not an increase in the nominal interest rate. But in both cases agents have more real balances than they would like to have.

after six months. The effects of the change in the nominal interest rate can be felt long after the shock.

The price level does not adjust instantaneously to the interest rate shock. This addresses the apparent price stickiness after a change in the nominal interest rate reported, for example, in Christiano, Eichenbaum and Evans (1999).

Grossman and Weiss (1983) and Rotemberg (1984) study the effects of monetary shocks in economies where agents transfer funds periodically. A more recent model with this feature is Alvarez, Atkeson and Edmond (2002). In these models, the interval between transfers is exogenously set²².

We now define the structure to study a nominal interest shock. We want to approximate a situation in which the economy is initially in the steady state and the interest rate changes unexpectedly. The money holdings at the time of the shock are equal to the values compatible with the economy in the steady state.

Suppose the existence of two possible states denoted by $s = 1, 2$. In state 1 the government sets the nominal interest rate at r_1 for all periods. In state 2, the nominal interest rate is set at r_2 . The realization of the state occurs at time zero.

Agents trade bonds contingent on the realization of the states. Denote by θ the probability of the state being equal to 1. Trade occurs through the brokerage account. Only deposits in the brokerage account change according to the state. Money holdings are not state contingent.

The maximization problem of each agent is given by

$$\max \theta \sum_{j=0}^{\infty} \int_{T_j(1)}^{T_{j+1}(1)} e^{-\rho t} u(c(t, 1)) dt + (1 - \theta) \sum_{j=0}^{\infty} \int_{T_j(2)}^{T_{j+1}(2)} e^{-\rho t} u(c(t, 2)) dt \quad (20)$$

²²The difficulty of allowing the interval between transfers to change optimally is the relation between aggregate variables and individual behavior. See, for example, Caplin and Leahy (1991, 1997).

subject to

$$\sum_{s=1,2} \left[\sum_{j=1}^{\infty} Q(T_j(s), s) \int_{T_j(s)}^{T_{j+1}(s)} P(t, s) c(t, s) dt + \sum_{j=1}^{\infty} Q(T_j(s), s) P(T_j(s), s) \gamma Y \right] = W_0 + \sum_{s=1,2} Q(T_1(s), s) K(s), \quad (21)$$

$$\int_0^{T_1(1)} P(t, 1) c(t, 1) dt + K(1) = M_0, \quad (22)$$

$$\int_0^{T_1(2)} P(t, 2) c(t, 2) dt + K(2) = M_0. \quad (23)$$

Where W_0 denotes deposits in the brokerage account,

$$W_0 \equiv B_0 + \sum_{s=1,2} \int_0^{\infty} Q(t, s) P(t, s) Y dt.$$

Agents differ by their money and bond holdings M_0 and W_0 .

Note that we have a single budget constraint after the first transfer. Agents are able to trade bonds using the resources of the brokerage account. But their money holdings do not change with the state.

In section 3, we start from an economy with constant nominal interest rate and we move backwards to find a distribution of M_0 and W_0 over consumers. This distribution is such that the economy is in equilibrium since time zero and hence it is in fact compatible with a constant nominal interest rate.

Following the same reasoning, we seek a distribution of M_0 and W_0 such that the economy is in equilibrium after the realization of the state. The difference is that now the price level will not increase at a constant inflation rate as in the stationary equilibrium. For either state 1 or state 2, the inflation rate will not be constant.

The task of find M_0 and W_0 for each agent is facilitated if we assume that the probability θ is arbitrarily close to one. In this case, M_0 and W_0 are very close to the values such that the economy is under state one without the possibility of starting

in state two. These are the amounts of money and bonds calculated in section 3. Moreover, the Lagrange multiplier associated to the present value budget constraint (21), denoted by λ , is the same as if the economy had only state 1 as the possible starting state. Recall that λ is the same across agents in the steady state.

To summarize, we first calculate the money holdings and bond holdings such that the economy is in the steady state under the nominal interest rate r_1 . With this, we calculate the value of λ associated to this state. We then calculate individual consumption using the first order conditions of the maximization given above, with the value of M_0 for each agent and the value of λ .

In the steady state, agents can be distributed over the interval $[0, N)$, where the interval between transfers N is given by the parameters ρ , σ and γ , and the nominal interest rate r . The values of M_0 and W_0 are then given by $\tilde{M}_0(n)$ and $\tilde{W}_0(n)$, for $n \in [0, N)$, stated in proposition 7.

The individual consumption depends on the price level $P(t)$. We need to find, for each agent, the optimal consumption and the optimal timing of a transfer. We have these values by the two following propositions²³.

Proposition 9. The timing of a transfer $T_j(n)$ of consumer $n \in [0, N)$ is given by

$$\begin{aligned} -\frac{e^{(1-1/\sigma)r_2N_j(n)} - 1}{1 - 1/\sigma} + \frac{\gamma Y [r_2 - \pi(T_j(n))]}{c^+(T_j(n))} \\ = -r_2 \frac{e^{\rho T_j(n)/\sigma}}{P(T_j(n))^{1-1/\sigma}} \int_{T_j(n)}^{T_{j+1}(n)} e^{-\rho t/\sigma} P(t)^{1-1/\sigma} dt, \end{aligned}$$

where $c^+(T_j(n)) = [\lambda e^{\rho T_j(n)} Q(T_j(n)) P(T_j(n))]^{-1/\sigma}$ and $j = 2, 3, \dots$. The consumption level $c(t, n)$ of agent n at time t is given by

$$c(t, n) = \left[\frac{e^{-\rho t}}{\lambda Q(T_j) P(t)} \right]^{1/\sigma}, \quad t \in (T_j(n), T_{j+1}(n)), \quad j = 1, 2, \dots$$

²³Recall that $Q(t) = e^{-rt}$ and that $T_j(n) \equiv N_1 + \dots + N_j$. We also have that $\lambda = \frac{1}{P_0 c_0^\sigma}$, where P_0 is the price level in the initial steady state (before the shock) and c_0 is the level of consumption just after a transfer in the initial steady state. See section 3 for notation.

For the logarithmic case, $\sigma = 1$, the values of $T_j(n)$, $j = 2, 3, \dots$, are given by

$$-r_2 N_j(n) + \frac{\gamma Y [r_2 - \pi(T_j(n))]}{c^+(T_j(n))} = -r_2 \frac{1 - e^{-\rho N_{j+1}(n)}}{\rho}.$$

Proof. See appendix.

Proposition 10. The first transfer $T_1(n)$ of consumer $n \in [0, N)$ is given by

$$\left[\left(\frac{\mu(n)}{\lambda} \right)^{1-1/\sigma} e^{(1-1/\sigma)r_2 T_1(n)} - 1 \right] \frac{\sigma}{1-\sigma} + \frac{\gamma Y [r_2 - \pi(T_1(n))]}{c^+(T_1(n))} - \frac{r_2 K(n)}{P(T_1(n)) c^+(T_1(n))} = -r_2 \frac{e^{\rho T_1(n)/\sigma}}{P(T_1(n))^{1-1/\sigma}} \int_{T_1(n)}^{T_2(n)} e^{\rho t/\sigma} P(t)^{1-1/\sigma} dt,$$

for $\sigma \neq 1$, and

$$-r_2 T_1 - \log \frac{\mu(n)}{\lambda} + \frac{\gamma Y [r_2 - \pi(T_1(n))]}{c^+(T_1(n))} - \frac{r_2 K(n)}{P(T_1(n)) c^+(T_1(n))} = -r_2 \frac{1 - e^{-\rho N_2}}{\rho},$$

for $\sigma = 1$. $\mu(n)$ is the Lagrange multiplier associated to the budget constraints (22) and (23).

Proof. See appendix.

We proceed numerically in order to obtain the values of the timing of the transfers and consumption. We assume that after the J th transfer each consumer chooses $N_{J+1} = N'$, where N' is the interval between transfers under the new nominal interest rate r_2 . We also start with a guess of the price level $P(t)$.

Therefore, propositions 9 and 10 imply a system of J equations and J unknowns given by the intervals N_1, \dots, N_J . Once we determine the values of the N_j 's, we also obtain the consumption levels of each agent.

We discretize the interval $[0, N)$ to $\{n_1, n_2, \dots, n_{\max}\}$ where $n_1 = 0$ and $n_{\max} < N$. For each time t we must have, in equilibrium,

$$\frac{1}{n_{\max}} \sum_n c(t, n) + \frac{1}{n_{\max}} \gamma Y \times \text{Number of Transfers}(t) = Y. \quad (24)$$

The left hand side of this equation is equal to aggregate demand. Their components are consumption and resources used to transfer assets from the brokerage account to the bank account. The right hand side is equal to aggregate supply.

If the equilibrium condition in (24) is not satisfied, we change the price level at time t and recalculate the consumption levels and transfer times. We continue to do this until the difference between demand and supply is smaller than a preestablished value for every t . If demand is higher than supply, we increase the price level at t . If demand is lower than supply, we decrease the price level.

The results reported in this paper are for the simulations with $J = 40$ and a number of consumers such that the difference $n_{i+1} - n_i$ is given by 0.20 days. This implies 1,047 consumers for the parameters used. In particular, we assume utility to be logarithmic and $\gamma = 1.791$. The nominal interest rate increases from 3% to 4% p.a. The number of transfers at t is calculated summing the agents with $T_j(n)$ such that $t \leq T_j(n) < t + 1$. The maximum difference between demand and supply after the last iteration is given by 2.2% for a total of 150 iterations^{24,25}.

Figure (8) shows the behavior of the price level after the shock for the whole simulation period. It shows that prices initially drop and, after about six months of relative low inflation, enter a cyclical process that gradually decreases in amplitude towards its steady state level.

The behavior of the price level obtained in this model is very different from the one obtained in Fusselman and Grossman (1989). They study a model with transfer cost in utility terms and logarithmic utility. In that case, the price level does not affect consumers because it disappears from the first order conditions in the logarithmic case. Only the higher nominal interest rate affects consumption. Therefore, the cyclical variation in prices is sustained indefinitely.

²⁴Several other simulations were performed with different number of intervals N_j (different J 's), number of consumers, transfer costs, and initial guesses for prices. These changes do not affect the qualitative behavior of the price level or of the other equilibrium variables.

²⁵Other parameters used in the simulation: $\gamma = 1.791$, $\sigma = 1$, $\rho = 3\%$ p.a. See chapter 3 for the calibration of γ . When we fix J we implicitly set the time for the steady state. For $J = 40$ we have that the steady state is assumed to be reached in 19.9 years.

What is making prices to smooth gradually in the present model is the change in behavior caused by the variation of prices. If prices are unusually high during a certain period, agents avoid making a transfer because they will pay a higher transfer cost. This redistributes consumers in a way that the number of transfers does not continue indefinitely to be concentrated over some intervals. This redistribution is slow, even five years after the shock the economy still experiences strong variation in the price level.

The rate of inflation after the shock implied by the price series is in figures (9) and (10)²⁶. The shock to the interest rate causes a strong variation in the inflation rate. Gradually, the peaks and troughs of the inflation rate converge towards its steady state level of 1% p.a.

Figures (11) and (12) show respectively the nominal demand and the money-income ratio after the shock. In the second steady state, the nominal money demand increases at the steady state rate of inflation, 1% p.a. According to the simulations, we find that the nominal money demand decreases about 13% during the first 6 months after the shock. We have then a long period of money demand cycles with high amplitude. The cycles of the nominal money demand seem to converge in around 14 years after the shock, after all consumers made about 28 transfers. We still find, however, a new series of cycles, with smaller amplitude after this period²⁷.

The money-income ratio or, equivalently, the real money demand, has a similar behavior as observed in figure (12). Note that the real money demand must eventually decrease with a higher nominal interest rate. According to the simulations, the real money demand decreases around 12% in the first six months. A little less than the

²⁶In order to filter the variation of the price level caused by the numerical simulation, the price series in this figure was filtered for the calculation of inflation. It was used the Hodrick-Prescott filter. Since the original series is in days, the parameter λ was set to 400×365 .

²⁷It is possible that these new cycles are caused by the numerical procedure, magnified by the continuous increase in the money demand of 1% p.a.

15% decrease predicted by the $-1/2$ elasticity after the 1 percentage point increase in the nominal interest rate²⁸. After that, the real money demand shows cycles of high amplitude for a long period, but apparently converge to a lower value, with cycles of smaller amplitude²⁹.

6. CONCLUSION

We present a Baumol-Tobin model in general equilibrium to measure the welfare cost of inflation and to calculate the effects on prices after an interest rate shock.

Spending is constant in Baumol and Tobin. But even when agents exhibit high elasticity of intertemporal substitution and consumption varies considerably within holding periods, the optimal choice of the interval and the aggregate money demand are close to the ones in Baumol and Tobin.

The monetary policy that maximizes welfare is to decrease money supply at the rate of intertemporal discounting, ρ . This sets the nominal interest rate to zero. When the nominal interest rate decreases towards zero, money demand increases to the present value of output, Y/ρ .

The welfare cost calculations could differ substantially according to the intertemporal elasticity. But the calculations yield that consumers are affected in the same way by inflation in the steady state, independently of their elasticity of intertemporal substitution.

The source of welfare loss caused by inflation in this model is the deviation of resources from consumption to the management of money holdings. But other sources

²⁸Since the beginning interest rate is equal to 3%. An increase to 4% is a 33% increase in the nominal interest rate. The real money demand decreases 13% after 194 days, the lowest point in the first year.

²⁹As for the nominal money demand, it is possible that these new cycles are caused by the numerical procedure.

of welfare loss may be more important, specially for high inflation rates. One of them is the difficulty in obtaining information through the price mechanism, as pointed out by Harberger (1998).

The model is suitable to study the effects of a change in the nominal interest rate when agents decide the moment to exchange bonds for money. The adjustment to the new steady state involves a slow increase in the inflation rate, as opposed to the instantaneous adjustment found in most models. The convergence to the new steady state is slow. According to the simulations with the calibrated model, inflation varies high above and below its steady state level after 5 years after the shock. The path of prices and consumption during the transition leads to the study of the gain in welfare of a price stabilization program.

APPENDIX A - FIRST ORDER CONDITIONS

The Lagrangian of the problem in (1), (4) and (5) is

$$\begin{aligned} \mathcal{L} = & \sum_{j=0}^{\infty} \int_{T_j}^{T_{j+1}} e^{-\rho t} u(c(t)) dt + \lambda(M_0, W_0) \left[W_0 + Q(T_1) K(M_0, W_0) \right. \\ & \left. - \sum_{j=1}^{\infty} Q(T_j) \int_{T_j}^{T_{j+1}} c(t, M_0, W_0) P(t) dt - \sum_{j=1}^{\infty} Q(T_j) P(T_j) Y \gamma \right] \\ & + \mu(M_0, W_0) \left[M_0 - \int_0^{T_1} P(t) c(t, M_0, W_0) dt - K(M_0, W_0) \right] \end{aligned}$$

where we use the fact that $M^+(T_j) = \int_{T_j}^{T_{j+1}} P(t) c(t, M_0, W_0) dt$ and $T_j = T_j(M_0, W_0)$.

The first order conditions are given by the equations below.

$c(t, M_0, W_0)$:

$$\begin{aligned} e^{-\rho t} u'(c(t, M_0, W_0)) &= \lambda(M_0, W_0) Q(T_j) P(t), \quad t \in (T_j, T_{j+1}), \\ e^{-\rho T_j} u'(c^+(T_j, M_0, W_0)) &= \lambda(M_0, W_0) Q(T_j) P(T_j), \\ e^{-\rho T_{j+1}} u'(c^-(T_{j+1}, M_0, W_0)) &= \lambda(M_0, W_0) Q(T_j) P(T_{j+1}), \end{aligned}$$

for $j = 1, 2, \dots$;

$$\begin{aligned} e^{-\rho t} u'(c(t, M_0, W_0)) &= \mu(M_0, W_0) P(t), \quad t \in (0, T_1), \\ u'(c^+(0, M_0, W_0)) &= \mu(M_0, W_0) P(0), \\ e^{-\rho T_1} u'(c^-(T_1, M_0, W_0)) &= \mu(M_0, W_0) P(T_1). \end{aligned}$$

T_1 :

$$\begin{aligned} &e^{-\rho T_1} u(c^-(T_1)) - e^{-\rho T_1} u(c^+(T_1)) \\ &= \lambda \left[-\dot{Q}(T_1) K + \dot{Q}(T_1) \int_{T_1}^{T_2} c(t) P(t) dt - Q(T_1) c^+(T_1) P(T_1) \right] \\ &\quad + \lambda Y \gamma \left[\dot{Q}(T_1) P(T_1) + Q(T_1) \dot{P}(T_1) \right] \\ &\quad + \mu P(T_1) c^-(T_1); \end{aligned}$$

$T_j, j = 1, 2, \dots$:

$$\begin{aligned} e^{-\rho T_j} u(c^-(T_j)) - e^{-\rho T_j} u(c^+(T_j)) &= \lambda \left[\dot{Q}(T_j) \int_{T_j}^{T_{j+1}} c(t) P(t) dt \right. \\ &\quad \left. - Q(T_j) c^+(T_j) P(T_j) + Q(T_{j-1}) c^-(T_j) P(T_j) \right] \\ &\quad + \lambda \gamma Y \left[\dot{Q}(T_j) P(T_j) + Q(T_j) \dot{P}(T_j) \right]. \end{aligned}$$

K :

$$Q(T_1) \lambda(M_0, W_0) - \mu(W_0, M_0) \leq 0 \quad (= 0 \text{ if } K > 0);$$

and the budget constraints.

With CRRA utility, $u'(c(t)) c(t) = (1 - \sigma) u(c(t))$ for $\sigma \neq 1$. Using this in the first order condition for $T_j, j = 2, 3, \dots$, we have, after simplification

$$\begin{aligned} \gamma Y [r(T_j) - \pi(T_j)] + \frac{1}{1 - \sigma} \left[\frac{Q(T_{j-1})}{Q(T_j)} c^-(T_j) - c^+(T_j) \right] \\ = \left[\frac{Q(T_{j-1})}{Q(T_j)} c^-(T_j) - c^+(T_j) \right] - r(T_j) \int_{T_j}^{T_{j+1}} \frac{P(t) c(t)}{P(T_j)} dt. \end{aligned}$$

If $\sigma = 1$, we obtain

$$\gamma Y [r(T_j) - \pi(T_j)] + c^+(T_j) \log \frac{Q(T_j)}{Q(T_{j-1})} = -r(T_j) \int_{T_j}^{T_{j+1}} \frac{c(t) P(t)}{P(T_j)} dt.$$

Using the budget constraint and the first order condition with respect to consumption, the value of $\lambda(M_0, W_0)$ is given by

$$\begin{aligned} \lambda(M_0, W_0) &= \sum_{j=1}^{\infty} \int_{T_j}^{T_{j+1}} e^{-\rho t} [c(t)]^{1-\sigma} dt \\ &\quad \times \left[W_0 + Q(T_1) K - \gamma Y \sum_{j=1}^{\infty} Q(T_j) P(T_j) \right]^{-1} \end{aligned}$$

for $\sigma \neq 1$, and

$$\lambda(M_0, W_0) = \frac{e^{-\rho T_1}}{\rho} \left[W_0 + Q(T_1) K - \gamma Y \sum_{j=1}^{\infty} Q(T_j) P(T_j) \right]^{-1} \quad (25)$$

for $\sigma = 1$. Working analogously for $\mu(M_0, W_0)$ using the budget constraint for $0 \leq t < T_1$, we obtain

$$\mu(M_0, W_0) = \frac{1}{M_0 - K} \left[\int_0^{T_1} e^{-\rho t} [c(t)]^{1-\sigma} dt \right] \quad (26)$$

for $\sigma \neq 1$, and

$$\mu(M_0, W_0) = \frac{1}{M_0 - K} \frac{1 - e^{-\rho T_1}}{\rho}$$

for $\sigma = 1$.

APPENDIX B - PROOFS

Proposition 1

Proof. Consider the solution $c(t, M_0, W_0; Y)$, $K(M_0, W_0; Y)$, and $N_j(M_0, W_0; Y)$ of a consumer with (M_0, W_0, Y) . Multiply (M_0, W_0, Y) by $h > 0$. The values $hc(t, M_0, W_0; Y)$, $hK(t, M_0, W_0; Y)$, and $T_j(M_0, W_0; Y)$ satisfy the budget constraint of the new problem. By the equations for the Lagrange multipliers, we see that

$\lambda(W_0, M_0; Y)$ and $\mu(W_0, M_0; Y)$ are homogeneous of degree $(-\sigma)$ in (W_0, M_0, Y) . Hence, $hc(t, M_0, W_0; Y)$, $hK(t, M_0, W_0; Y)$, and $T_j(M_0, W_0; Y)$ satisfy the first order conditions for consumption, for T_j and for K . For money holdings, we have $M^+(T_j, hM_0, hW_0; hY) = hM^+(T_j, M_0, W_0; Y)$ using

$$M^+(T_j, M_0, W_0; Y) = \int_{T_j}^{T_{j+1}} P(t) c(t, M_0, W_0; Y) dt. \blacksquare$$

Lemma 1

Proof. Consumption within a holding period is given by $c(t) = c_0 e^{-\frac{r}{\sigma}(t-T_j)}$, $T_j \leq t < T_{j+1}$ using its first order condition. In the steady state, total consumption at each time t is given by

$$\frac{1}{N(r, \rho, \sigma, \gamma)} \int_0^{N(r, \rho, \sigma, \gamma)} c_0 e^{-\frac{r}{\sigma}x} dx.$$

Hence the market clearing condition in the steady state implies

$$\frac{1 - e^{-\frac{r}{\sigma}N(r, \rho, \sigma, \gamma)}}{rN(r, \rho, \sigma, \gamma)/\sigma} c_0 + \frac{\gamma Y}{N(r, \rho, \sigma, \gamma)} = Y$$

$$\Rightarrow c_0 = Y \left(1 - \frac{\gamma}{N(r, \rho, \sigma, \gamma)} \right) \frac{rN(r, \rho, \sigma, \gamma)/\sigma}{1 - e^{-rN(r, \rho, \sigma, \gamma)/\sigma}}. \blacksquare$$

Proposition 2

Proof. The first order condition with respect to T_j in the steady state implies, after simplification,

$$-\frac{\sigma}{1-\sigma} [c^+(T_j) - c^-(T_j) e^{rN_j}] = -r \int_{T_j}^{T_{j+1}} c(t) \frac{P(t)}{P_0 e^{\pi T_j}} dt + \gamma Y (r + \pi).$$

Using $c(t) = c_0 e^{-\frac{r}{\sigma}(t-T_j)}$, $j = 1, 2, \dots$, $r = \rho + \pi$, and simplifying, yields

$$-\frac{1}{1-1/\sigma} [1 - e^{rN_j(1-1/\sigma)}] - \rho \gamma \frac{Y}{c_0} = r \frac{1 - e^{-\rho N_{j+1}} e^{rN_{j+1}(1-1/\sigma)}}{\rho - r(1-1/\sigma)}.$$

With $N_j = N_{j+1} = N$ we have the desired result. The steps for $\sigma = 1$ are analogous.

Note also that

$$\lim_{\sigma \rightarrow 1} -\frac{1}{1-1/\sigma} [1 - e^{rN(1-1/\sigma)}] = rN. \blacksquare$$

Proposition 3

Proof. Define the functions $a, b, G : (\gamma, +\infty) \rightarrow R$ for $\sigma \neq 1$ by

$$a(N) = \left(\frac{1 - e^{-rN/\sigma}}{rN/\sigma} \right) \left(1 - \frac{\gamma}{N} \right)^{-1},$$

$$b(N) = \frac{\sigma}{1 - \sigma} \left(1 - e^{-rN(1-\sigma)/\sigma} \right) - r \frac{1 - e^{-rN(1-\sigma)/\sigma} e^{-\rho N}}{\rho + r(1 - \sigma)/\sigma},$$

and

$$G(N) = b(N) - \rho\gamma a(N).$$

Note that $a = Y/c_0$. The optimal interval N^* is such that $G(N^*) = 0$.

We have $\lim_{N \rightarrow \gamma_+} a(N) = +\infty$ and $\lim_{N \rightarrow \gamma_+} b(N) = 0$. Therefore,

$$\lim_{N \rightarrow \gamma_+} G(N) = -\infty.$$

We also have that $\lim_{N \rightarrow \infty} a(N) = 0$. Intuitively, if $N \rightarrow \infty$ then the consumer is consuming almost his total present value in the beginning of the holding period. So c_0 is very large, and $a = Y/c_0 \rightarrow 0$. Moreover, $b'(N) > 0$, and $a'(N) < 0$ because $e^{rN/\sigma} > 1 + Nr/\sigma - \gamma r/\sigma$. Hence,

$$G'(N) = b'(N) - \rho\gamma a'(N) > 0.$$

Even though G is always increasing, it can be the case that $\lim_{N \rightarrow +\infty} G(N) < 0$. This possibility is ruled out for $\sigma \geq 1$ because $\lim_{N \rightarrow \infty} b'(N) = +\infty$ for $\sigma > 1$ and $\lim_{N \rightarrow \infty} b'(N) = r$ for $\sigma = 1$. On the other hand, $\lim_{N \rightarrow \infty} b'(N) = 0$ for $0 < \sigma < 1$. In this case, $\lim_{N \rightarrow +\infty} G(N) = \lim_{N \rightarrow +\infty} b(N) = \sigma\rho > 0$. ■

Proposition 4

Proof.

(i) $\partial N/\partial r = -(\partial G/\partial r)/G'(N)$. We know that $G'(N) > 0$. On the other hand,

$$\frac{\partial b(N, r)}{\partial r} = N e^{-rN(1-\sigma)/\sigma} \left(1 - \frac{r(1-\sigma)/\sigma e^{-\rho N}}{\rho + r(1-\sigma)/\sigma} \right) > 0$$

and

$$\frac{\partial a(N, r)}{\partial r} = \left(\frac{(1 + rN/\sigma) - e^{rN/\sigma}}{e^{rN/\sigma} r^2 (N/\sigma)} \right) \left(1 - \frac{\gamma}{N} \right)^{-1} < 0.$$

Therefore, $\partial G/\partial r = \partial b(N)/\partial r - \rho\gamma \partial a(N)/\partial r > 0$ and then $\partial N/\partial r < 0$.

(ii) $\partial G/\partial \gamma = -\rho a(N) - \rho\gamma \partial a(N)/\partial \gamma$, $\partial a(N)/\partial \gamma = \left(\frac{1 - e^{-rN/\sigma}}{rN/\sigma} \right) \frac{N}{(N-\gamma)^2} > 0$. Thus, $\partial G/\partial \gamma < 0$ and $\partial N/\partial \gamma = -(\partial G/\partial \gamma)/G'(N) > 0$.

(iii) For $\sigma = 1$,

$$\rho \frac{\partial G}{\partial \rho} = \frac{r}{\rho} (1 - e^{-\rho N}) - rN e^{-\rho N} - \rho\gamma \left(\frac{1 - e^{-rN}}{rN} \right) \left(1 - \frac{\gamma}{N} \right)^{-1}.$$

Use the value of the third term in the left hand side implied by $G(N^*) = 0$ to obtain

$$\rho G_\rho = 2 \frac{r}{\rho} (1 - e^{-\rho N}) - rN (1 + e^{-\rho N}).$$

Then, $G_\rho < 0 \iff -2(1 - e^{-\rho N}) + \rho N (1 + e^{-\rho N}) > 0$. Define

$$f(x) = -2(1 - e^{-x}) + x(1 + e^{-x}).$$

We have $f(0) = 0$ and $f'(x) = 1 - e^{-x}(1 + x) > 0$. Therefore, $G_\rho < 0$ and $\partial N/\partial \rho = -G_\rho/G'(N) > 0$. As $G(\sigma, N)$ is continuous in σ , we also have $\partial N/\partial \rho > 0$ for σ sufficiently close to 1. Moreover, numerical simulations yield that $\partial N/\partial \rho > 0$ for any $\sigma > 0$.

(iv) The optimal N is given by $G(N) = 0$. As G is increasing and continuous in N , and $\lim_{N \rightarrow \gamma_+} G(N) = -\infty$ then it must be the case that the optimal N is higher than γ .

(v) We saw that N decreases when γ decreases and that $N > \gamma$. The equation that determines N converges to

$$-\frac{1}{1 - 1/\sigma} (1 - e^{rN(1-1/\sigma)}) = r \frac{1 - e^{-N(\rho - r(1-1/\sigma))}}{\rho - r(1 - 1/\sigma)}$$

when $\gamma \rightarrow 0$. This expression holds if and only if $N = 0$.

(vi) When r decreases, N increases. But $\lim_{r \rightarrow 0} rN$ is bounded because $|\partial N / \partial r| < 1$. Define $x \equiv \lim_{r \rightarrow 0} rN$. Using the equation that defines N and $\lim_{r \rightarrow 0} N = +\infty$, the value of x is given by

$$-\frac{1}{1-1/\sigma} (1 - e^{x(1-1/\sigma)}) = \rho\gamma \left(\frac{1 - e^{-x/\sigma}}{x/\sigma} \right).$$

In order to have an idea of the magnitude of this number, consider this equation for $\sigma = 1$,

$$x = \rho\gamma \left(\frac{1 - e^{-x}}{x} \right).$$

This value of x is approximately equal to $\rho\gamma / (1 + \rho\gamma/2)$. ■

Proposition 5

Proof. By the first order condition for T_1 and $e^{-\rho t} u(c) = c\mu(M_0, W_0)P(t) / (1 - \sigma)$.

We obtain, after rearranging,

$$\begin{aligned} & \frac{\sigma}{1-\sigma} \frac{\mu}{Q(T_1)} c^-(T_1) - \frac{\sigma}{1-\sigma} \lambda c^+(T_1) \\ &= \lambda \left[-\frac{\dot{Q}(T_1)}{Q(T_1)} \frac{K}{P(T_1)} \frac{\dot{Q}(T_1)}{Q(T_1)} \int_{T_1}^{T_2} c(t) \frac{P(t)}{P(T_1)} dt \right] + \lambda Y \gamma \left[\frac{\dot{Q}(T_1)}{Q(T_1)} + \frac{\dot{P}(T_1)}{P(T_1)} \right]. \end{aligned}$$

In the steady state, we know that $c(t) = c_0 e^{-\frac{r}{\sigma}(T_2-t)}$, $T_1 \leq t < T_2$, $c^-(T_1) = c_0 e^{-\frac{r}{\sigma}N}$, and $c^+(T_1) = c_0$. Also, inflation is constant and $r = \rho + \pi$. Then, with $K = 0$ and after simplification

$$\frac{\sigma}{1-\sigma} e^{rT_1} e^{-\frac{r}{\sigma}N} - \frac{\sigma}{1-\sigma} \frac{\lambda}{\mu} = \frac{\lambda}{\mu} \left[-r \frac{1 - e^{-(\rho-r(\sigma-1)/\sigma)N}}{\rho - r(\sigma-1)/\sigma} - \rho \frac{Y}{c_0} \gamma \right].$$

With the optimality condition for N , we obtain

$$e^{rT_1} e^{-\frac{r}{\sigma}N} = \frac{\lambda}{\mu} e^{-\frac{r}{\sigma}N} e^{rN}.$$

Taking logs on both sides yields the desired result. ■

Proposition 6

Proof. We provide two proofs for this proposition.

(1) For $t > jN$, consumers will be in their j th or $(j + 1)$ th holding period, $j = 1, 2, \dots$. Individual money demand is given by

$$M(t, n) = \begin{cases} \int_t^{T_{j+2}(n)} P(t) c_0 e^{-\frac{r}{\sigma}(t-T_{j+1}(n))} dt, & n \in [0, t - jN), \\ \int_t^{T_{j+1}(n)} P(t) c_0 e^{-\frac{r}{\sigma}(t-T_j(n))} dt, & n \in [t - jN, N). \end{cases}$$

where $T_j(n) \equiv n + (j - 1)N$. Solving the integrals and with a change of variables, aggregate money demand is given by

$$M(t) = \frac{1}{N} \int_{t-N}^t P_0 c_0 \frac{e^{\pi(s+N)} e^{-rN/\sigma} - e^{\pi t} e^{-r(t-s)/\sigma}}{(\pi - r/\sigma)} ds$$

or,

$$\frac{M(t)}{P(t)} = \frac{c_0}{N} \int_0^N \frac{e^{(r-\rho-r/\sigma)N} e^{-(r-\rho)x} - e^{-rx/\sigma}}{(r - \rho - r/\sigma)} dx.$$

Solving the integral, with the value of c_0 given by Lemma 1 and with $m \equiv M(t) / P(t)$, we obtain the real money demand in the text. ■

(2) Real money demand for any consumer in the stationary equilibrium is such that

$$\dot{m}^n(t) = -c^n(t) - \pi m^n(t),$$

where m denotes real money demand and the superscript refers to the consumer with $n \in [0, N)$. For $n = 0$, the boundary condition for this differential equation is $m(N) = 0$. Solving the differential equation, we obtain

$$m^0(t) = c_0 e^{-rt} \frac{e^{\rho(t-N)} e^{r(1-1/\sigma)N} - e^{r(1-1/\sigma)t}}{r(1-1/\sigma) - \rho}.$$

By the symmetry of the steady state, aggregate real money demand is given by

$$m(t) = \frac{1}{N(r, \rho, \sigma, \gamma)} \int_0^{N(r, \rho, \sigma, \gamma)} m^0(t) dt.$$

Substituting $m^0(t)$ and solving the integral yields the desired result. ■

Proposition 7

Proof. $\tilde{M}_0(n)$ is exactly enough to allow the consumer to consume at the steady state rate in the interval $[0, n)$. This value is such that

$$\tilde{M}(n) = \int_0^n P(t) c(t) dt.$$

$c(0)$ is not necessarily equal to the level of consumption when a transfer is made. This is only true for the consumer $n = 0$. We know that

$$c^-(n) = c_0 e^{-\frac{r}{\sigma} N},$$

for all $n \in [0, N)$, and that

$$\frac{\dot{c}}{c} = -\frac{r}{\sigma}.$$

Solving this differential equation yields

$$c(x, n) = c_0 e^{\frac{r}{\sigma} n} e^{-\frac{r}{\sigma} N} e^{-\frac{r}{\sigma} x}, \quad 0 \leq x < n.$$

Therefore, solving the integral

$$\tilde{M}(n) = \int_0^n P_0 e^{\pi t} c_0 e^{\frac{r}{\sigma} n} e^{-\frac{r}{\sigma} N} e^{-\frac{r}{\sigma} t} dt$$

we have the desired result for $\tilde{M}(n)$.

For $\tilde{W}_0(n)$. First, the value of money needed in each holding period is given by

$$M_j = \int_{n+(j-1)N}^{n+jN} P(t) c_0 e^{-\frac{r}{\sigma}(t-T_j)} dt,$$

where $j = 1, 2, \dots$ and $T_j = n + (j-1)N$. So,

$$M_j = P_0 c_0 e^{\pi n} \frac{e^{(\pi-r/\sigma)N} - 1}{(\pi - r/\sigma)} e^{\pi(j-1)N} \equiv \bar{M} e^{\pi(j-1)N}$$

The value at $t = n$ of these transfers is $A_M \equiv \bar{M} \frac{1}{1-e^{-\rho N}}$. For the transfer cost, we have

$$\begin{aligned} TC_j &= \gamma Y P(n + (j-1)N), \quad j = 1, 2, \dots \\ &= P_0 \gamma Y e^{\pi(n+(j-1)N)}. \end{aligned}$$

Working analogously, $A_{TC} \equiv P_0 \gamma Y e^{\pi n} \frac{1}{1-e^{-\rho N}}$. Finally, the value of $\tilde{W}(n)$ is given by

$$\tilde{W}(n) = e^{-rn} A_M + e^{-rn} A_{TC}. \blacksquare$$

Proposition 8

Proof. Substitute the value of c_0 given by equation (14) in the integral in the text. Use this calculation in equation (18) to obtain the desired result. For $\sigma = 1$, follow the same steps with $u(c) = \log c$. \blacksquare

Proposition 9

Proof. From the first order conditions of the utility maximization problem with respect to $c(t)$ and T_j we have

$$c^+(T_j) \left[\frac{Q(T_{j-1}) c^-(T_j)}{Q(T_j) c^+(T_j)} - 1 \right] \frac{\sigma}{1-\sigma} + \gamma Y [r_2 - \pi(T_j)] = -r_2 \int_{T_j}^{T_{j+1}} \frac{P(t) c(t)}{P(T_j)} dt.$$

Where we used the fact that $\left(\frac{c^+(T_j)}{c^-(T_j)}\right)^{-\sigma} = \frac{Q(T_j)}{Q(T_{j-1})} = e^{-r_2 N_j}$, $j = 2, 3, \dots$ and $\frac{c(t)}{c^+(T_j)} = \left[\frac{e^{-\rho T_j} P(t)}{e^{-\rho t} P(T_j)}\right]^{-1/\sigma}$, and substituted $Q(t) = e^{-r_2 t}$. With further algebraic manipulation we obtain the formula in the body of the text. The logarithmic case is analogous. \blacksquare

Proposition 10

Proof. From the first order conditions of the utility maximization problem with respect to $c(t)$ and T_j we have

$$c^+(T_1) \left[\frac{\mu(n)}{\lambda} \frac{1}{Q(T_1)} \frac{c^-(T_1)}{c^+(T_1)} - 1 \right] \frac{\sigma}{1-\sigma} - r_2 \frac{K}{P(T_1)} = -r_2 \int_{T_1}^{T_2} \frac{P(t) c(t)}{P(T_1)} dt - \gamma Y [r_2 - \pi(T_1)].$$

Where we used the fact that $\left(\frac{c^+(T_1)}{c^-(T_1)}\right)^{-\sigma} = \frac{\lambda}{\mu(n)} Q(T_1) = \frac{\lambda}{\mu(n)} e^{-r_2 N_1}$. With further algebraic manipulation we obtain the formula in the body of the text. The logarithmic case is analogous. \blacksquare

APPENDIX C - DATA

I am using a similar data set as the one used in Lucas (2000) to facilitate comparisons between the two models.

GDP

From 1900 to 1928 it is from the Bureau of the Census (1975), *Historical Statistics of the United States: Colonial Times to 1970*. Series F1, Nominal GDP. From 1929 to 2000 it is from NIPA, Tables 1.1.5, 1.1.6.

Interest Rate

The nominal interest rate is the short commercial paper rate. From 1900 to 1975 it is from Friedman and Schwartz (1982), *Monetary trends in the United States and the United Kingdom: their relation to income, prices and interest rates, 1875-1975*, Chicago: University of Chicago Press, Table 4.8, column 6, p. 122, "Interest Rate, Annual Percentage, Short-Term, Commercial Paper Rate". From 1976 to 1997 it is from the *Economic Report of the President*, Table B-73 "Bond Yields and Interest rates". In Friedman and Schwartz, "these are annual averages of monthly rates on sixty-to-ninety-day, through 1923, since four-to-six-month commercial paper in New York City, based on weekly figures of dealers' offering rates until 1944, thereafter, on daily figures". In the Economic Report of the President, data are for commercial paper 6 months, and 4 to 6 months commercial paper prior to November 1979. The last value of the series available is for 1997.

Money

From 1900 to 1913, it is from the Bureau of the Census (1960), *Historical Statistics of the United States: colonial times to 1957*, Series X-267, "demand deposits adjusted plus currency outside banks". From 1914 to 1958 it is from Friedman and Schwartz (1963), *A Monetary History of the United States, 1867-1960*, December of each year, seasonally adjusted. For M1, I used column 7, sum of currency and demand deposits.

For M2, I used column 8, sum of currency, demand and time deposits. From 1959 to 1997 it is from the Federal Reserve Bank of St. Louis, FRED Database. Series M1SL and M2SL for M1 and M2 respectively, December of each year, seasonally adjusted.

APPENDIX D - FIGURES

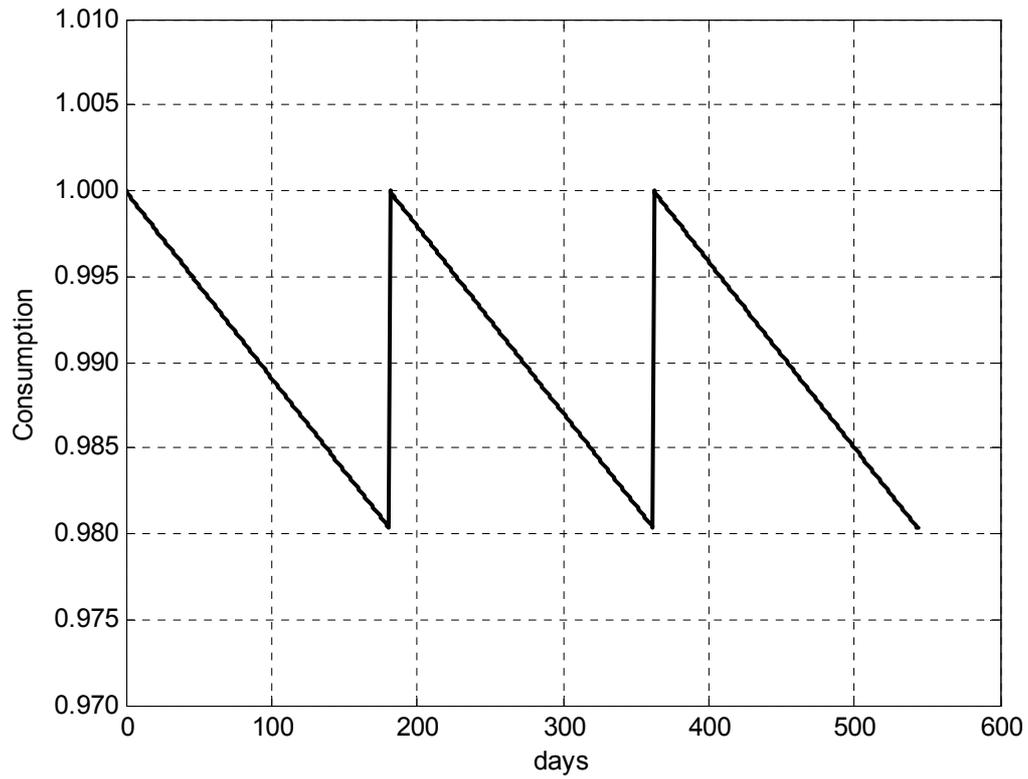


FIG. 1. Consumption within a holding period. $r = 4\%$ p.a., $\rho = 3\%$ p.a., $\sigma = 1$, $\gamma = 1.791$, $Y = 1$. For these parameters, $N = 181$ days.

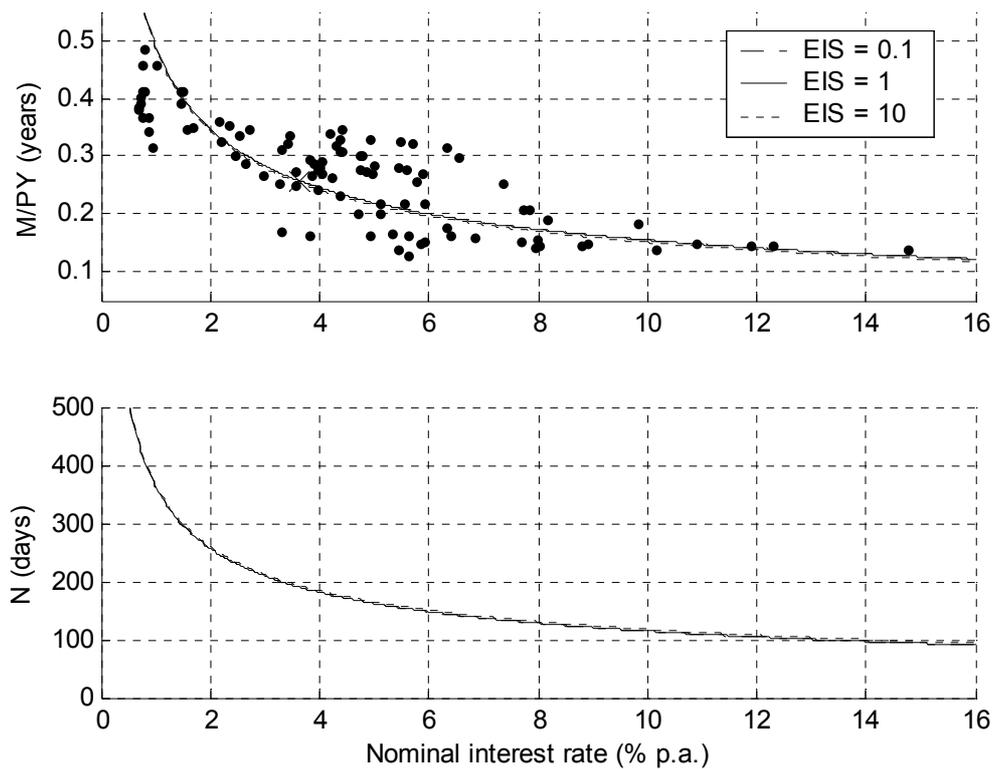


FIG. 2. Money-income ratio and optimal interval between transfers. The data points are for the U.S. economy in the period 1900-1997. The \times marks the geometric mean of the data. EIS stands for Elasticity of Intertemporal Substitution.

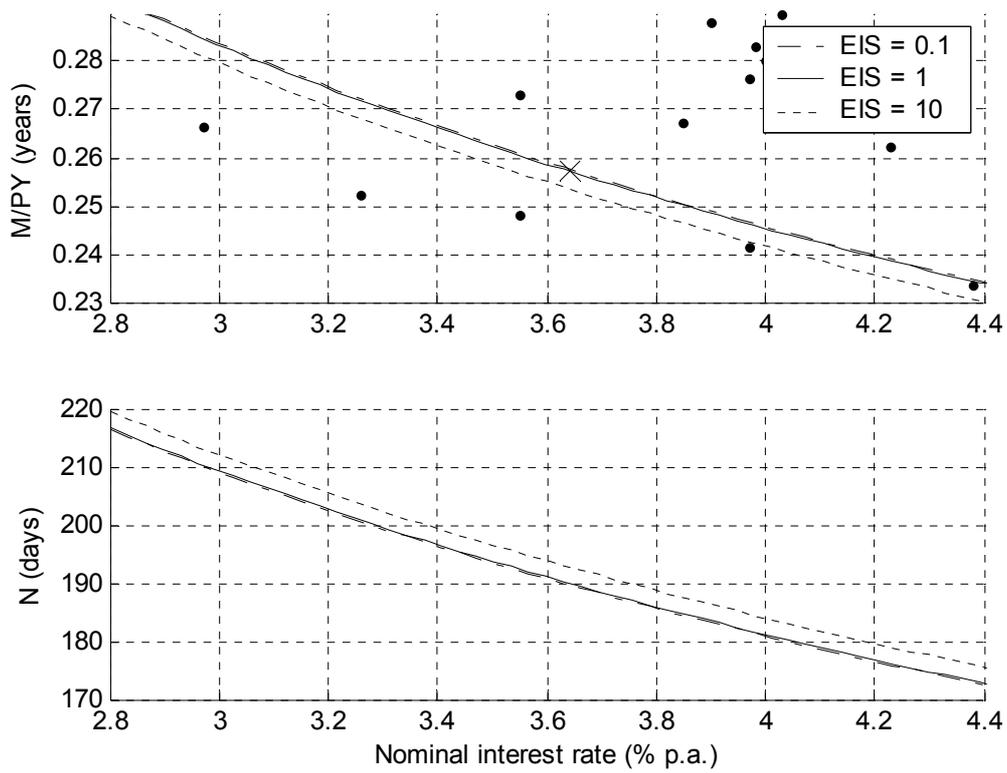


FIG. 3. Zoom in the graphs of figure 2.

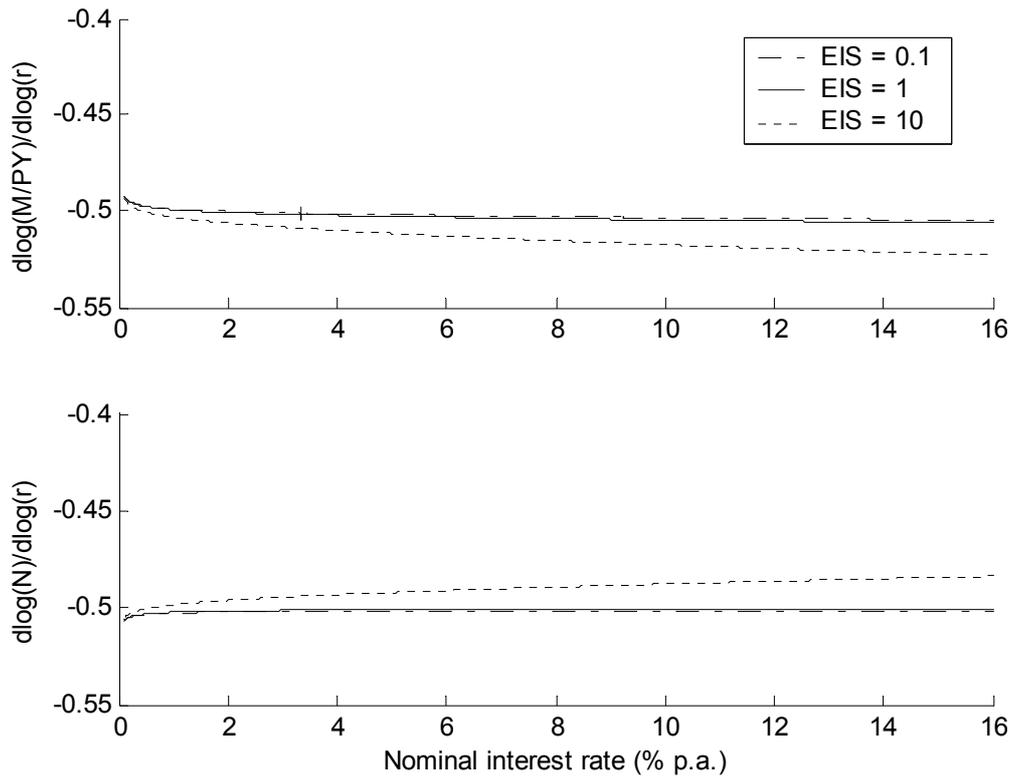


FIG. 4. Elasticities of the real money demand and of the interval between transfers with respect to the nominal interest rate. The interval $[0.1\%, 16\%]$ was divided in 50 thousand points and the elasticities approximated by the discrete calculations.

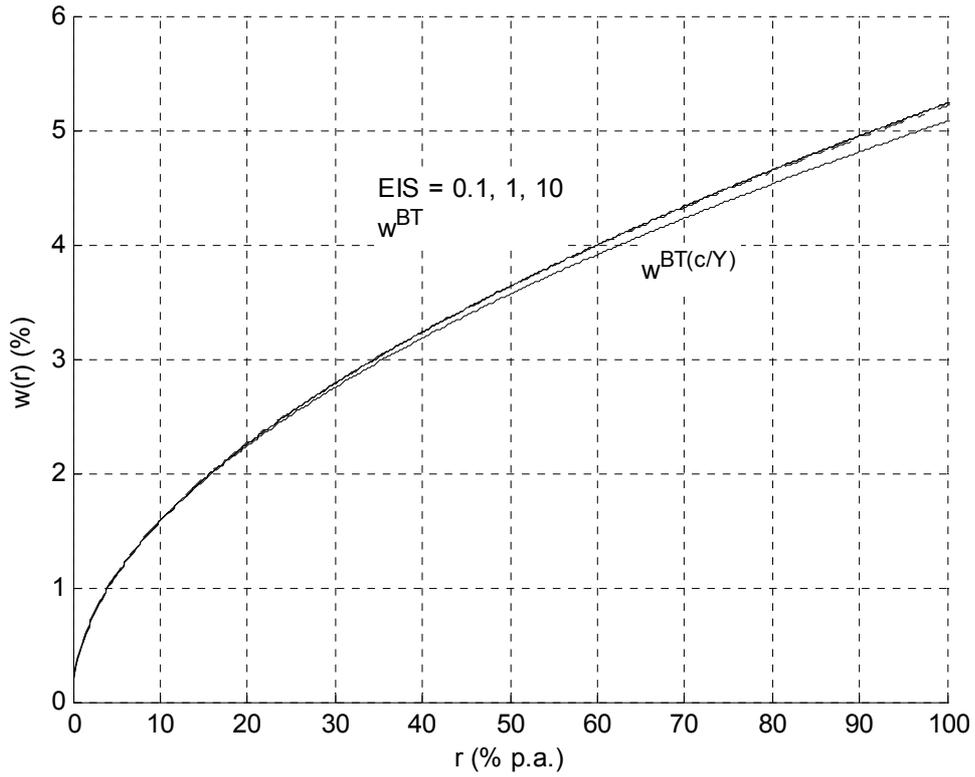


FIG. 5. Welfare cost of inflation, $w(r)$. The curves for $EIS = 0.1, 1, 10$, and for the welfare cost derived with the Baumol-Tobin money demand are indistinguishable. The baseline nominal interest rate was set to $\bar{r} = 0.001\%$ p.a.

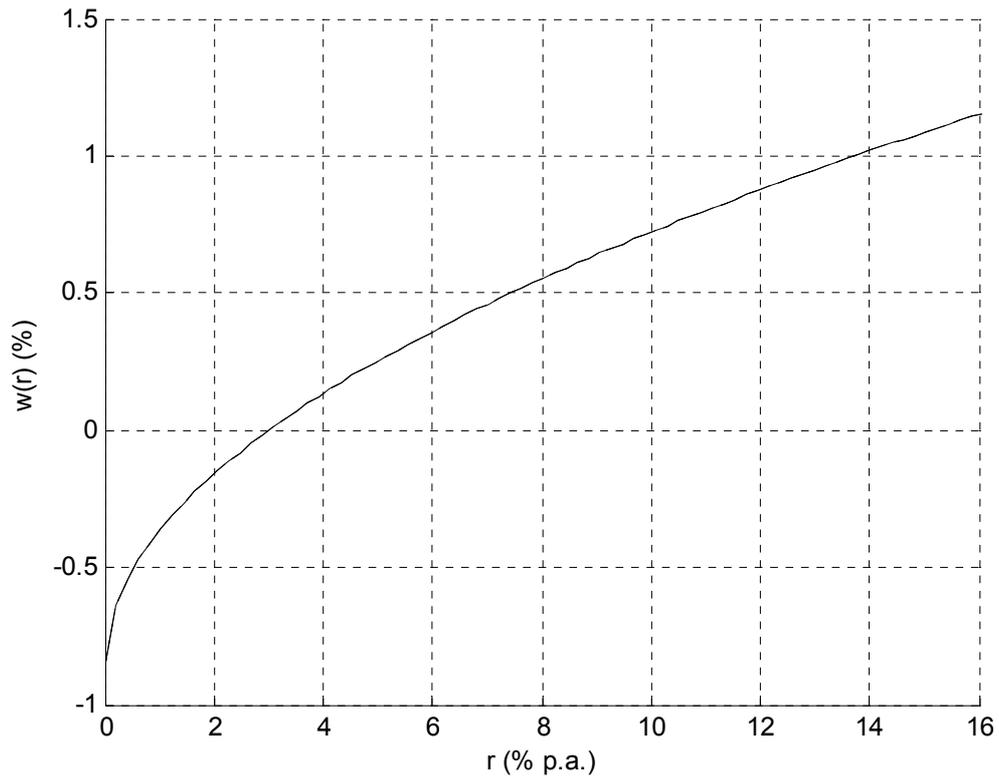


FIG. 6. Welfare cost of inflation. The compensation $w(r)$ is calculated relative to $r = 3\%$ p.a., i.e., the nominal interest rate that implies zero inflation. The curves for $EIS = 0.1, 1, 10$ are indistinguishable.

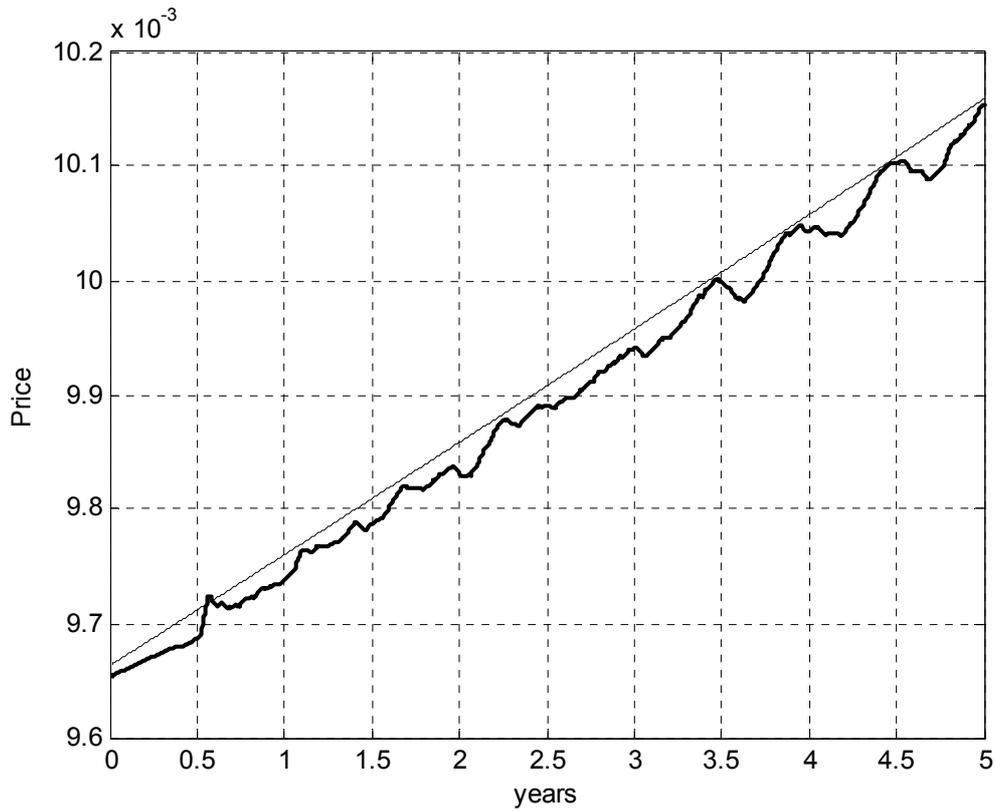


FIG. 7. Price level after an increase of 1 percentage point in the interest rate. The straight line indicates a constant 1% p.a. inflation, with the initial price equal to the price before the shock. The price level drops -0.11% at the moment of the shock. The steady state interval between transfers decreases from 209 days to 181 days.

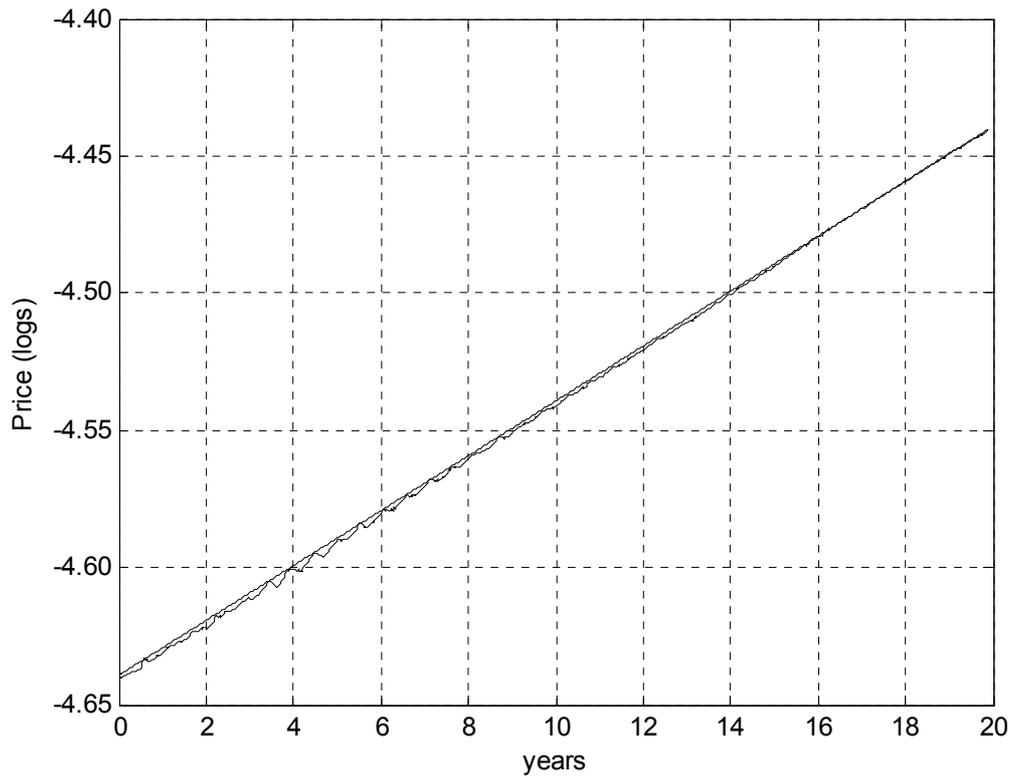


FIG. 8. Price level after an increase of 1 percentage point in the nominal interest rate, in logs. The straight line indicates a constant 1% p.a. inflation, with the initial price equal to the price before the shock.

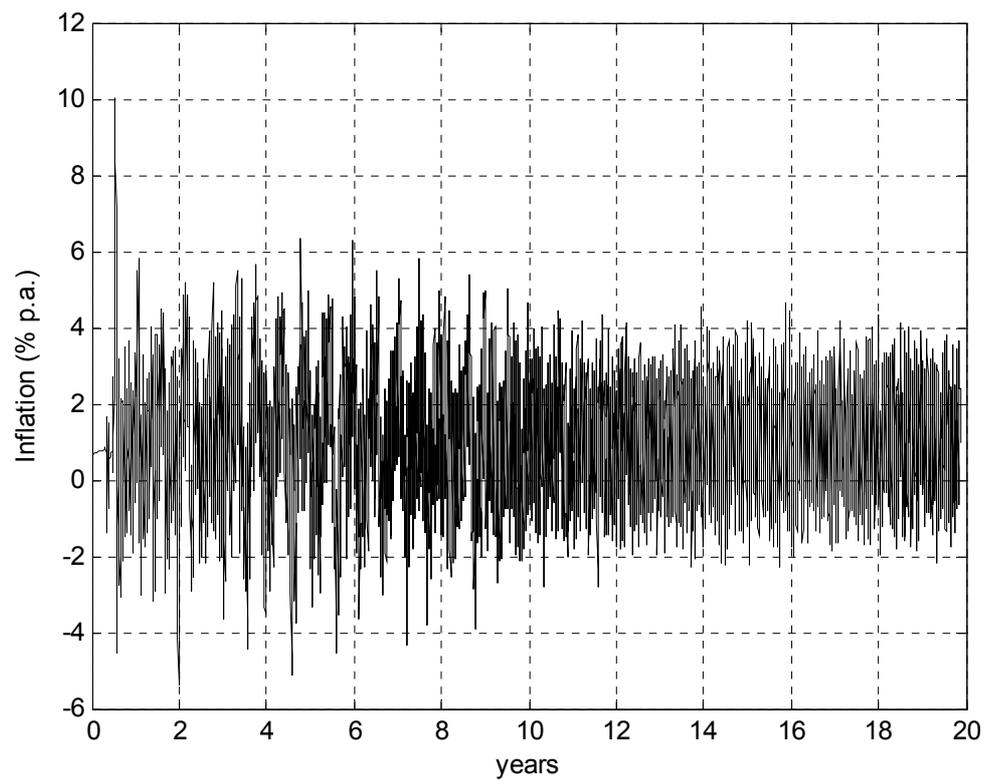


FIG. 9. Inflation after a 1% shock to the nominal interest rate.

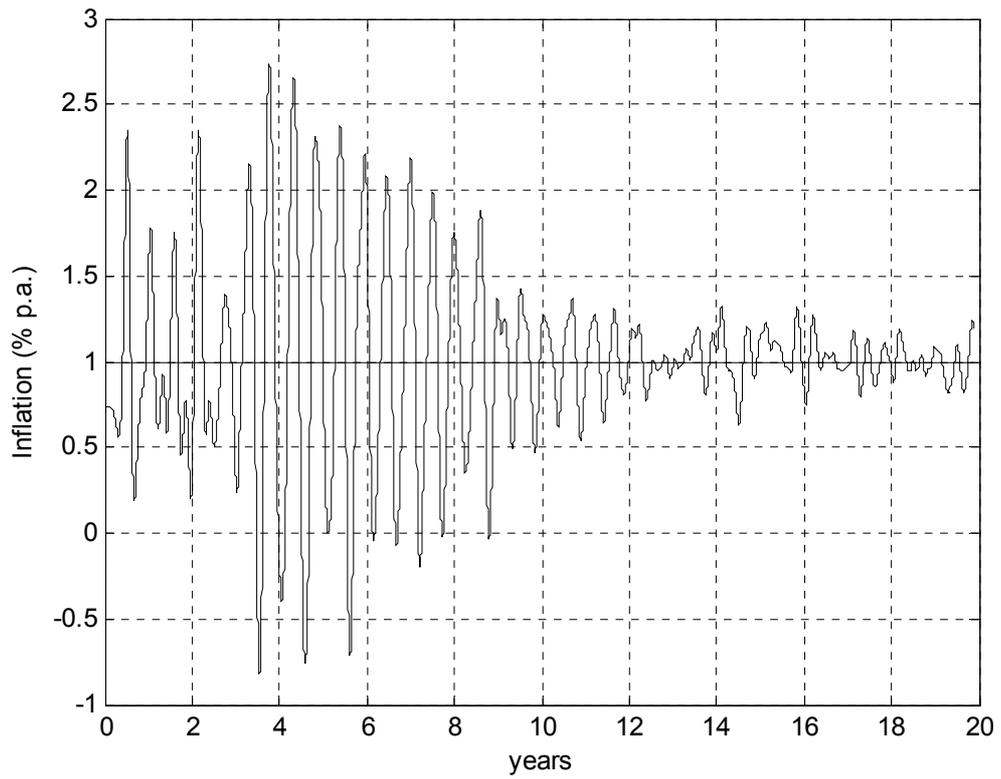


FIG. 10. Inflation after a 1% shock to the nominal interest rate. Price series filtered before the calculation of the inflation rate.

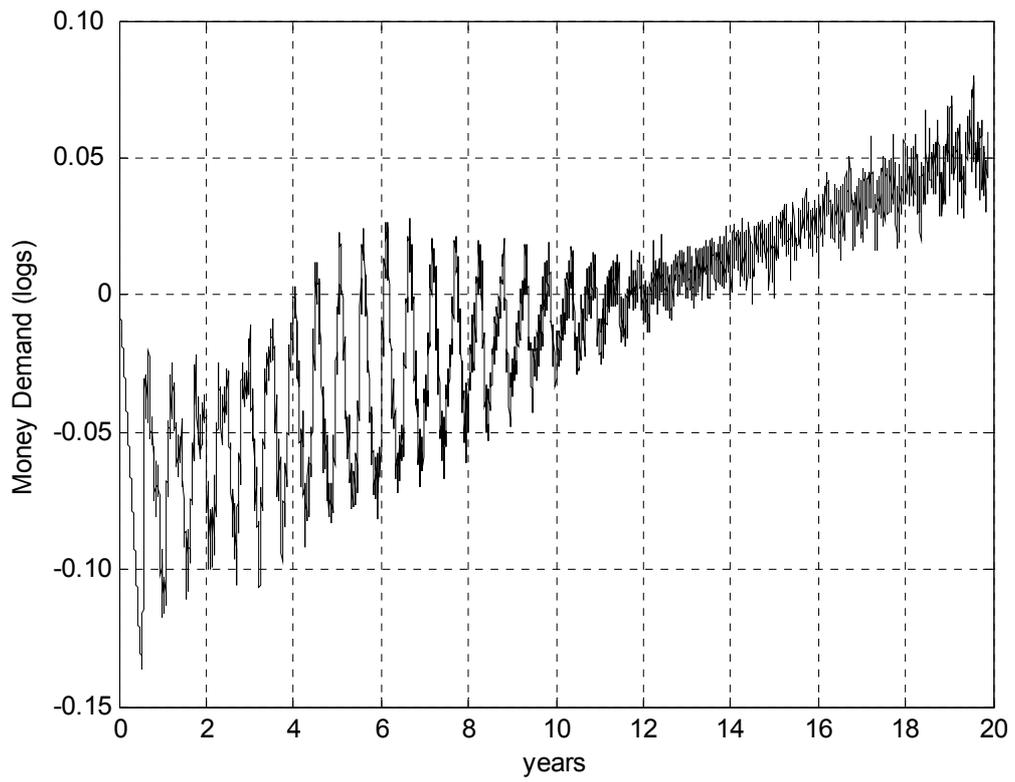


FIG. 11. Nominal money demand. Economy assumed to be in the steady state after 20 years.

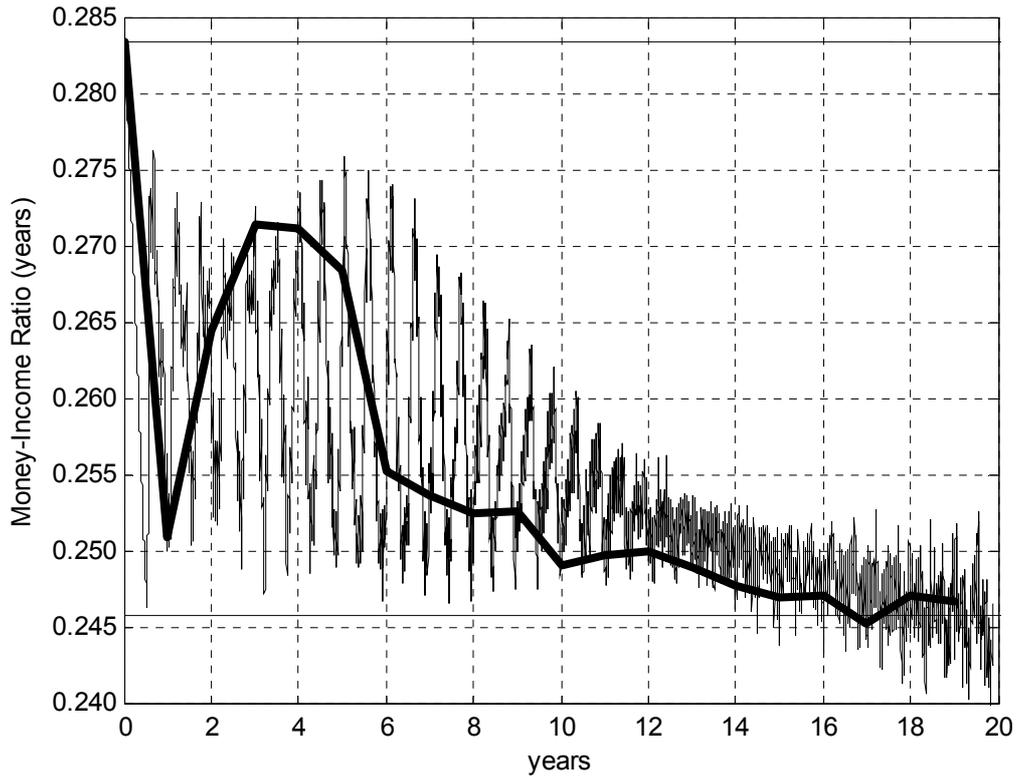


FIG. 12. Money-income ratio, $M/(PY)$, in years. The thick line is the money-income ratio of the last day of the year, with the exception of the first point, given by the money-income ratio before the shock. The vertical lines are the first and the second steady state values obtained analytically.

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