Optimal Growth in a Two-Sector Model without Discounting: A Geometric Investigation*

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August 17, 2002

Abstract

We characterize optimal policy in a two-sector growth model with fixed coefficients and with no discounting. The model is a specialization to a single type of machine of a general vintage capital model originally formulated by Robinson, Solow and Srinivasan, and its simplicity is not mirrored in its rich dynamics, and which seem to have been missed in earlier work. Our results are obtained by viewing the model as a specific instance of the general theory of resource allocation as initiated originally by Ramsey and von Neumann and brought to completion by McKenzie. In addition to the more recent literature on chaotic dynamics, we relate our results to the older literature on optimal growth with one state variable: specifically, to the one-sector setting of Ramsey, Cass and Koopmans, as well as to the two-sector setting of Srinivasan and Uzawa. The analysis is purely geometric, and from a methodological point of view, our work can be seen as an argument, at least in part, for the rehabilitation of geometric methods as an engine of analysis.

1 Introduction

In the late sixties, under the general heading of "technical choice under full employment in a socialist economy," Robinson [33, pp. 38-56], [34], Okishio [29] and Stiglitz [43], [44], studied the problem of optimal economic growth in a model of an economy originally formulated by Robinson [33], Solow [39] and Srinivasan [41] (henceforth RSS model).[1] The work generated controversy. Stiglitz [43, Paragraph 1], [44, p. 421] argued that the Robinson-Okishio assumption of a fixed labor allocation between the consumption and investments sectors had no place in an exercise that sought to determine the optimal

*The project of which this paper is a part has had a long gestation period – it began in Illinois in 1986, received invaluable impetus from Bob Solow's presentation at the Srinivasan Conference held at Yale in March 1998, and research support while Khan was at EPGE, Fundação Getulio Vargas. It is a pleasure to acknowledge the encouragement of Professors Sen, Solow and Srinivasan. This paper represents work in progress and is not to be quoted without the authors' permission – it is intended for Ron Jones' seventieth birthday – to the authors, a teacher, friend and a geometer *par excellence.*

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[1]In [11, p.3], the model is referred to as the Solow-Srinivasan model; also see [39, concluding three paragraphs] and [11, Footnote 12] for the way it is referred to in earlier work.

growth path, and thereby an optimum time-path of the allocation of labor.[2] On her part, Robinson criticized the assumption of a positive discount rate, continuous time and the linearity assumption in the specification of the planner's felicity function.[3] Stiglitz's [44] response is important for the record.

> Even if there is a minimum consumption constraint and a finite gestation period, the path of development will, after an initial "adjustment" period, look exactly as I have described it. [Unlike] long-run neoclassical models with malleable capital [where] the optimal policy is always of the so-called bang-bang variety – if the initial capital labor ratio is less than its long-run equilibrium value there is always a period of zero consumption, after which consumption jumps to its long-run equilibrium value, whereas in our *ex-post* fixed coefficients model consumption increases steadily to its long-run value.

In his recent revisit of Srinivasan [41], Solow [40] asks for a solution to the "Ramsey problem for this model." Since Stiglitz had a already provided a solution with "linear utility and positive time preference", the open question concerns "a strictly concave social utility function for current per capita consumption." Indeed, Solow also mentions that "adoption of this criterion can indeed lead to unjustifiable neglect of early consumption," and if one were to share "Ramsey's belief that the only ethically defensible social rate of time preference is zero, a sufficiently sharply-concave utility function would enforce a closer approach to intergenerational inequality." In short, the generalization of Stiglitz's work in this direction of a non-linear felicity function remains yet to be accomplished.[4]

In this paper, we do not report any results regarding this open problem,[5] but rather reconsider Stiglitz' analysis with a linear felicity function, but without discounting and in discrete time. Furthermore, we totally abstain from the question of the choice of technique and consider an economy with only *one* type of machine. Indeed, with all these simplifications – the model's parameters are reduced to two real postive numbers – it is natural to wonder whether there is very much left to determine; the problem seems at first glance to have been simplified into a triviality. In the light of all this, the results go against Stiglitz' intuitions in a way that is nothing short of dramatic. We identify parameter values under which the optimum programs consist of two-period cycles around a unique golden-rule stock of machines, and the amplitude of each cycle is different for different values of the initial capital stock lying in an identifiable interval. Outside this interval, there is convergence in finite time to one of these cycles, but one that is not necessarily of a monotonic type. Moving to other parameter values, we show convergence to the golden-rule stock, again not necessarily of a monotonic type, and one that takes finite or infinite time depending on the initial stock of capital the economy is endowed with. Thus, in several instances, Ramsey optimality requires over-building and under-depreciating relative to the golden-rule stock. In particular, the dynamic system fails in a particularly sharp way the recent criterion of "history independence" proposed in [26].

Though the results appear surprising in terms of the volatility of the optimal programs when viewed against the backdrop of earlier work, they seem tame in the light of more recent work on chaotic dynamics. Our model, despite its simplicity, is a *bona fide* two-sector model, and recent work of Nishimura, Sorger and Yano presents robust methods which allow one to construct two-sector models of the standard Srinivasan-Uzawa type (see [42], [49]) whose optimal programs exhibit chaos under its various

[2] Stiglitz [44, p. 421] writes "There may be some special situations ... where the employment allocation is the same for all steady-state paths, but even then, in going from one steady-state path to another, one cannot infer that the employment allocation is unchanged – and it is this dynamic problem that we are discussing."

[3] We restate Robinson's criticisms in our own terminology; she phrases them in terms of a "discount rate chosen once and for all, ... negligible gestation periods, ...[and] ceasing to consume and living on air during the first phase of the plan."

[4] For some partial attempts at solution, see [45]. In [11, p. 15], this is expressed as "The loose end remains loose."

[5] We reserve this for another place; see [13].

definitions. It is these programs that really merit the term "volatile", and the results presented here can be alternatively viewed as showing how optimal programs loose their volatility when there is discounting and one of the sectors, the investment sector, uses only one factor, namely labor. However, it is worth observing that in either case, the full implications of all of this work have yet to be drawn out for the field which served as their primary motivation – development economics.[6]

Whereas the results presented in this paper constitute its primary motivation, a secondary concern is to report what we see as a rehabilitation of geometric methods as an engine of analysis. rather than simply as an instrument of illustration. Our methods allow us to go beyond a statement of mere convergence, and characterize the structure of the optimal policy in some detail, including its dynamics in the short-run. In particular, we can compute the number of periods it takes for an optimal policy to converge to the golden-rule stock, and what it implies for the the range in which the initial stocks of capital must lie. The geometrical techniques that we elaborate rest on two basic mathematical results: the Kuhn-Tucker theorem of optimization theory, and the Brock theorem on the existence of optimal programs obtained as those minimizing value-loss at suitably defined prices; see [48], [3] respectively. By showing how basic and beautiful ideas in the general theory of intertemporal resource allocation are amenable to geometric manipulation, albeit in the specific context of our simple model, we hope to make them accessible to a wider audience.[7] In this move beyond calculus to a recourse to global methods, we are very much in the tradition of the classical theory of international trade.[8] We are also in tune with a similar move in recent investigations in economic dynamics, as for example in the work of Nishimura-Yano [27].

The paper proceeds as follows. In Section 2, we develop the model and place it in the context of the two-sector neoclassical setting originally associated with Srinivasan and Uzawa.[9] Section 3 and 4 lay out the basic geometrical apparatus. Despite our best efforts, this material may perhaps prove to be difficult and tedious reading as a result of the modern reader's rusty recall of Euclidean geometry. To ease this, we have summarized at the end each section all that we hope to take from it, and the reader willing to accept these assertions can proceed directly to the characterization and discussion of optimal policies in Sections 5, 6 and 7. Section 8 considers extensions and comments on issues having to do with comprative dynamics. Section 9 concludes the paper with an overview and a listing of some open questions.

2 The Model and its Antecedents

Given its familiarity, we introduce the RSS model in terms of the standard notation of the two-sector setting especially familiar to students of (old) trade and growth theory.[10] Let there be two sectors pro-

[6] We make a beginning in [13].

[7] For a quick list of the concepts exposited in this paper, the reader can see the last but one paragraph of the next section. Given the authors' emphasis on recursive methods, none of these concepts make it to the textbook of Stokey-Lucas [46], for example.

[8] Such a tradition of exposition and investigation of course begins with Marshall's "Pure Theory of Foreign Trade" and continues with Samuelson, Meade, Jones, Johnson and others. In the context of growth theory, see Koopmans [14]. For a succinct statement on the advantages of the geometric method, see for example [17, p. 5] and [10, pp. 9-10]. For differing reactions to the use of geometry, see Kurz's [15] and Shackle's [37] reviews of Meade's work.

[9] Also see Shell [38] and Haque [7] in the context of optimum growth. The use of the two-sector model in the theory of international trade has a long tradition; see [9], [10] and their references.

[10] In addition to Srinivasan [42], Uzawa [49], Shell [38] and Haque [7] in the case of optimum growth with discounting; also see [25] for references to modern work. As is well known, the model has been used and exposited by Uzawa, Solow, Meade, Johnson and others in the context of so-called descriptive growth. We also give no references to the long tradition

ducing two distinct commodities, a consumption good and an investment good, without joint production and through the use of two factors capital and labor. For each date t, $t = 0, 1, \cdots$, let the technologies be stationary and given by

$$C(t+1) = F_C(K_C(t), L_C(t)) = \min\{K_C(t), L_C(t)\},$$

$$Z(t+1) = F_Z(K_Z(t), L_Z(t)) = (1/a)L_Z(t) \text{ where } a > 0,$$

and the allocation of factors $(K_C(t), L_C(t), K_Z(t), L_Z(t))$ is non-negative. Let $K(t) > 0$ and $L(t) > 0$ denote the amounts of capital and labor that are available in period t. Then we have have the following (material balance) constraints on labor and capital use in each period:

$$L_C(t) + L_Z(t) \leq 1 \text{ and } K_C(t) + K_Z(t) \leq K(t).$$

Since the investment good sector does not use capital at all, we are working under the extreme case of the assumption that the consumption good sector is more capital-intensive than the investment good sector.[11] In this two-sector setting, $Z(t+1)$ constitutes the gross investment during period $(t+1)$ and under the assumption that capital depreciates at the rate $d \in (0, 1)$, we obtain for each time-period

$$Z(t+1) = K(t+1) - K(t) + dK(t).$$

It follows that "investment is irreversible"; by the very specification of the model, gross-investment is restricted to be non-negative at each date. Under the assumption of a positive discount factor $\delta < 1$, and a stationary felicity function $U(\cdot)$, leading to the maximization of $\sum_{t=0}^{\infty} \delta^t U(C(t))$ subject to a given initial stock of labor and capital $(K(0), L(0)) \geq 0$, we obtain a special case of the two-sector optimal growth model exposited, for example, in Mitra [25].

We now further specialize this model by assuming that the discount factor δ is unity,[12] the felicity function $U(\cdot)$ is linear and normalized so that $U(C(t)) = C(t)$, and that labor is stationary and normalized to unity.[13] To reflect the latter assumption, we shall use lower-case letters $c(t)$ and $z(t)$ for the consumption and investment good, and $x(t)$ for the stock of capital. Since the capital-labor in the consumption good sector is unity, labor use in that sector equals the amount of capital being used which in turn equals the amounts of output and utility produced in each period. We shall denote the common value of all these four quantities by $y(t)$, but taking it in the first instance to refer to the amount of machines being used in the consumption goods sector.

In supposing that future welfare levels are treated like current ones in the planner's objective function, we of course take our lead from Ramsey [31], but rather than the assumption of a "bliss point" or that of "capital saturation", as in [31], [36] and [35], we follow the literature and work with the "overtaking criterion of optimality" associated with is due to Atsumi [2] and von Weiszäcker [51]. Thus, given an initial capital stock $x_0 \geq 0$, we work with *programs* starting from x_0. These are simply sequences of capital stocks (non-negative numbers) $\{x(t)\}_{t=0}^{\infty}$ such that $x(0) = x_0$ and which satisfy the technological

of the use of this model in Heckscher-Ohlin-Samuelson trade theory, as exposited for example in [9] and [10].

[11] We invite the interested reader to redo the geometric analysis of [8] under the factor intensity assumptions of this paper.

[12] From now on, δ will have a totally different meaning in the sequel.

[13] The latter assumption seems to be in keeping with the tone of the time; see [1, Section 1.2] for the exposition of what is termed there as the Cass-Koopmans-Ramsey model. Such an assumption is also made by Srinivasan [41] and by Stiglitz work [43], [45], [47].

and material balance constraints laid out above. A program $\{x^*(t)\}_{t=0}^{\infty}$ starting from $x(0)$ is said to be *optimal* if for any other program $\{x(t\}_{t=0}^{\infty}$ starting from $x(0)$

$$\liminf_{T\to\infty} \sum_{t=1}^{T} (c(t) - c^*(t)) \leq 0,$$

where $\{c(t)\}_{t=1}^{\infty}$ and $\{c^*(t)\}_{t=1}^{\infty}$ represent the consumption sequences associated with each of the two programs under consideration. Note that the lim inf operation can alternatively stated to say that for the program $\{x(t)\}_{t=0}^{\infty}$, given any $\varepsilon > 0$, there exists a time period t_ε such that

$$\sum_{t=1}^{T} (c(t) - c^*(t)) \leq \varepsilon \text{ for all } T \geq t_\varepsilon.$$

Thus an optimal program is one in comparison to which no other program from the same initial stock is eventually significantly better, for any given level of significance. A program[14] $\{x(t\}_{t=0}^{\infty}$ is said to be *stationary* if it is constant over time, i.e., $x(t+1) = x(t)$ for all $t = 0, 1, \cdots$. A program is said to be a *stationary optimal program* if it is stationary and optimal.

So far, in emphasizing the maximization of a planner's preferences, we have not focussed sufficiently on the assumption of a fixed coefficients technology in our model. It is this assumption that looks towards the multi-sectoral setting and thereby takes its lead from Von Neumann [50]. The work of Gale [5], Brock [3] and McKenzie [18] can be seen as stemming from von Neumann's paper, and finds its culmination in a general so called "reduced form" model[15] summarized by two basic parameters: a period-to-period technology set in a (product) space of capital stocks initial and terminal to the period, and a planner's utility function defined on this set.[16] Such a formulation is flexible enough to yield as special cases the variety of growth models studied in the literature and thereby qualify as a general theory of intertemporal resource allocation.[17] It is through the vocabulary of this theory that we analyze the simple two-sector model studied here.[18] In addition to Brock's theorem, such a formulation leads us to discuss in the sequel the golden-rule capital stock and its associated golden-rule price system, value-loss per period and its aggregate over time, the von Neumann facet and its privileged subset, good programs, and the average turnpike property. In terms of a comparison with the methods of Pontryagin, as used in our context in [43], perhaps the crucial difference is the possibility of a complete analysis based on the golden-rule price system from the very start, rather a price system that corresponds to the optimal program, and is then typically shown to converge to it.[19]

[14]If we do not specify the starting point of a program, the reader should take it to mean that it starts from its value at time zero.

[15]In [23, p. 389], McKenzie dates this "reduced form" model to 1964, and describes the work of Gale, McFadden and himself as a "fusion of the Ramsey and von Neumann models. Malinvaud's 1953 paper is also an important step in this evolution.

[16]In his work, McKenzie does not give a name for the analogue in a general setting of the set that we denote by Ω. In [25], it is referred to as the *transition possibility set.* The evolution of this set is of some interest: in [18], it includes values of utilities, while [19] retains the notational convention of [4] where positive numbers stand for outputs and negative ones for inputs.

[17]For statements of the theory and its scope, see [21], [22], [23] and [25].

[18]As we shall indicate in the sequel, the relevant theorems of this theory do not directly apply, but the methods leading to the proofs of these theorems extend to our simple setting in a straightforward way.

[19]This is of course not to say that a price system associated with any optimal program has no role to play in the theory – only that we make no recourse to it.

We conclude this section with the summary statement that the version of the RSS model that we study in the subsequent sections is summarized by two parameters: positive real numbers a and d with $d < 1$.

3 The Basics of the Geometry

We begin with Figure 1 in which the 45^o-degree line serves as an important benchmark. The amount of capital stock (number of machines),[20] $x(t)$, available today is measured on the X-axis and that available next period, say tomorrow, $x(t+1)$, on the Y-axis. Figure 1 highlights the fact that we work in discrete time; when the particular time period is not to be emphasized, we shall use the symbols (x, x') for $(x(t), x(t+1))$.

Next, we focus on the line OD lying strictly within the cone formed by the X-axis and the 45^o-degree line. The slope of OD measures the rate of depreciation:[21] a unit of today's capital stock depreciates to $(1-d)$ units tomorrow. The line represents a constraint embodying a precise time-invariant form of depreciation, one that does not distinguish on this score between machines produced today and those produced in the past, as well as the fact that capital cannot be disposed off through a market or in any other way. Thus it already alerts the reader to the fact that we work with irreversible investment and that there is a bound to disinvestment. Any point, say S_d, on OD represents a situation involving zero net investment, and with negative gross investment being equal to the depreciation on the capital stock.

The line ML is drawn parallel to OD and represents the constraint of the availability of labor in each period.[22] The length[23] OM represents the maximal amount of capital stock available tomorrow if there is no available capital stock today and all of the labor force is employed in the investment sector. Since a units of labor are required to produce one unit of the capital stock (machine), this length equals $(1/a)$. Since the lines ML and OD are parallel, the distance[24] between them is a constant equal to OM, the constant maximal amount of net investment that can be undertaken at any period. Its constancy reflects both the stationarity of labor and the fact that machines are not needed to manufacture machines.

The fact that one machine is used along with a unit of labor to produce one unit of the consumption good, the technological specification of the consumption sector, is not explicitly graphed in Figure 1.[25] Given this, the technology is represented by the "open" parallelogram Ω enclosed by the lines OM, ML and OD.[26] We shall refer to elements of this period-to-period production set Ω as one-period plans.[27] Note that unlike [4], these are not simply production plans but also contain a specification of consumption levels that they permit. This will become clearer as we proceed. For the moment, two features of Ω ought to be noted in our context: (i) the absence of free disposal, already mentioned above,

[20] We shall use these two interpretive phrases interchangeably.

[21] The equation of this line is simply $x' = (1-d)x$.

[22] The equation of this line is simply $x' = (1-d)x + 1/a$.

[23] The reader is warned to be alert to identical symbolism being used to designate lengths and lines.

[24] Note that we depart from the conventional notion of (Euclidean) distance between two sets which involves the length of the projection of a point one line to the other.

[25] In [20, p. 846], Mckenzie writes "When our interest is an asymptotic property of the path of capital stocks, there is no need to show how utility depends on production and consumption during the period ... the significant choice from the viewpoint of the intertemporal maximization problem is the choice of terminal stocks given initial stocks. This fixes the contribution of the period to the optimal program." However, as we shall see below, in our specific context, the diagram can be useful even for the delineation of consumption levels and of dynamics in the short-run.

[26] Formally, $\Omega = \{(x, x') \in \mathbb{R}_+ \times \mathbb{R}_+ : x' - (1-d)x \geq 0 \text{ and } a(x' - (1-d)x) \leq 1\}$.

[27] As such, when we refer to them in the sequel, we shall not resort to the qualifying adjective "feasible".

and (ii) a zero initial stock (input) allows a positive terminal stock (next period's output), specifically OM, next period.[28]

Before we turn to the depiction of preferences in the (x, x')-plane represented by initial and terminal capital stocks, we consider the line segment MV. This proves to be a (most) valuable construct for the geometric development of the model; it delineates several important aspects of the model and we turn to the first of them. Note, to begin with, that the slope of the line MV is given by

$$-\frac{MV'}{V'V} = (1-d) - 1/a, \text{ to be designated by } \xi. \tag{1}$$

Since the coordinates of the points M and V are known to be $(0, 1/a)$ and $(1, 1-d)$ respectively, it is now a simple matter to get the equation of the line MV. However, it is more useful first to reflect on what such an equation equates.[29] Towards this end, note that unlike the investment sector which does not use capital and is dependent solely on labor, the consumption sector uses both labor and capital, and is thereby subject to two constraints – of labor as well as that of capital. Since the only production technique available to this sector entails a unit capital-labor ratio, we can use the same symbol y to represent the amount of the consumption good being produced in a given period, as well as the amounts of capital and labor being used. (Since the marginal utility of the consumption good is also unity, y is simultaneously a proxy for the number of utils that the planner obtains, but this can be bracketted for the moment.) Thus, any one-period plan (x, x') constrains y by the capital availability x and by the residual labor available once the labor requirements of the investment sector have been met. Thus, in symbols,

$$y \leq x \text{ (the capital constraint)}; \quad y \leq 1 - a(x' - (1-d)x) \text{ (the labor constraint)}. \tag{2}$$

The line MV then represents a situation where the capital and labor amounts available to the consumption sector are identical and therefore identically equal to the amount of consumption good being produced. As such, MV is the locus of initial and final terminal stocks for which there is full employment and no excess capacity of capital. It can be referred to as the full employment, no excess-capacity line. At a point of Ω "below" the line MV, the points S_3 and S_d for example, all machines are being utilized but there is surplus labor. Similarly, at a point of Ω "above" MV, points S_1 and S_ℓ for example, there is full employment but idle machines and excess capacity.

Since this is a crucial consideration of the model, a further elaboration of this fact is perhaps warranted. Note that MV divides Ω into two parts: the triangle MOV and the "open" remainder $LMVD$. Consider any one-period plan in the triangle MOV, say S represented by the generic coordinates (x, x'). At the initial capital stock x corresponding to S, a net investment of SS_d leads to full employment of labor. On appealing to properties of similar triangles,[30] we obtain

$$\frac{SS_d}{MO} = \frac{x' - (1-d)x}{1/a} = \frac{VS_d}{OV} = 1 - x. \tag{3}$$

Since the capital-labor ratio in the consumption sector is unity, x also represents the employment in this sector, and hence (3) furnishes the characterization (equation) of the line MV as the full-employment line when it is rewritten as

$$1 - a(x' - (1-d)x) = x \Longrightarrow x' = (1 - d - 1/a)x + (1/a) = \xi x + (1/a). \tag{4}$$

[28]This does not of course imply a "free lunch" – labor which is kept in the background, is being used to produce the output. McKenzie's intuition of this set in [23, Figures 7 and 8], for example, bears comparison with Figure 1.

[29]The equation of the line extending the segment MV is given by $x' = \xi x + 1/a$; see (4) below.

[30]This constitutes a basic prerequisite for understanding our exposition, and we shall assume it.

We can now use this line to determine the surplus labor associated with any one-period plan in MOV. At the point S_2, for example, the amount of terminal capital stock required to generate full employment is given by the ordinate of the point S, and since net investment at S_2 is only S_2S_d, the ordinate of S_2 falls short by the amount SS_2. This segment SS_2 then is a measure of surplus labor. Similarly, at the one-period plan, S_3, there is surplus labor in the amount $S_1'S_3$. Thus the triangle MOV represents plans with no excess capacity of capital but surplus, unemployed labor.

When we turn to a one-period plan, say S_1 in $LMVD$, the situation is reversed. At the initial capital stock x corresponding to S_1, we have already seen that a net investment of S_dS supports full employment of labor. But at S_1, there is more investment than this, and consequently less labor to man all the available stock of capital. At this stage, note an alternative interpretation of the lines OD and ML; they are iso-employment lines pertaining to the consumption sector. ML represents the labor constraint in (2) above with y equated to zero, and OD with it equated to unity. Thus the line passing through S and parallel to ML and OD represents the employment in the consumption sector at the initial capital stock x. Its interesection with the line MV furnishes us with the capital stock that this reduced amount of labor can use, and at S_1, it is given by the abscissa of the point S_1'. This can be seen again by considering similar triangles.

$$\frac{S_\ell S_1}{S_\ell S} = \frac{MS_1'}{MS} = \frac{S_1'S_1''}{x'S} = \text{ capacity utilization rate, to be designated by } \alpha. \tag{5}$$

Thus at the plan S_1, there is full employment of labor, but idle capacity of the amount $x - S_1'S_1''$. Only part of the stock of available capital, as measured by $S_1'S_1''$ is used. Similarly, at the one-period plan, S_ℓ, the capacity utilization rate α is zero while at S it is unity with all the available capital stock being used. We thus have the characterization (equation) of the line MV as the no excess capacity line. In summary, $LMVD$ represents plans with excess capacity of capital but full employment of labor.

It is worth noting at this stage that at a given initial capital stock x, as in Figure 1, the movement from S_ℓ to S_d traces out a production possibility surface in the space of consumption and investment goods in Figure 2. It is useful to see how the arguments furnished above in the context of Figure 1 translate to Figure 2. If all labor is employed in the investment sector, the economy is at S_ℓ in Figure 2, with complete specialization, full employment and zero utilization rate α. As we move down the production possibility surface, decreased investment and consequent release of labor leads to the production of the consumption good, and to an increase in α. Beyond S, there is no further increase in the consumption is blocked by the capital constraint, and the only effect is an increase in unemployment.[31]

We can now underscore the fact that a particular one-period production plan in Figure 2 represents more than simply initial and terminal capital stocks. Given the parameters we work with, for any plan in the triangle MOV, its abscissa (first coordinate), in furnishing the fully utilized stock of capital being used in the consumption sector, also furnishes the amount of labor that is being used there, as well as the amount of the consumption good and the utility that it provides. On the other hand, for any plan in $LMVD$, as discussed above, one can determine the capital stock that is being used, and from it all of the other amounts.[32]

All these considerations are useful when we turn to the depiction of the planner's preferences. In Figure 3, consider the point I_1 on the MV line. By specification and construction, the amounts of labor and capital being used in the consumption good sector are all identical to the amount of the

[31] Note how Rybczynski's theorem pertaining to an equilibrium without specialization holds in our setting. As x increases, the capital intensive consumption good increases and the investment good (with zero capital intensity) decreases. This conclusion is reversed with an increase of labor. For details, see for example [8, 9], and also [10].

[32] Footnote 25 is also relevant in this connection.

consumption good and the level of utility being produced. Now move vertically downwards from I_1. This implies a decrease in the terminal capital stock x' and a consequent increase in the labor available to the consumption good sector. But this sector is constrained by capital and hence no increase in the consumption good, and thereby in utility, is possible. In terms of our previous discussion, labor is being made increasingly available in a zone in which there is already surplus labor. On the other hand, move outwards from I_1 in a direction parallel to ML or OD. This move represents an increase in the initial capital stock x and a consequent increase in the capital available to the consumption good sector. But no labor is being released and this sector now faces a labor constraint. Hence, again no increase in the consumption good, and thereby in utility, is possible. In terms of our previous discussion, capital is being made increasingly available in a zone in which there is already idle capacity. Thus we obtain an indifference curve $I_1I_1I_1$ of the Leontief type with a kink at its intersection with MV. If the initial capital stock is increased and terminal capital stock decreased, relative to I_1, as at I_2 or I_3 for example, an increased amount of both capital and labor is made available to the consumption sector, resulting in a higher level of the consumption good, and thereby of utility. The line MV thus takes on a new identity having solely to do with preferences; it pegs a map of kinked indifference curves in the initial, terminal capital plane. The planner's preferences are complete in the sense that we now have an indifference map over all of the period-to-period production set Ω with OML marking the zero-utility indifference curve. We shall denote the utility function[33] corresponding to this indifference map is by $(x, x') \longrightarrow u(x, x')$.

The basics of our geometrical apparatus are now all laid out: the 45^o-line, the lines ML and OD constituting Ω, the point V, and the line MV delineating both a privileged subset of Ω and the indifference map corresponding to u.[34]

4 Determination of the Benchmarks

The first unknown to be determined is a standard benchmark in the theory of intertemporal resource allocation – the level of the capital stock that allows a maximal sustainable utility level, the so-called golden-rule stock $\hat{x}$. In the words of [23, p. 389], "at a point of maximal sustainable utility, the terminal capaital stocks must be as large as the initial stocks, and the utility must be as large as possible given this condition."[35]

It is easily seen in Figure 4 to be the unique one-period plan G obtained by the intersection of the 45^o-degree line with MV. But the geometry actually enables us to compute it once we recall the slope of the line MV identified in (2) above.

$$\frac{MG'}{G'G} = \frac{(1/a) - \hat{x}}{\hat{x}} = -\xi \Longrightarrow \frac{1/a}{\hat{x}} = 1 - \xi = d + 1/a \Longrightarrow \hat{x} = \frac{1}{1 + ad}. \tag{6}$$

The golden-rule stock is solution to a maximization problem in which the objective function u maximized over the constraint set Ω and the additional sustainability constraint that terminal stocks are not less that the initial one.[36] As such, there is a shadow price $\hat{p}$ associated with the sustainability

[33]Analytically, the utility function is given by $u(x, x') = \max\{y \in \mathbb{R}_+ : 0 \leq x \text{ and } y \leq 1 - a(x' - (1 - d)x)\} = \min[1 - a(x' - (1 - d)x), x]$.

[34]The period-to-period production set Ω has been specified in Footnote 26 and the utility function u in Footnote 33.

[35]McKenzie [23, p. 389, paragraph 3] also refers to this as a "von Neumann point". For a formal definition in the context of this model, see [12], and more generally [5], [3].

[36]Analytically, we maximize $u(x, x')$ subject to $x' \geq x$ for all $(x, x') \in \Omega$.

constraint. On appealing to Uzawa's version of the Kuhn-Tucker theorem [48],[37] we can write

$$u(x, x') + \hat{p}(x' - x) \leq u(\hat{x}, \hat{x}) \text{ for all } (x, x') \in \Omega. \tag{7}$$

We can now follow Radner [30] and define the value-loss $\delta_{(\hat{p},\hat{x})}(x, x')$ at the golden-rule price system $\hat{p}$ associated with the one-period plan (x, x') by re-writing the above as[38]

$$\delta_{(\hat{p},\hat{x})}(x, x') = u(\hat{x}, \hat{x}) - u(x, x') - \hat{p}(x' - x) \text{ for all } (x, x') \in \Omega. \tag{8}$$

All this is a standard rehearsal of a key concept in the general theory of intertemporal resource allocation.[39] What is possibly new is that we have all the machinery we need to furnish a clear geometrical representation of this idea.

We begin with the determination of the golden-rule price system $\hat{p}$. Towards this end, consider Figure 4, and note that the zero net investment one-period plan V, given by $(1, (1 - d))$, can be substituted in (7) to yield

$$1 - \hat{p}(1 - d - 1) \leq \hat{x} \Longrightarrow \hat{p} \geq \frac{1 - \hat{x}}{d} \Longrightarrow \hat{p} \geq \frac{V'V}{PV}. \tag{9}$$

By the same token, the maximal net investment, zero consumption one-period plan M, given by $(0, 1/a)$, can be substituted in (4) to yield

$$0 + \hat{p}(1/a - 0) \leq \hat{x} \Longrightarrow \hat{p} \leq \frac{\hat{x}}{(1/a)} \Longrightarrow \hat{p} \leq \frac{OG''}{MO}. \tag{10}$$

Now, on using the value of $\hat{x}$ in (3), an easy computation yields that

$$\frac{1 - \hat{x}}{d} = \frac{1 - \frac{1}{1+ad}}{d} = \frac{ad}{d(1 + ad)} = \frac{\frac{1}{1+ad}}{1/a} = \frac{\hat{x}}{(1/a)}.$$

Hence the weak inequalities are all equalities in the following expression

$$\frac{V'V}{PV} = \frac{1 - \hat{x}}{d} \leq \hat{p} \leq \frac{\hat{x}}{(1/a)} = \frac{OG''}{MO}. \tag{11}$$

But then in terms of the geometrical development, we have shown that the angle $\angle OMG''$ equals the angle $\angle V'PV$. Since the triangle MOG'' is congruent to the triangle $OM'G''$,

$$\angle OM'G'' = \angle OMG'' = \angle V'PV = \angle PV'G,$$

which implies that the line $M'O$ is parallel to the PV'. We have thus shown that the golden-rule price-system is given by the slope of either of the lines $M'O$ and PV'.[40]

[37] Note that all of the conditions for the invoking of this theorem are satisfied: u is a concave function, Ω is a convex set, and the positivity of a leads to Slater's form of the Kuhn-Tucker qualification being satisfied. Also see Gale [6] for a generalization relevant to context of growth theory.

[38] Note that we are defining a function $\delta_{(\hat{p},\hat{x})} : \Omega \longrightarrow \mathbb{R}_+$. We depart from the literature in retaining the subscript $(\hat{p}, \hat{x})$ in $\delta_{(\hat{p},\hat{x})}(x, x')$ even though the golden-rule price system will remain fixed in the sequel.

[39] As surveyed for example in [21]. One may quote here Gale's [6, p. 22] statement that the necessity and sufficiency of (7) for the $\hat{x}$ to be the golden-rule stock "provides the single most important tool in modern economic analysis both from the theoretical and computational point of view".

[40] Note that it is the slope as measured relative to the Y-axis.

All that remains is the determination of the zero value-loss line, which is to say, the locus of all one-period production plans for which $\delta_{(\hat{p},\hat{x})}(x,x') = 0$ where from (5),

$$\begin{aligned} \delta_{(\hat{p},\hat{x})}(x,x') &= u(\hat{x},\hat{x}) - u(x,x') - \hat{p}(x'-x) \\ &= [u(\hat{x},\hat{x}) - x - \hat{p}(x'-x)] + [x - u(x,x')] \\ &= \text{shortfall from golden-rule utility level + idle capacity} \end{aligned} \tag{12}$$

The first point to be noticed in this connection is that the slope of the line MV which was identified to be ξ in (1) turns out to be characterized also by the golden-rule price system in a particularly simple form. Working with Figure 4, we obtain

$$\xi = -\frac{MG'}{G'G} = -\frac{MO - G'O}{OG''} = -\frac{1 - \frac{G'O}{MO}}{\frac{OG''}{MO}} = -\frac{1-\hat{p}}{\hat{p}}, \tag{13}$$

where the last equality follows from the above indentification of the golden-rule price system. Once the slope of the line MV is determined, we need only its intercept for a full identification. Since MV passes through the golden-rule stock point $\hat{x}$ on the 45^o-degree line, we obtain its equation as

$$x' = -\frac{1-\hat{p}}{\hat{p}}x + C \Longrightarrow \hat{x} = \frac{\hat{x}}{\hat{p}} + C \Longrightarrow \hat{p}x' + (1-\hat{p})x = \hat{x} = u(\hat{x},\hat{x}). \tag{14}$$

We have already seen the line MV serving three distinct roles: as the full employment line, as the no excess capacity line and and as benchmark line for delineating the planner's preferences. The latter yielded, in particular, the conclusion that the utility level, $u(x,x')$, of any one-period production plan (x,x') on MV is x. On substituting (14) in (12), we obtain

$$\hat{p}x' + x - \hat{p}x = u(\hat{x},\hat{x}) \Longrightarrow u(\hat{x},\hat{x}) - u(x,x') - \hat{p}(x'-x) = 0 = \delta_{(\hat{p},\hat{x})}(x,x'), \tag{15}$$

and thereby discover a fourth identity of the line MV in this geometric development: it is the zero value-loss line with value-loss governed by the golden-rule price system. It is worth pointing out here that all the one-period production plans on the zero value-loss line constitute what is referred to as the von Neumann facet.[41]

But now, we are in a position, through Figure 5, to determine the value-loss of any one-period production plan, which is to say, of any point $(x,x') \in \Omega$. As (12) makes clear, this value-loss is *not* determined solely by the difference in intercepts (say, on the 45^o-degree line) of MV and a line parallel to it and passing through the one-period production plan. For one thing, this would lead us to conclude that there is value-gain as the line MV moves outwards. Lines parallel to MV are indeed iso-value-loss lines, but they depict the value-loss after taking excess capacity into account. Of course, for all one-period production plans, say S_1, in the surplus labor triangle MOV, there is no excess capacity of capital and hence its utility is furnished by its first coordinate, leading to the second term in (12) being zero. Hence its value-loss consists only of its shortfall from golden-rule utility level, the first term in (12). This is given by the difference between $\hat{x}$ and the abscissa of the point of intersection S of a line $M'V'$ parallel to MV and passing through S_1. To see this, let the coordinates of S be given by $(\bar{x},\bar{x}')$,

[41] See McKenzie [21], [22], [23]. In [23, p. 391], McKenzie writes "The von-Neumann facet plays a crucial role in the multi-sector Ramsey model once the assumption of strict concavity of the reduced utility function is dropped." As the reader has noticed, this assumption does not hold for the RSS model.

and hence the equation of the line $M'V'$ is given by[42]

$$x' = -\frac{1-\hat{p}}{\hat{p}}x + C \Longrightarrow \bar{x}' = -\frac{1-\hat{p}}{\hat{p}}x + \left(\bar{x}' + \frac{1-\hat{p}}{\hat{p}}\bar{x}\right). \tag{16}$$

If we now denote the abscissa of S as x^o, and recalling that S is the intersection of $M'V'$ with the 45^o-line, we obtain the shortfall from the golden-rule utility level that we seek:

$$\begin{aligned} \frac{x^o}{\hat{p}} = \left(\bar{x}' + \frac{1-\hat{p}}{\hat{p}}\bar{x}\right) \Longrightarrow u(\hat{x},\hat{x}) - x^o &= u(\hat{x},\hat{x}) - \hat{p}\bar{x}' - (1-\hat{p})\bar{x} \\ &= u(\hat{x},\hat{x}) - \bar{x} - \hat{p}(\bar{x}' - \bar{x}) = \delta_{(\hat{p},\hat{x})}(\bar{x},\bar{x}'). \end{aligned} \tag{17}$$

In this demonstration, we have also shown that any one-period plan on $M'V'$ has the same value loss.

Next, we turn to one-period plans in the "open" parallelogram $LMVD$. In this case, value-loss stems from both excess capacity and from the negative shortfall from the golden-rule utility level. We have already seen that this shortfall is the same for all plans on the line S_2S_5 parallel to MV, and is given by the difference between $\hat{x}$ and the abscissa of the point of intersection S_3 of S_2S_5 and the 45^o-line. In order to show that S_2S_5 is an iso-value-loss line, all that remains is for us to show that the excess capacity associated with any one-period production plan on it, say S_2, S_3, S_4, S_5, is identical. But this is easy from our procedure for computing excess capacity: all of the triangles with vertices S_2, S_4 and S_5 exhibited in Figure 5 are congruent, and hence their bases are equal.

Next, we have to show that the value losses increase as iso-value loss lines move "away" from the zero-value loss line MV in either direction. This is clear when we limit ourselves to the full capacity, surplus labor triangle MOV. The difficulty concerning one-period plans in the full employment, excess capacity area $LMVD$ lies in the fact that as MV moves outwards, both the negative shortfall from golden-rule utility as well as the excess capacity increase. However, the latter increases more than the former. To see this, consider the parallel lines $M'V'$ and $M''V''$ in Figure 6. The increase in the shortfall amounts to x_1x_2, whereas the increase in the excess capacity is the amount W_1W_2. To see that W_1W_2 is always greater than x_1x_2, draw a line $M'F$ parallel to the 45^o-line, and simply observe that the difference in the abscissae of the points F and M' (which is x_1x_2 since triangles with vertices F and F' are congruent) is greater than W_1W_2. And this is always so by virtue of the fact that the slope of the 45^o-line is steeper than the slope of OD, which is another way of saying that the rate of depreciation d is always less than unity.

We now have a basic geometric tool, iso-value-loss lines, to characterize optimal policies, and we turn to providing such a characterization.

5 Optimal Policies ($\xi = -1$): Equilibrium Cycles

Return to Figure 1, and note an incidental aspect that we have ignored so far in our discussion: the line MV is orthogonal to the 45^o-line. This orthogonality is simply a consequence of how the two parameters, the input-output ratio a and the depreciation rate d relate to each other. In particular, from (1), and the fact that the $\angle OMV$ is 45^o,

$$-\xi = 1/a - (1-d) = \frac{MV'}{V'V} = 1 \Longrightarrow a = \frac{1}{2-d}. \tag{18}$$

[42]The procedure is identical to that already used in (14) in the derivation of the equation of the line MV.

For later reference we graph this relationship in Figure 7, and simply note here that it depicts the range of parameter values of the model, albeit singular, under which we work in this section.[43]

Returning to Figure 4, we note that the orthogonality of MV and the 45^o-degree line implies that the triangles GPV and GPP' are congruent to each other, and therefore the segment PV equals PP', which implies that $PP'QV$ is a square with the golden-rule stock point G as its center. With this, the geometry brings to light interesting invariants pertaining to the case that we are considering. Irrespective of how a and d change, but provided that they continue to relate to each other as graphed in Figure 6, the golden rule stock $\hat{x}$ is given by $1 - (d/2)$, the abscissa of Q is $(1 - d)$, and finally, the golden rule price system, $\hat{p}$, as given by the ratio, $V'V$ to PV, is always one half and thereby independent of a and d.[44]

We now turn to the characterization of optimal policies by considering the square $PP'QV$ in Figure 8. For this, we need to recall Brock's Theorem [3] which guarantees that any program (trajectory) that minimizes the aggregate of the sequence all (non-negative) value-losses over the long run is indeed an optimal trajectory, provided the golden-rule stock is unique and some additional (convexity) assumptions concerning Ω and $u(\cdot, \cdot)$ are satisfied. We ask the reader to accept that this theorem applies in our case.[45] But this makes it evident that the optimal program starting from any initial stock in the interval QV cycles around the golden-rule stock. It bears emphasis that this consequence of Brock's theorem follows only from the characterization of MV as the zero-value-loss line (the von-Neumann facet); the fact that losses increase as we move away from MV in either direction is only used when we want to assert that the identified program is uniquely optimal. In any case, it is a result that has a surprise to it, ruling out as it does, other plausible intuitions about the optimal policy. Thus, it rules as non-optimal, say starting from $x(0)$, the entirely feasible trajectory of maintaining current consumption levels in excess of what optimality requires and investing less so as to have the golden-rule stock of machines tomorrow.

An important aspect of these optimal programs deserves to be highlighted. The geometry, in particular the fact that $PP'QV$ is a square with G as its center, makes it evident that the average of the optimal program starting from any initial stock in the interval QV is precisely the golden-rule stock. This is precisely Brock's average turnpike property which asserts that the average of the capital stocks of all "good programs" and not only optimal ones converge to the golden-rule stock.[46] This is precisely the case when the von Neumann facet is not stable.[47]

When we turn to initial capital stocks that lie outside the interval QV, we retain this aspect of optimality under which development planning entails over-building and then over-depreciating relative to the golden-rule stock $\hat{x}$. The exceptions to this are situations when the the initial capital stock is $\hat{x}$ itself, or $x(0)''$ in the interval between 1 and 2, or other identifiable and computably characterizable initial stocks in the intervals $[2^{n-1}, 2^n]$, where $n = 2, 3, \cdots$. We leave it to the reader to use Figure 4 to show for herself that for all initial capital stocks less than that corresponding to the point Q, or for those in $]1, 2]$, the optimal policy converges to the two-period equilibrium cycle in two periods. In general, for an initial capital stock in $[2^{n-1}, 2^n]$, it takes $n + 1$ periods to converge to the equilibrium cycle, where $n = 1, 2, \cdots$. In general, optimal policy is to follow the path charted by the two lines MV

[43] Note also the function relating a to d is an infinitely differentiable convex function.

[44] Of course, we could have equally well expressed all these values in terms of the input-output ratio a; we leave this to the interested reader. Once the geometry allows us to see these invariants, it is a simple matter to also obtain them from the formulae presented in (1), (3) and (8).

[45] One of the assumptions in [3] is that Ω be compact which is clearly not fulfilled in our case. It is not a difficult matter to show that Brock's methods extend to yield a theorem that does apply to our situation; for details, see [12].

[46] See [5] for a definition of "good programs", and [3] for details pertaining to the claim. In particular, note the importance of the assumption that the golden-rule stock is unique.

[47] It is also the case when Inada's 1964 assumption that "all paths that remain on the facet forever converge uniformly to a maximal stationary path" does not hold; see [22, p.15] for a discussion.

and OD, which is to say, the path tracked by the following difference equation:

$$x(t+1) = \max[-x(t) + 1/a, (1-d)x(t)] \quad \text{for all } t = 0, 1, \cdots. \tag{19}$$

We are now in a position to make two observations. The first relates to what is termed as "history independent" dynamical systems in [26]. The optimal policy function underlying the optimal program leads to a dynamical system which is anything but history independent. The speed of convergence to the equilibrium, but perhaps more importantly, the particular equilibrium and its amplitude, are all dependent on the initial stock of machines the economy is endowed with. Second, note that the set of equilibria to which the optimal programs converge are of uncountable cardinality; and whereas it does not make sense to say that this cardinality is inversely related to the rate of depreciation, we can say that the length of the interval QV serving as a proxy for the set increases as we decrease both d and a in a way that is consistent with the relationship pictured in Figure 7.

We conclude this section with the natural question as to how the optimal policies characterized here change when the parameter values a and d are not functionally related as shown in Figure 6. We turn to this in the next section.

6 Optimal Policies $(\xi < -1)$: Convergence in Finite Time

A reader familiar with only the basics of the theory of difference equations will appreciate the fact that the qualitative properties of the optimal program analyzed in the previous section hinge on the unit root of the underlying characteristic equation corresponding to the line MV – the only complication relates to the fact that there is a kink in the difference equation.[48] In this section we grapple with the difficulties arising from the fact that the absolute value of this root is greater than one.[49] What is interesting is that despite this, perhaps because of it, rather than a divergence from the golden rule stock, we obtain particularly fast convergence to it golden-rule stock. However, the geometric analysis becomes somewhat more intricate.

The basic intuition proceeds as follows: One can fall back on the previous case to suppose that the optimal policy is tracked by the locus MVD. This is easily seen to be false by considering a one-period production plan such as S. The optimal policy cycles around the golden rule. Not only is the average turnpike prpoerty negated but also that program is bad. Thus the optimal policy is to jump off the path tracked by MVD. The question is when and to where. The answer to the second part is obvious – since the value losses are zero at the golden rule stock, and it is sustainable through time, it is clear that optimal program must end there. And once the solution is phrased this way, first part of the question also admits the only possible answer: namely, "as soon as possible." The only reason why this cannot be accomplished in one period is because of the irreversibility of capital – the economy may be so capital-rich that it takes time to dispose of its capital. Where this intuition fails is in situations where the initial stock of capital is less than the golden-rule stock – the capital-poor economy. In this case, there is always over-building of capital followed by subsequent depreciation to the desired stock. As we shall see below, this is related to the desirability of always maintaining full employment. In any case, our methods enable us not only to show convergence in finite time but to be able to compute the number of time periods required to attain convergence, and to be able say how they are related to

[48]In [22, p. 714], McKenzie writes "The behavior of paths on [the von Neumann facet] F may be studied by means of difference equations" and refers to his 1963 paper where this was "done explicitly for the generalized Leontief model.

[49]Readers not aware of difference equation may simply ignore these remarks.

determinable ranges on initial capital stock. Simply stated, the action takes place on the OD arm of the locus MVD, and this is easy to analyze.

The first point to be established is that optimality requires, for any time period t, $t = 0, 1, \cdots$, and for all initial capital stocks $x^*(t)$ falling in the interval $[0, \hat{x}]$, the period's terminal capital stock is chosen to be on the no value-loss line MV. This is simply to say that for all such initial capital stocks, full employment and full capacity be maintained the next period. If this was not so, and there was an optimal program $\{x^*(t)\}_{t=0}^{\infty}$ such that for some time period $\bar{t}$, $x^*(t \bar{+} 1)$ was strictly below MV, the reader can check for herself in the context of Figure 8, or of Figure 6, that there exists an alternative program which differs from the optimal only at time $t \bar{+} 1$, and in having a stock higher than $x^*(t \bar{+} 1)$ then. But such a program has a lower value loss, and therefore contradicts the optimality of the given optimal program. Note that this argument has nothing to do with the slope of the MV line, and that is why we can work with Figure 6 or Figure 8.

We can now gain more intuition into the basis of our claim by considering Figure 9. Full employment guarantees that for all stocks in the interval $[0, \hat{x}$ no one-period production plan is chosen in the surplus labor zone. The iso-value-loss line guarantees that no choice be made off the segment MG or V_1D. Thus all that remains is the optimality of the segment GV_1, so to speak.

Next, we begin with an overview of the optimal program, and once we have a claim, we can furnish its substantiation. Towards this end, we begin with an identification of benchmarks for values of the initial capital stock. First, in Figure 8, read off some invariants labelled $\zeta_1, \zeta_2, \cdots, \zeta_{\underline{n}}$ in the interval $[0, \hat{x}]$, where $\underline{n}$ is such that $\zeta_{\underline{n}-1} > 0$ and $\zeta_{\underline{n}} < 0$. Since the zero-value-loss line intersects with the Y-axis (at M), and OD has a positive slope, such a $\underline{n}$ is well-defined.[50] As can be seen from Figure 8, these benchmarks correspond, indeed are determined by,[51] capital stock values $1, 2, 4, \cdots, 2^n$, $n = 1, 2, \cdots$.

Next we turn to series of values of capital stocks to the right and left of the unit capital stock corresponding to the kink V at the termination of the zero-value-loss line. The ones on the right are given by[52] γ_n^r, $n = 1, 2, \cdots$, and the ones on the left, by γ_n^ℓ, $n = 1, 2, \cdots, \bar{n}$, where $\bar{n}$ is such that $\gamma_{\bar{n}-1}^\ell > 0$ and $\gamma_{\bar{n}}^\ell < 0$. Again, since the zero-value-loss line intersects with the Y-axis (at M), and OD has a positive slope, such a $\bar{n}$ is well-defined.[53] As can be seen from Figure 8, all of these benchmarks correspond, indeed are determined by,[54]

We have now all that is necessary to show that the optimal policy is to follow the path charted by the three lines MV, GV_1 and OD, which is to say, the path tracked by the following difference equation:

$$x(t+1) = \max[-\xi x(t) + 1/a,\ \hat{x},\ (1-d)x(t)] \quad \text{for all } t = 0, 1, \cdots. \tag{20}$$

We shall develop the argument in a series of steps. First, we show that the proposed policy is optimal for a particular value of an initial capital stock lying in the interval $[\hat{x}, 1]$. This is the crux of the argument, and the next step is to use it to show that the proposed policy is optimal for any initial capital stock lying in the interval $[\hat{x}, 1]$. Successive steps are to work through successive intervals, alternating between left and right, until the intervals on the left are exhausted, and the focus shifts only to intervals on the right.[55]

[50] This is simply to say that a finite number of such intervals $[\zeta_n, \zeta_{n+1}]$, $n = 1, \cdots$, cover the interval $[0, \hat{x}]$.

[51] The specific formulae are as follows: $\zeta_n = (1/\xi)(2^{n-1} - (1/a))$, $n = 1, 2, \cdots, \bar{n}$.

[52] The superscript r in γ_n^r stands for right, and the superscript ℓ in γ_n^ℓ for left.

[53] A version of Footnote 50 is also relevant here.

[54] The specific formulae are as follows: $\gamma_n^r = \hat{x}/(1-d)^n$, $n = 0, 1, 2, \cdots$, and $\gamma_n^\ell = (1/\xi)((1/a) - \gamma_n^r) = (1/\xi)((1/a)\hat{x}/(1-d)^n)$, $n = 0, 1, 2, \cdots, \bar{n}$.

[55] $[\zeta_1, \hat{x}]$, $[1, \gamma_1^r]$, $[\zeta_2, \zeta_1]$, $[\gamma_1^r, 2]$,], $[\gamma_1^\ell, \zeta_1]$, $[2, \gamma_2^r]$, $[0, \gamma_1^r]$, $[\gamma_2^r, 4] \cdots, [2^n, \gamma_n^r]$, $[\gamma_n^r, 2^{n+1}], \cdots, n =$.

For concreteness, let the initial capital stock be unity. We shall show that the optimality requires that the next period's choice be the golden-rule capital stock, which is to say, the one-period production plan J in Figure 9. Towards this end, consider the vertical line going through the points P and V to which the planner is limited in his choice of next period's capital stock. All one-period production plans on this line higher than J have greater value losses and are therefore ruled out.

We now magnify the part of Figure 9 around G as Figure 10, and construct three production plans J_1, J_2 and J'. J_1 is a one-period plan such that maintenance of full employment next period leads to an unchanged initial stock of capital. Thus, J_1 is a point on the line PV such that it yields a unit capital stock next period. J' is a one-period production plan such that zero employment in the consumption sector yields the same value-loss as the plan J. Thus, it is the intersection of the line OD with the a line parallel to MV and going through the point J. Finally, J_2 is a one-period plan with an initial stock of capital equal to unity and such that maintenance of full employment next period leads to the initial stock of capital as the plan J'. Thus, J_2 is a point on PV such that it yields a capital stock next period identical to the initial capital stock of J'. It is clear that the ordinates of J, J_1 and J_2 are strictly decreasing.

We now adopt a convention that shorten descriptions: when we say that a planner chooses a particular point in Ω, we are (loosely) referring to a one-period plan whose initial stock is the abscissa of the point and which the planner takes as given, and whose terminal stock is the ordinate of the point that she chooses.

All one-period production plans on PV equal to or lower than J_2 have a lower value-loss this period than J, but a higher value loss next period, and therefore cannot be part of an optimal program. Hence, all that remains for consideration is the interval of one-period production plans strictly between J and J_2.

Suppose the planner chose J_1. Since she can also choose it next period, and since this was her optimal choice this period, stationarity of our model dictates that it is also her optimal choice next period. But this leads to a program that keeps accumulating value losses *ad infinitum* and is not even a good program, leave alone an optimal one.

Thus suppose that the planner's optimal choice is a one-period production plan between J_1 and J_2. Full employment implies that next period's initial stock is higher than unity, which is to say that next period's plan is limited to a vertical line to the right of PV. But any point chosen on such a line gives a higher value-loss than before, and these losses will keep on accumulating unless the planner puts a stop by choosing a plan on the extension of GJ. But she could have done this in an earlier period, contradicting the optimality of the choice of a plan between J_1 and J_2.

Thus all that remains is to consider the choice of a production plan between J and J_1 – the darker trajectory in Figure 9. Towards this end, we first observe that the defining characteristic of the case that we analyze is that the no-value-loss line MV is steeper than the 45^o-line, and this fact has not been explicitly used so far. We do so now in Figure 11; this also allows us to appreciate what is special about the case of the previous section. What is true here is that GJ is strictly greater than JJ_1. To see this, construct the line $P'S$ orthogonal to the 45^o-line, recall how J_1 was derived – from J to P to P_2 to P_2'. Since MV is steeper than $P'S$, PP_2 is less than PS, this completes the proof of the claim.

The above fact allows us to assert that the line parallel to MV from any point in the segment JZ intersects the segment GJ.

Consider a plan, say Z, chosen between J and J_1. By the choice of Z, as opposed to J, the planner saves value not exceeding WJ. Given feasibility and the maintenance of full-employment, the choice of Z takes the economy to a capital stock given by the vertical line P_1P_1' through Z_0, Z_1 and Z_2, which is to say, a capital stock closer to the golden-rule stock than in the previous period. Since the triangles

$W_1Z_0Z_1$ and WJZ are congruent, the saving in value does not outweigh by the subsequent value-loss. But any point higher than Z_0 on P_1P_1' has even greater value-loss than at Z_0.

Thus suppose the planner chooses a point in the segment Z_0Z_1. Under such a choice, the least value-loss in the period would be at Z_1, but again, since W_1Z_0 equals WJ, would still be greater than initially choosing J.

Thus all that remains is the consideration of a choice on P_1P_1' below Z_1. But Z_1 is playing the role for Z that J_1 played for J, and so by an ragument identical to what we have seen before, the value-loss at Z_0 is less.

Again, as in the case of $\xi = -1$, we over-build and under-depreciate, but in a way that the discrepancies go to zero in a finite number of time periods, this number depending on the magnitude of the stock of machines that we initially start with. But there are exceptions to this statement where there is monotonic convergence.

The golden-rule stock $\hat{x}$ and the golden-rule price system are no longer invariants. By inspection of Figure 8, we can see that they are inversely related to the rate of depreciation d and directly related to the input-output ratio a. And in these parametric changes we are no longer on the knife-edge pictured in Figure 6 but have "room to manoeuvre".

The dynamical system is "history independent" in terms of [26]. The equilibrium is unique, and optimal programs irrespective of the initial capital stock they begin with, converge to it. To be sure, the speed of convergence, and the qualitative feature of the optimal program (amplitude, monotonic versus non-monotonic convergence etc.), does depend on the initial capital stock, but these features are not given prominence in the definition of "history independent" systems that we are working with.

7 Optimal Policies ($\xi > -1$): Monotonic Convergence

In this section, we turn to the case grapple with the difficulties arising from the fact that the absolute value of the root is greater than one.[56]

Three sub-cases, none of them analytically difficult but perhaps of some interest from an economic point of view: $1- < \xi < 0$, $\xi = 0$, $0 < \xi < 1 - d$. Essentially a modification of figures sent earlier. But a slightly disorientation when MV has a positive slope.

8 Possible Extensions

9 Conclusion

Summarize and go back to Stiglitz' quote in the introduction. No living on air. Bang-bang. monotonicity. Fixing the rate of profit to be zero. relate to the rate of discount. labor surplus. labor allocation.

Open problems.

[56] 49 is also relevant here.

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