

A TIME-VARYING FISCAL REACTION FUNCTION FOR BRAZIL¹

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Preliminary Version

This paper evaluates the sustainability of public debt in Brazil using monthly data from the period between January 2003 and June 2016, based on estimation of fiscal reaction functions with time-varying coefficients. Three methods are considered: Kalman filter, penalized spline smoothing, and time-varying cointegration. Although the Kalman filter shows the best statistical results, all methods indicate that the fiscal reaction declined over almost the entire period considered, and lead to the conclusion that the Brazilian public debt, observing the parameters then in force, reaches an unsustainable trajectory in the last years of the study.

Key words: public debt, sustainability, fiscal reaction function, time-varying coefficients, Kalman filter, penalized spline smoothing, time-varying cointegration.

1. Introduction

Public debt in Brazil has been growing rapidly in recent years. For example, between the fourth quarter of 2013 and the second quarter of 2016, general government gross debt rose from 51.7% to 68.5% of the country's Gross Domestic Product (GDP). Consolidated public sector net debt rose from 30.6% to 42% of GDP (BCB, 2016). Factors that contributed to this increase are not only the high and successive primary deficits, but also an interest rate systematically higher than the product growth rate.

The more the debt grows in relation to output, the more fiscal effort is needed to reverse its trajectory, since the interest expense and the risk premiums rise, feeding back into debt and debt charges. Thus, it is evident that the primary balance needed to stabilize it tends to become increasingly higher (Cysne and Gomes, 2016).

¹ Preliminary Version.

² ENCE and FGV/EPGE

³ FGV/EPGE

A condition established in a common definition of public debt is that the sum of the anticipated future primary surpluses (revenues minus expenses, excluding interest payments), duly discounted to present value, be sufficient to offset the debt amount and interest payments at the present time (Blanchard et al, 1990). For the size of the country's economy and its evolution to be considered, both public debt and primary surpluses are usually expressed proportionally to output.

This definition, although simple, involves uncertainty not only in relation to future surpluses, but also regarding the appropriate discount factor, since - among other sources of uncertainty - future interest rate and product growth rate are unknown. This leads to the need of a statistical treatment for the problem.

One approach to investigating debt sustainability in an environment of uncertainty is based on the application of unit root tests to the debt series based on Hamilton and Flavin's (1986) seminal paper. Another approach involves the cointegration analysis between income and expenditure series (Treham and Walsh, 1991). In Brazil, some papers have used these techniques to investigate debt sustainability, including Rocha (1997) and Issler and Lima (2000), Simonassi (2007) and Ourives (2002). This approach, however, came to be questioned after Bohn's paper (2007), which demonstrates that the concept of sustainability is compatible with a series of integrated debt of any finite order, not necessarily unitary. Through this result, the above-mentioned tests become insufficient to conclude that the debt trajectory is unsustainable¹.

An alternative approach - proposed by the same author in 1998 - gained prominence in the economic literature, becoming dominant in the discussions on sustainability. This approach is based on the concept of fiscal reaction function, which establishes a relationship between primary surpluses and changes in public debt stock. The idea is to verify if, and to what extent, the fiscal authority responds to increases in debt, through the generation of primary surpluses.

Among the publications on fiscal reaction function, besides the article by Bohn (1998), we can

¹ Other criticisms of Bohn regarding the unit root testing are its low power, sensitivity to sample size, possible non-stationarity related to factors that are not directly related to the fiscal policy carried out, and the fact that in its conventional formulation, structural breaks and product fluctuations and government spending - quite frequent in emerging economies do not adjust adequately. It should be noted that some of these criticisms may also apply to the unit root tests we carried out in this paper. These are, however, only tests that are subsidiary to the analysis, and are not central to obtaining the results.

highlight Canzonerie (2001), Davig and Leeper (2006), Penalver and Thwaites (2006), Uctum, Thurston and Uctum (2006), Adedeji and Williams (2007), Khalid et al. (2007), Greiner and Kauermann (2007), Nguyen (2007), Budina and Wijnbergen (2008), Turrini (2008), Afonso and Hauptmeier (2009), Song (2009), Egert (2010), Stoica and Leonte (2010), Burger et al. (2011), Fincke and Greiner (2015) and Mutuku (2015); and, in Brazil's case, Luporini (2001), De Mello (2005), Lima and Simonassi (2005), Aguiar (2007), Lopes (2007), Mendonça, Santos and Sachsida (2009), Cavalcanti and Silva (2010), Dill (2012), Jesus (2013), Simonassi (2013) and Luporini (2015).

In Brazil, the issue of the sustainability of public debt has aroused strong interest amongst economic analysts, since it plays a central role in the discussions involving fiscal adjustment. This paper aims to contribute to this discussion, evaluating the sustainability issue from the estimation of fiscal reaction functions appropriate to the Brazilian reality, for the period between January 2003 and June 2016. The proposed specifications are inspired by the original work of Bohn (1998), but other important variables for the Brazilian case are included in the fiscal reaction function, investigating its impact empirically.

Another recent trend in the literature is to assume that both the fiscal authority's attitude and the effectiveness of its policies may vary throughout the study period, due to changes in the macroeconomic scenario. For example, Uctum, Thurston and Uctum (2006) analyze the implications of structural breaks for the estimation of fiscal reaction functions for some countries in Asia and Latin America (but not for Brazil); Greiner and Kauermann (2007) apply semi-parametric methods for estimating models with time-varying coefficients for some OECD countries. For the Brazilian case, more recently, Mendonça, Santos and Sachsida (2009) apply a Markovian transition model to identify regime changes; Simonassi (2013) uses a fiscal reaction function incorporating structural breaks; and Luporini (2015) estimates ordinary least squares regressions using moving windows.

Following this practice, this paper considers three possible specifications for models with coefficients that may vary throughout the study period. An additional advantage of this approach is to enable the investigation of sustainability in sub periods of interest. In the Brazilian case, this interest lies at the end of the sample, where an intense and growing fiscal deterioration is verified. The results obtained provide evidence that Brazilian public debt reaches an unsustainable trajectory as of January 2014.

2. Sustainability and Fiscal Reaction Function

2.1 Government Budget Constraint and the Sustainability Condition

The government budget constraint, in nominal terms, is represented as follows:

$$B_t = G_t - T_t + (1+i_t)B_{t-1} \quad (2.1)$$

where B_t is the net debt stock ¹, G_t are the government's primary expenditures (consumption, investment and transfers, not including interest payments), T_t are the primary revenues (tax plus other net current revenues) - all computed at the end of time t - and i_t is the nominal interest rate, associated with a security purchased at time $t-1$ and remunerated at t .

A public debt series – or, accordingly, the fiscal policy associated with it - is characterized as sustainable if the present value of future surpluses is sufficient to offset the present debt value. To formalize this condition, the budget constraint in (2.1) must be solved iteratively for $t = 1, 2, \dots, T$ (it is considered that $i_t = i$, for simplicity purpose):

$$B_t = (1+i)^t B_0 + \sum_{k=1}^t (1+i)^{t-k} (G_k - T_k), \text{ or even: } B_0 = \frac{B_t}{(1+i)^t} + \sum_{k=1}^t \frac{S_k}{(1+i)^k},$$

where $T_k - G_k = S_k$ is the primary surplus at instant $t = k$.

The condition of sustainability of debt B in the form of equality is given by:

$$\lim_{t \rightarrow \infty} \frac{B_t}{(1+i)^t} = 0 \quad (2.2)$$

At (2.2), $B_0 = \sum_{k=1}^t \frac{S_k}{(1+i)^k}$, i.e., the discounted sum of primary surpluses at present value is equal

to the current debt, thus satisfying the definition of sustainability presented.

2.2 Fiscal Reaction Function and the Sustainability Test

¹ Consider B_t as the gross debt would equal to disregard the government assets and the remuneration thereof, which would result in the equation (2.1) becoming just an approximation for the debt evolution. This occurs because the nominal deficit is not opposed to a variation of same value in the gross debt. This ratio only applies to the net debt.

Bohn (1998) presents a sustainability test whose importance would become more pronounced in 2007, when Bohn (2007) establishes the test formally, making it less interesting - for this purpose – testing for the presence of a unit root and/or cointegration, which was the previously predominant method in the literature on statistical treatment of sustainability of public debt.

This section is dedicated to presenting Bohn's test, which establishes the theoretical basis of this work. Initially, it is convenient to rewrite (2.1) in relation to the nominal product Y . This is because the analysis of a country's debt and its evolution is better analyzed when considering the size of the economy, which affects both the insolvency risk and the potential surplus.

To do this, both sides of (2.1) are divided by Y_t , to obtain:

$$\frac{B_t}{Y_t} = \frac{G_t - T_t}{Y_t} + (1 + i_t) \frac{B_{t-1}}{Y_t}.$$

Then, $(1 + i_t) \frac{B_{t-1}}{Y_t}$ is multiplied by $\frac{Y_{t-1}}{Y_{t-1}}$, to arrive at:

$$\frac{B_t}{Y_t} = \frac{G_t - T_t}{Y_t} + (1 + i_t) \frac{B_{t-1}}{Y_{t-1}} \frac{Y_{t-1}}{Y_t}.$$

The following notation is now defined: let X be any variable (representing B , G , or T). Then $x = X/Y$. This notation is applied to the previous equation, which is therefore rewritten as:

$$b_t = g_t - t_t + (1 + i_t) b_{t-1} \frac{Y_{t-1}}{Y_t} \quad (2.3)$$

where $b = B/Y$ is the debt expressed as product ratio, denominated Debt-to-GDP ratio.

The variation rate of the product θ_t is defined so that:

$$Y_t = (1 + \theta_t) Y_{t-1} \quad (2.4)$$

Using (2.4) in (2.3), and defining $s_t = t_t - g_t$ as the primary surplus in relation to the product, the equation (2.3) can be rewritten as follows:

$$b_t = -s_t + \frac{(1 + i_t)}{(1 + \theta_t)} b_{t-1} \quad (2.5)$$

Bohn (1998) establishes a fiscal reaction mechanism, defined as follows:

$$s_t = \rho b_{t-1} + \gamma X_t \quad (2.6)$$

where X_t is a vector of control variables¹.

In the following theoretical developments, with the purpose of evaluating the sustainability condition for the simplest case, the ρ coefficient, the nominal interest i and the product growth rate θ are considered constant. It is assumed a priori that $\rho > 0$, in the sense that increases in the Debt-to-GDP Ratio in a period tend, in the subsequent period, to reduce the deficit or raise the surplus².

Then, the sustainability condition is derived from the fiscal reaction function. Specifically, the conditions that must be satisfied by the fiscal reaction coefficient ρ will be developed in such a way as to ensure that the public debt trajectory is sustainable.

Replacing (2.6) in (2.5)³:

$$b_t = \left(\frac{1+i}{1+\theta} - \rho \right) b_{t-1} \quad (2.7)$$

Solving (2.7) iteratively, it is obtained:

$$b_t = \left(\frac{1+i}{1+\theta} - \rho \right)^t b_0 \quad (2.8)$$

For the debt-to-GDP ratio ($b_t = B_t/Y_t$), the sustainability condition is given by:

$$\lim_{t \rightarrow \infty} \frac{b_t}{(1+i)^t} = 0 \quad (2.9)$$

instead of (2.2). Thus, the direct conclusion from (2.8) and (2.9), considering the approximation

¹ Bohn (1998) specifies the function with b_t , and not b_{t-1} , on the right side. However, to circumvent the problem of simultaneous causality, several applied works consider the debt-to-GDP ratio to be out of date, which makes even more practical sense. Another option is to consider a bivariate model (it will be discussed in section 4.3).

² As we use here a partial equilibrium model of debt evolution, there is a symmetry with respect to the cause of the surplus generated. The reaction function does not establish whether surpluses are generated by an increase in revenue or a containment of expenses. As a possible alternative, for example, Nguyen (2007) and Jesus (2013) specify their fiscal reaction functions with revenue and expenditure as dependent variable, respectively.

³ For simplicity, the term γX_t is omitted, which does not imply loss of generality.

$\frac{1+i}{1+\theta} \cong 1+i-\theta$, is that the following condition is necessary:

$$\rho > i - \theta \quad (2.10)$$

3. Data

Monthly data from January 2003 to June 2016 was used. The concept of debt adopted was that of consolidated public sector net debt ¹ (federal, state, and municipal spheres, social security, Central Bank and government-controlled companies - except Petrobrás and Eletrobrás). For surplus S_t , the consolidated primary result of the public sector accumulated for the previous 12 months was used, which is the reference used in the Budget Guidelines Law for the elaboration of the annual primary income targets.

To calculate the Debt-to-GDP Ratio, $b_t = B_t/Y_t$ and $s_t = S_t/Y_t$ it was considered that Y_t = monthly nominal GDP estimated by the central bank - based on IBGE quarterly data – also accumulated for 12 months. It should be noted that the use of accumulated variables for 12 months attenuates the impact of seasonality, which would be accentuated for primary surplus and GDP.

Figure 3.1 below shows the evolution of b_t and s_t over the study period:

Figure 3.1 - Monthly Evolution of b_t = Debt-to-GDP Ratio (values on the left axis) and s_t = Primary Surplus/GDP (values on the right axis) - January 2003 to June 2016

¹ It should be noted that working with the concept of net debt requires that the quality of remuneration of government assets does not differ much from those associated with government's liabilities, which hypothesis is questionable according to recent fiscal studies.



Source: prepared independently, based on Central Bank website data.

Despite the increase in net debt, in monetary terms, throughout the study period, the Debt-to-GDP ratio, an indicator adopted in this study, shows a downward trend until the beginning of 2014, when its trajectory reverses. The primary surplus-to-product ratio, on the other hand, is generally decreasing over the whole period, which trend was accentuated in the final part of the sample, corresponding to the period of greatest fiscal deterioration of the Brazilian economy.

Additionally, the series appear to have some correlation and evolve together most of the period. Bohn (1998) suggests as control variables the output gap, to capture the effect of oscillations in economic activity, and a variable indicative of sudden rises in spending (periods of war).

To calculate the output gap h_t , the monthly estimated GDP Y_t^R provided by IBRE/FGV GDP monitor¹ was taken, and the potential product Y_t^* , obtained via Hodrick-Prescott filter, applying the formula: $h_t = (Y_t^R - Y_t^*) / Y_t^*$. The cycles of sudden rise in expenditures (as periods of war in Bohn (1998)) were represented by specifying variables indicating election years.

One of Bohn's criticisms of sustainability analysis based on unit root tests is that these tests did not incorporate other variables affecting the debt trajectory, making it difficult to detect the reversion to the mean. The fiscal reaction function, on the other hand, allows to incorporate these variables, allowing to estimate in isolation the fiscal authority's response to debt increases. In this sense, in addition to the variables proposed in the original specification, others were considered

¹ Only estimate of real monthly GDP available in Brazil. It should be noted that some studies use the industrial production index or IBC-Br of the Central Bank, however these series are only proxies for the Real GDP.

particularly important for Brazil, which are listed below¹.

i_t : economy basic interest rate (Selic)

i_t^* : implicit interest rate, which is charged on the cost of debt load. In Brazil, this rate is quite different from the Selic, and the difference is accentuated when considering net debt².

r_t : debt Risk - Measure of risk perception associated with debt insolvency, calculated as a ratio between EMBI+ (monthly average) and the rating risk assigned by Standard & Poors'. The EMBI+ is an index based on debt securities issued by emerging countries, reflecting the difference between the rate of return on these securities and the US Treasury³.

π_t : inflation - monthly series obtained as the IPCA relative variation for the previous 12 months⁴.

se_t : deficit in current account (or foreign savings) and tt_t : terms of trade = ratio between export prices and import prices, both representing the effect of the external sector.

4. Methodology

To estimate the function and reaction proposed by Bohn (1998), specified by equation (2.6), it is considered that the fiscal reaction function parameters may vary over time, thus allowing for structural changes and discretionary policy, in addition to checking sustainability for sub periods of interest. Three modeling strategies and estimation methods are used, presented below.

¹ Sources - Net consolidated debt of the public sector, nominal gross domestic product, annualized implicit interest rate, export volume and net external liabilities: <http://www.bcb.gov.br>; consolidated primary surplus of the public sector: <http://www.tesouro.fazenda.gov.br>; EMBI: <http://www.ipeadata.gov.br>; S&P ratings: <https://www.standardandpoors.com>; Inflation: <http://www.ibge.gov.br> and exchange terms: www.funcex.org.br.

² It is assumed that i_t is the gross rate on public debt, i.e., without deducing the portion that returns to the government in the form of taxes on interest, such taxes being included in the variable T_t . The alternative of considering i as net rate, not including in T_t the taxes on interest, would not change the results of the work at all.

³ Lopes (2007) *apud* Megale (2003). The classifications have been converted into a numerical variable as follows: D (defaulter) = 0; SD = 1; CC = 2; CCC- = 2.5; CCC = 3; CCC+ = 3.5; B- = 4; adding 1 point for each promotion. For the positive (negative) concepts attributed by S&P, an increase (decrease) of 0.25 is considered.

⁴ Inflation is not significant in any work on fiscal reaction function in Brazil contemplating the period after 1994 monetary stabilization. Nevertheless, it was decided to evaluate its significance in this work.

4.1 State Space Modeling and Kalman Filter

The state space representation (Harvey, 1989) is a way of expressing a linear statistical model, which allows the estimation of the parameters of this model for each instant of time. This representation consists of two equations. The first one is the observation equation, which represents the evolution of the series s_t in time:

$$s_t = z_t' \alpha_t + d_t + \varepsilon_t, \quad (4.1)$$

where z_t is a vector ($m \times 1$), α_t is a vector ($m \times 1$), called the state vector, d_t is a scalar and ε_t is a white noise term with zero mean and variance σ_ε^2 , for $t = 1, 2, \dots, T$, where T is the total of observations. The second equation is state transition:

$$\alpha_t = T_t \alpha_{t-1} + c_t + R_t v_t, \quad (4.2)$$

where T_t is a matrix ($m \times m$), called state transition matrix, c_t is a vector ($m \times 1$), R_t is a matrix ($m \times g$) and v_t is a vector ($g \times 1$) of white noise terms, with zero mean and covariance matrix Q_t . The vector of states at instant $t = 0$, α_0 , has a_0 mean and covariance matrix P_0 . Additionally, ε_t and v_t satisfy $E(\varepsilon_t v_s') = 0$, $\forall t, s = 1, 2, \dots, T$.

In this paper, the equation (4.1) represents the fiscal reaction function, where s_t is the surplus-to-GDP ratio, $z_t = (1 \quad b_{t-1} \quad X_{t-1})$, $\alpha_t = (\mu_t \quad \rho_t \quad \gamma_t)$ - where μ_t is the intercept, ρ_t is the tax reaction coefficient and γ_t is a vector that contains the coefficients of the variables in X_{t-1} , including s_{t-1} - and $d_t = 0$. The equation (4.2) represents the evolution of μ_t , ρ_t and γ_t , where $c_t = 0$ and T_t and Q_t are diagonal matrices, so that all elements of α_t follow - by hypothesis - autoregressive processes of first order and mutually independent. Therefore, (4.1) and (4.2) become:

$$s_t = (1 \quad b_{t-1} \quad X_{t-1}) \begin{pmatrix} \mu_t \\ \rho_t \\ \gamma_t \end{pmatrix} + \varepsilon_t \quad (4.3)$$

$$\begin{pmatrix} \mu_t \\ \rho_t \\ \gamma_t \end{pmatrix} = \begin{pmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & \Phi_3 \end{pmatrix} \begin{pmatrix} \mu_{t-1} \\ \rho_{t-1} \\ \gamma_{t-1} \end{pmatrix} + \begin{pmatrix} \eta_t \\ v_t \\ \kappa_t \end{pmatrix} \quad (4.4)$$

where Φ_3 is also a diagonal submatrix, whose elements are the specific coefficients of the

autoregressive processes obeyed by the controlling variables at X_{t-1} in the reaction function.

These equations are estimated using a method called Kalman filter (Kalman (1960), Kalman and Bucy (1961)). This method consists of predictive and updating equations. The predictive equations represent the expected value and the variance of the state vector at the time t , subject to the available observations up to the instant $t-1$, $S_{t-1} = \{s_1, s_2, \dots, s_{t-1}\}$:

$$a_{t|t-1} = \mathbf{E}(\alpha_t | \mathbf{S}_{t-1}) = \mathbf{T}_t a_{t-1|t-1} + c_t \quad (4.5)$$

$$P_{t|t-1} = \mathbf{V}(\alpha_t | \mathbf{S}_{t-1}) = \mathbf{T}_t P_{t-1|t-1} \mathbf{T}_t' + \mathbf{Q}_t \quad (4.6)$$

The update equations - or filtering - represent the expected value and the variance of the state vector at t , subject to the available observations until time t , $Y_t = \{y_1, y_2, \dots, y_t\}$:

$$a_{t|t} = \mathbf{E}(\alpha_t | \mathbf{S}_t) = a_{t|t-1} + \mathbf{K}_t (s_t - z_t a_{t|t-1}) \quad (4.7)$$

$$P_{t|t} = \mathbf{V}(\alpha_t | \mathbf{S}_t) = P_{t|t-1} - \mathbf{K}_t z_t' P_{t|t-1} \quad (4.8)$$

where the expression $\mathbf{K}_t = P_{t|t-1} z_t' (z_t P_{t|t-1} z_t' - \sigma_\varepsilon^2)^{-1}$ is called Kalman gain. For the estimation of coefficients of the fiscal reaction function, we used smoothing equations, which consider the information of the whole sample to estimate the coefficients at each moment, allowing a more efficient estimation. These equations are shown below:

$$a_{t|T} = \mathbf{E}(\alpha_t | \mathbf{S}_T) = a_t + P_t \mathbf{T}_t' P_{t+1|t}^{-1} (a_{t+1|T} - \mathbf{T}_t a_t) \quad (4.9)$$

$$P_{t|T} = \mathbf{V}(\alpha_t | \mathbf{S}_T) = P_t + P_t \mathbf{T}_t' P_{t+1|t}^{-1} (P_{t+1|T} - P_t) [P_t \mathbf{T}_t' P_{t+1|t}^{-1}]' \quad (4.10)$$

The coefficients of \mathbf{T}_t in (4.2), shown in equation (4.4), which govern the evolution of each coefficient, are considered constant over time, as well as the variances of error terms. These fixed parameters (hyperparameters) are estimated by maximum likelihood method (Harvey, 1989).

4.2 - Penalized Spline Smoothing

An n -degree spline is a piecewise continuous function that joins multiple n -degree polynomials to generate a smooth curve using a finite set of points. In this paper, the coefficient of fiscal reaction $\rho(\cdot)$ is represented by a spline whose components are weighted in such a way as to maximize the fit to the surplus observed. Thence:

$$s_t = \rho(t)b_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \text{iid}(0, \sigma^2) \quad (4.11)$$

$\rho(\cdot)$ is estimated based on the following criterion:

$$\min_{\rho} \sum_{t=1}^n (s_t - \rho b_{t-1})^2 + \lambda \int \rho''(t)^2 dt \quad (4.12)$$

That is, the quadratic error of the regression is minimized – first term of (4.12) - and the curve is penalized - second term of (4.12). Then $\rho(t) = \sum_{j=1}^d \gamma_j B_j(t)$ is defined, where $\{\gamma_j\}$ are real numbers. Now being:

$$\Omega_{ij} = \int B_i''(t) B_j''(t) dt \quad (4.13)$$

and the matrix Ω , with each element (i,j) given by (4.13). Thus, the problem in (4.12) becomes:

$$J(\beta(\cdot), \alpha, \lambda) = \min_{\beta} \sum_{t=1}^n (s_t - \gamma B(t) b_{t-1} - \alpha X_t)^2 + \lambda \gamma \Omega \gamma^T \quad (4.14)$$

whose solution is given by the following system of equation:

$$\frac{\partial J(\beta(\cdot), \alpha, \lambda)}{\partial \alpha_i} = \sum_{t=1}^n 2(s_t - \gamma B(t) b_{t-1} - \alpha X_t) X_{it} = 0, \quad \forall i = \langle 1, \dots, p \rangle \quad (4.15)$$

$$\frac{\partial J(\beta(\cdot), \alpha, \lambda)}{\partial \gamma_j} = \sum_{t=1}^n 2(s_t - \gamma B(t) b_{t-1} - \alpha X_t) B_j(t) b_{t-1} = \lambda \gamma_j (\Omega_j + \Omega_j^T), \quad \forall j = \langle 1, \dots, d \rangle \quad (4.16)$$

where Ω_j denotes the j th line of Ω . The above expressions represent a system of linear equations of $p + d$ order. The parameter λ governs the data penalization - in such a way to avoid overstepping - being defined by cross validation as in Hastie and Tibshirani (1990) and Craven and Wahba (1979) and/or information criteria as in Akaike, 1969. (For a survey of the methods for model selection see Hastie et. al. (2008))

With respect to hypothesis tests, Cantoni and Hastie (2002) present tests involving degrees of freedom, which in the case of this work are particularized to test the linearity of the empirical model. More specifically, the test statistic defined in the article, corresponding to the test of $H_0: \lambda = \lambda_0$ versus $H_A: \lambda < \lambda_0$, has asymptotic distribution F with degrees of freedom depending on λ_0 and the data. The case of interest involves testing the hypotheses $H_0: df = df_0$ versus $H_A: df > df_0$, where linearity corresponds to $df_0 = 2$. The results of this test will be reported in section 5.3.

4.3 - Time-Varying Cointegration

Johansen (1988) suggests a method based on the following autoregressive vector model:

$$\Delta Z_t = \mu + \Pi' Z_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \Delta Z_{t-j} + \varepsilon_t, t = 1, 2, \dots, T, \quad (4.17)$$

where $Z_t = (Z_{1t}, Z_{2t}, \dots, Z_{kt})$ is a vector ($k \times 1$) of observations for each series at the instant t , μ is a vector ($k \times 1$) of intercepts, Γ_j , $j = 1, \dots, p$, are vectors ($k \times 1$) of coefficients of ΔZ_{t-j} , and ε_t is a vector ($k \times 1$) of errors, so that $\varepsilon_t \sim N(0, \Omega)$. The Johansen test is based on the rank of Π . If the hypothesis that the matrix has an r rank is not rejected, we conclude that there are r cointegration vectors, where $r < k$. In this case, we can write $\Pi' = \alpha \beta'$, where β is a matrix ($k \times r$) whose columns are the cointegration vectors and α is the vector ($1 \times k$) whose components are the coefficients measuring the long-term equilibrium adjustment speed.

In this paper, we consider a cointegration ratio that may vary over time. Park and Hahn (1999) present a method where the evolution of cointegration vector components are defined by a Fourier series. Such procedure applies only if a single cointegration ration exists between the variables. Bierens and Martins (2010) suggest a more general procedure extending the Johansen method, allowing the incorporation of multiple cointegration relationships considering the model:

$$\Delta Z_t = \mu + \Pi_t' Z_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \Delta Z_{t-j} + \varepsilon_t, t = 1, 2, \dots, T, \quad (4.18)$$

the only difference between (4.18) and (4.17) being the fact that matrix Π varies over time. Two aspects should be highlighted in (4.18): the vector of intercepts μ does not varies over time e $\Pi_t = \alpha \beta_t$, so that only β varies over time, with vector α kept constant. The evolution of β over time is represented by Chebyshev polynomials, defined as: $P_{0,T}(t) = 1$, $P_{i,T}(t) = 2^{1/2} \cos(i\pi(t-0.5)/T)$, $t = 1, 2, \dots, T-1$. It is proved that any $g(t)$ time function can be represented as a linear combination of $T-1$ Chebyshev polynomials (Hamming, 1973).

In this paper, statistical criteria (Bierens and Martins, 2010) are used to define the number m of polynomials that satisfactorily approximates the trajectory of the coefficient β . Therefore, the evolution of β over time can be represented as:

$$\beta_t = \sum_{i=0}^m \xi_{i,T} P_{i,T}(t). \quad (4.19)$$

The higher the value of m , the more precise, however, the less smooth the approximation. A small value of m imposes a smooth behavior for β_t , approaching the invariant case. Therefore, the methodology allows to contemplate different patterns of behavior in the cointegration vector over time, capturing possible long term nonlinear relationships (see Granger(1988)). Substituting (4.19) into (4.18), we have:

$$\Delta Z_t = \mu + \alpha \xi' Z_{t-1}^{(m)} + \Gamma X_t + \varepsilon_t, \quad (4.20)$$

where $\xi' = [\xi_0', \xi_1', \dots, \xi_m']$ is a matrix $[r \times (m+1)k]$, of rank r , $Z_{t-1}^{(m)} = (Z_{t-1}', P_{1,T}(t)Z_{t-1}', P_{2,T}(t)Z_{t-1}', \dots, P_{m,T}(t)Z_{t-1}')$ and $X_t = (\Delta Z_{t-1}', \dots, \Delta Z_{t-p+1}')$. To investigate the hypothesis of time-varying cointegration, one should conduct the test: $H_0: \prod_t' = \alpha \beta'$ x $H_1: \prod_t' = \alpha \beta_t'$. Under H_0 (restrict model), $\xi' = (\beta', O_{r,km})$, where β is a matrix $(k \times r)$ whose columns are invariant cointegration vectors, so that, in (4.18), $\xi Z_{t-1}^{(m)} = \beta' Z_{t-1}^{(0)}$, com $Z_{t-1}^{(0)} = Z_{t-1}'$. Thus, at H_0 , all the coefficients of the Chebyshev polynomial are null, except for the first, which corresponds to $m = 0$, in which case the cointegration is invariant. On the other hand, H_1 postulates that at least some of the coefficients in (4.19) are nonzero, and thus the cointegration relationship varies over time. To investigate such hypotheses the wild and sieve bootstrap methods were used (Martins, 2013).

5. Results

In this paper, we use three estimation strategies for the fiscal reaction function, all considering linear specifications whose coefficients can vary in time, with the variables described in section 3. In all the proposed approaches, the coefficients of all the variables involved are considered to vary over time. This allows not only the response of the primary-result-to-GDP ratio to the Debt-to-GDP ratio, but also the partial effects of the other remaining variables, to be time-varying¹.

Two approaches were considered for short term estimation - Kalman filter and penalized spline smoothing - and the method of time-varying cointegration. In the case of the latter method, its conventional version was initially implemented with constant coefficients, using the model in (4.17). This version is not directly comparable with the other two methods because it specifies a static relation. On the other hand, in the version with time-varying coefficients, the cointegration

¹ Greiner and Finckler (2015), for example, implement the penalized spline smoothing method with restrictions, considering that only the fiscal reaction coefficient varies over time, keeping the remaining ones constant.

vector, according to Bierens and Martins (2010), can be interpreted as both a possible approximation for a long-term nonlinear relationship, as well as a linear relation whose coefficients may vary over time. This last interpretation is common to all three approaches, making it possible to compare them.

5.1 Long-Term Model

Firstly, unit root tests were implemented to investigate the non-stationarity hypothesis of the concerned series. The results are shown in annex I. It is concluded that, except for the output gap series, all other series show non-stationary behavior, the unit root hypothesis is not rejected by the ADF and Philips-Perron tests, and is accepted by the KPSS test at the usual levels. Thus, a natural approach is to investigate the existence of a long-term relationship between s_t , b_t and, possibly, other variable(s) and, thence, establish an error correction model to estimate together the short and long-term relationships between the variables involved.

The results obtained through the model estimation in (4.15) are presented below:

Table 5.1 – Conventional Error Correction Model

Cointegrating Eq:	CointEq1		
s(-1)	1.000000		
b(-1)	-0.054835 (0.00702) [-7.81054]		
se(-1)	2.11E-05 (7,2E-06) [2.93056]		
C	-0,023867 (0,00935) [-2.55262]		
Error Correction:	D(s)	D(b)	D(se)
CointEq1	-0.024363 (0.01179) [-2.06715]	-0.127823 (0.03499) [-3.65341]	-0.005615 (0.00587) [-0.95744]
D(s(-1))	0.066173 (0.08126) [0.81431]	-0.382277 (0.24124) [-1.58464]	-0.035221 (0.04044) [-0.87096]
D(b(-1))	-0.036445 (0.02831) [-1.28738]	0.036465 (0.08404) [0.43390]	0.012412 (0.01409) [0.88100]
D(se(-1))	2.60E-09 (7.8E-08) [0.03353]	8.99E-09 (2.3E-07) [0.03958]	0.409966 (0.07435) [-5.51383]
h(-1)	0.046021	-0.115643	0.008571

	(0.01201) [3.83039]	(0.03567) [-3.24227]	(0.00598) [1.43347]
D(r(-1))	0.009336 (0.02008) [0.46482]	-0.122574 (0.05962) [-2.05581]	0.002388 (0.00999) [0.23889]

The selected specification indicates the existence of a single cointegration vector between Primary Surplus/GDP, debt-to-GDP (b) and foreign savings (se), which corresponds to the following long-term relationship between those variables: $s_t = (0.054835)b_t - (2.11 \times 10^{-5})se_t$. The positive sign of the coefficient b_t is consistent with the idea of a fiscal deficit reducing as the net-debt-to-GDP ratio rises. The negative sign of the current account deficit ratio is justified by its use to finance the primary deficit.

As for the short-term model, the significant exogenous variables were output gap (h), which is stationary, and debt risk (r, in difference), both lagged in one unit of time. It should be noted that, in addition to the output gap and the correction term of errors, no other coefficients are significant in the equation for $D(s)$. This reflects the fact that the primary result, being the main fiscal policy instrument, played an endogenous role in successive corrections of the fiscal system towards long-term equilibrium.

5.2 Time-Varying Methods

The time-varying cointegration method was implemented using the same specification selected in section 5.1. EasyReg software, version 2015, was used¹. To implement the Kalman filter and penalized spline smoothing, the `dlm` and `mgcv` functions of R software, respectively, were used².

Table 5.2 below presents the averages of the fiscal reaction function coefficient estimates over time, obtained through the methods described in sections 4.1-4.3:

Table 5.2 - Fiscal Reaction Function - Average Coefficients Over the Period³

** - significant at level 0,05, * - significant at level 0,1.

Variable-dependent: consolidated primary result of the public sector in relation to GDP (s_t)			
control variables	Kalman filter	Penalized spline smoothing	Time-varying cointegration
s_{t-1} : primary result/GDP (at t-1)	0.9318**	0.9975**	1.0662**
b_{t-1} : debt-to-GDP ratio (at t-1)	0.0567**	0.0624**	0.0527**
h_{t-1} : output gap (at t-1)	0.0434**	0.0194**	0.0580**
se_{t-1} : current transaction deficit (at t-1)	-1.81×10^{-5} **	-1.14×10^{-5} *	-2.32×10^{-5} **
i^*_{t-1} : implicit interest rate (at t-1)	-0.0282*	-0.0877*	-
r_{t-1} : debt risk = EMBI/S&P (at t-1)	-	-	-0.0093

The Debt-to-GDP Ratio coefficient is positive and significant at 0.05 for all methods adopted. This indicates that a change in the net debt of 1% of GDP corresponds to a change in the primary surplus of approximately 0.05% of GDP. If, however, this fiscal reaction leads to a sustainable trajectory for public debt, it is an additional question, which involves the condition set out in section 2.2, equation (2.10), and will be investigated in chapter 6.

The lagged surplus coefficient (s_{t-1}) is significant, indicating a strong inertial component of the primary outcome series, as expected. The output gap coefficient h_t is positive and significant,

¹ <http://personal.psu.edu/hxb11/ERIDOWNL.HTM>

² The hyperparameters (coefficients of matrix T_t in equation (4.2)) estimates are reported in annex II.

³ The significance of the estimates was tested from 90 and 95% confidence intervals, for the Kalman filter, from variance estimates provided by equation (4.10); and for penalized spline smoothing and time-varying cointegration, by bootstrap method. In the latter case, we used the approach wild and sieve bootstrap proposed by Martins (2013).

indicating that, in periods of expansion a larger primary surplus is generated, either by increasing revenues or reducing public spending (for example, unemployment insurance). Obviously, the opposite occurs in the case of recession (negative output gap).

The coefficient of the deficit variable in current transactions is negative and significant in all methods, albeit at different levels. The negative coefficient occurs because when there is a greater acquisition of foreign savings by the economy it becomes easier to finance government's deficit. The reduced magnitude of the estimated coefficient reflects the units where the variables are represented: surplus as a percentage of GDP and foreign savings in millions of reais.

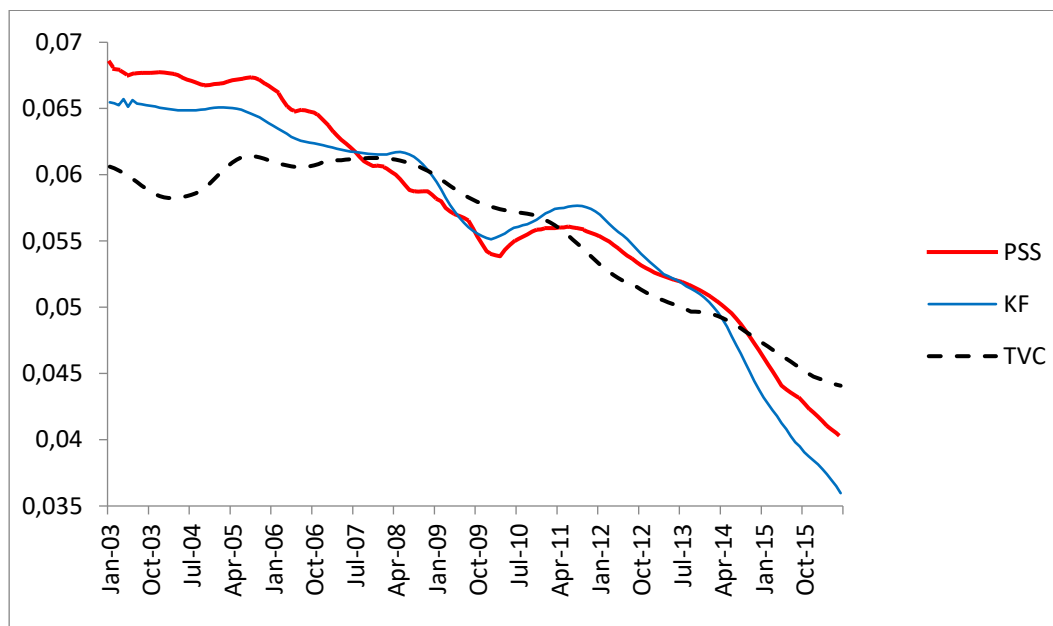
The simultaneous use of the implicit interest rate and the debt risk was avoided, since this leads to a pronounced inaccuracy of the estimates (high standard errors), due to the strong correlation between them. Accordingly, isolated specifications were estimated for each of them, each method leading to the selection of a different variable for the final model¹. The same occurs with external sector variables (deficit in current transactions and terms of trade), however in this case the results with the variable are higher for all methods.

The inflation variable is not significant for any of the methods, which is in line with the fiscal-reaction literature for the case of Brazil in the post-stabilization period (1994). One might expect an inflation effect on tax collection (Tanzi effect) or seigniorage. In the first case, there would be a negative impact on the fiscal reaction. In the second case, a higher inflation, if kept at interest, would affect the real value of the debt. In this case, the impact would be of a greater fiscal reaction. Apparently the two effects were either insignificant or offset, for the levels of inflation observed throughout the study period. Also, the effect of electoral years were not significant.

Figure 5.1 below shows the evolution of the fiscal reaction coefficient over time, according to the three methods adopted in this paper for its estimation (PSS = penalized spline smoothing, KF = Kalman filter and TVC = time-varying cointegration).

Figure 5.1 - Estimated Fiscal Reaction Coefficient by the Three Methods

¹ Both variables reflect the increased perception of insolvency risk, which, in turn, makes government bonds less attractive, which becomes an obstacle to debt growth. Indeed, it is noted that, in the bivariate model of table 5.1, these variables are not significant in the D(s) equation, but they are in the D(b) equation.



All methods point to a fiscal reaction with a declining trend that has become more pronounced in recent years, especially since 2014, with the Kalman filter indicating a steeper slope. This can be explained by the greater adaptive capacity of this method, despite the use of smoothing equations, which partially attenuate this property¹.

5.3 Constant Coefficient Hypothesis Tests

According to figure 5.1, the PSS and TVC methods present greater estimate stability over the period, in relation to the KF. Thus, it is important to test the hypothesis that this coefficient is constant over time. Fortunately, both methods allow us to test this hypothesis, according to the methodology presented in sections 4.2 and 4.3. The penalized spline smoothing method provided an estimate of $\lambda = 0,253$ - chosen by the Cross Validation procedure described in section 4.2 - for the fiscal reaction coefficient evolution. This value is lower than usual values in empirical papers for other countries, and evidences a strong sinuousness of fiscal reaction in relation to its mean value for the period, which provides indications for using a time-varying model. However, a hypothesis is necessary to investigate formally if this model presents statistical gain in relation to the conventional approach, i.e., the use of a linear model whose coefficients are time-invariant. The test that compares the estimated model with the linear model ($\lambda = \infty$) uses a statistics that is

¹ It should be clear, however, that the use of filtering or forecasting equations could make comparisons difficult with the other two methods considered, which consider the entire sample to be estimated at each instant.

based on the deviations of the estimates at each moment in relation to the average value in the period, whose distribution, for the reference model, has 2 degrees of freedom. Thus, H_0 is tested: $df = 2$ against $H_1: df > 2$, where df stands for degrees of freedom. The results indicate the rejection of the null hypothesis for the evolution of the coefficient of fiscal reaction and the deficit in current account, at a significance level of 0.05.

In the case of the time-varying cointegration method, the test proposed by Martins (2013), described in section (4.3), led to the rejection of the constant cointegration hypothesis, indicating a time-varying cointegration vector at 0.05 level.

5.4 Comparison Between Methods

This section presents a comparison between the three methods used to generate the estimates of sections 5.2 and 5.3. The objective is to choose the most appropriate method, so that the corresponding results are used to evaluate the sustainability of the debt in the period of study.

The reaction function adjusted by each method was used in this comparison, considering its estimated coefficients at each point in time, and its comparison with the effective primary surplus observed at each instant, using the mean square error and the mean absolute error.

Table 5.3, below, presents the results obtained, showing, notwithstanding the good results from the three methods, the slightly superiority of the Kalman filter in relation to the other.

Table 5.3 - Adjustment of Each Model to the Fiscal Reaction Effectively Observed

	Mean square error	Absolute mean error
Kalman filter	1.1388×10^{-5}	0.0029
Penalized spline smoothing	9.7451×10^{-5}	0.0046
Time-varying cointegration	1.8219×10^{-4}	0.0068

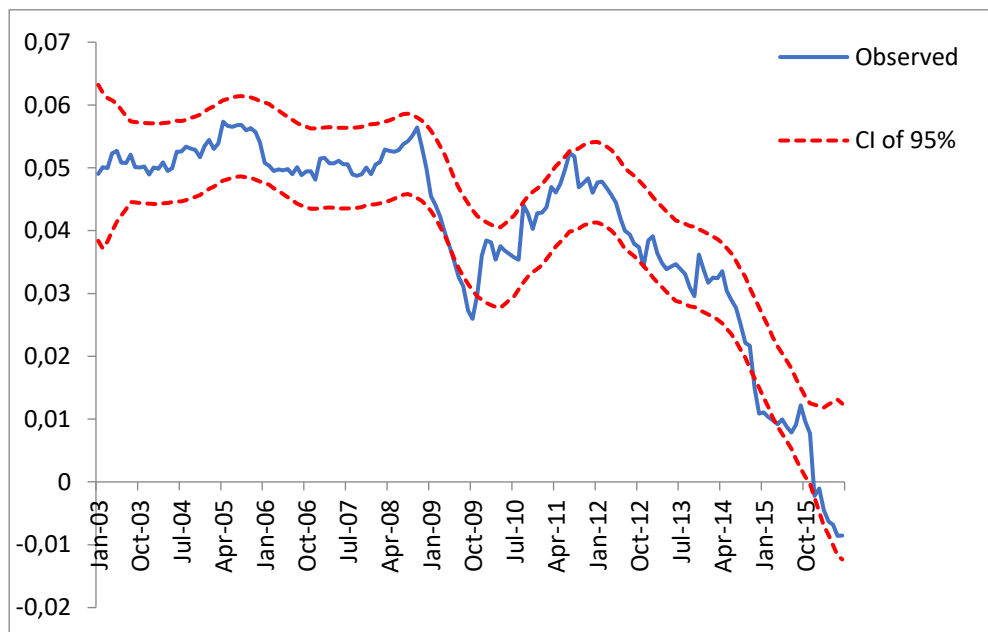
Another method used for comparison was the comparison of the probabilities of nominal coverage with the real coverage, also adopting as reference the primary surplus values effectively observed throughout the period. The table 5.4 below illustrates the results:

Table 5.4 – 95% CI Effective Coverage Probabilities for Each Method

	Observations outside	Actual/real coverage
Kalman filter	11	93.21%
Penalized spline smoothing	13	91.98%
Time-varying cointegration	16	90.12%

This criterion also leads to choosing the Kalman filter as the best method, with 11 out of the 162 real values being excluded from the estimated confidence interval, resulting in an effective coverage probability of 93.21%. The graph below illustrates this result.

Figure 5.2 - Observed Surplus vs. 95% Confidence Interval via Kalman filter

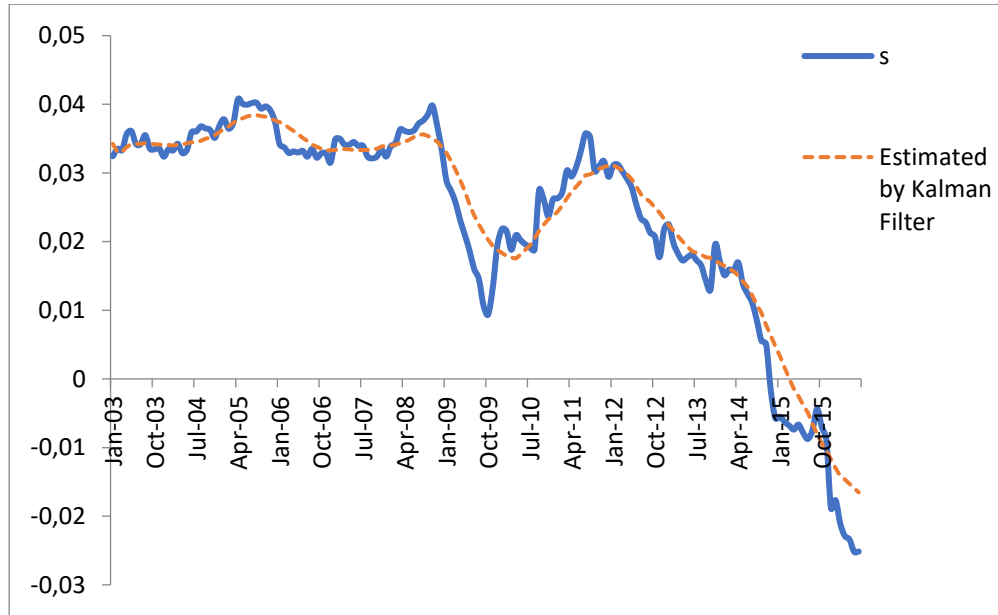


Points falling outside the range correspond to critical periods from November 2014 to March 2015 and June to October 2009, plus a single point in November 2012.

5.5 Model Adjust and Implications in Monetary Terms

The Figure below illustrates a comparison between the fiscal reaction estimated by the smoothing algorithm of Kalman filter and the primary surplus observed.

Figure 5.3 - Observed vs. Kalman Filter Estimated Primary Surplus



For practical illustration, we present in the remaining part of this section and in the following section (Section 6) some data based on the estimation by Kalman filter.

On average, in the period reviewed, the consolidated net debt of public sector increase of approximately 5% of GDP (around BRL 316 billion, for a GDP of around BRL 6.32 trillion) led to an increase in the consolidated primary surplus of the public sector around 0.28% (BRL 17.92 billion) of the GDP. A direct conclusion deriving from this result is that it is well below what the country needed to stabilize the net-debt-to-GDP ratio at the end of 2016.

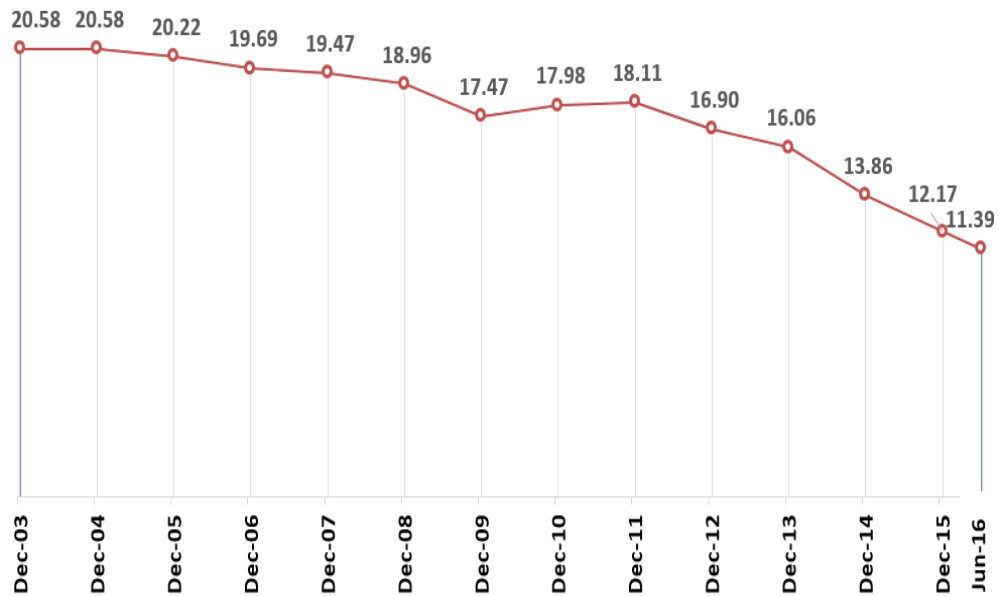
In fact, as the equation (2.5) shows, and considering that the real interest rate at the end of 2016 far exceeded the growth rate of the product, the stabilization of the net debt-to-GDP ratio would require a primary surplus (-s) sufficient to pay at least part of the actual interest on the debt (i.e.,

$$-s_t > \left(\frac{1+i_t}{1+\theta_t} - 1 \right) b_{t-1}).$$

However, in 2016 there was no primary surplus, but rather a deficit of approximately BRL 157 billion. Accordingly, the average reaction of BRL 17.92 of primary surplus to a rise in net debt of 5% of GDP is hardly a relieving data.

Let us now move from the average value of the fiscal reaction to the value estimated by the model, which may vary from period to period. Figure 5.4 shows the estimated changes in the primary surplus, period-to-period, in Reais of average purchasing power of 2016, when net debt is 5% of GDP:

Figure 5.4 - Fiscal Reaction Over Time, in Monetary Terms



Source: prepared independently.

As one might predict, based on the previous theoretical results, the fiscal reaction has declined over time, which is not something positive in the context of an attempt to stabilize the net-debt-to-GDP ratio.

6 Sustainability Analysis

Although, as mentioned earlier, the results presented at the end of the previous section and in this section are based only on the Kalman filter method (which provided the best results) the conclusions of the other methods adopted were basically the same. This applies with respect to the sustainability of debt in each period of interest.

In this section, we will use two different concepts of interest rates, the Selic rate and the implicit interest rate on net debt. It is important to note that the calculation of the previous fiscal reaction does not depend on this choice, since the variable interest rate was not statistically relevant for the estimation. Such a choice, however, will be important for the conclusion about debt sustainability, as we will see below.

An important advantage of estimation with variable coefficients is that it allows subsets of values

of the relevant variables (here, interest and GDP growth) to be used for specific periods. This is important because, as we shall see, sustainability analysis can lead to a result in the whole analysis period (2003-2016) distinct from that for a given sub period.

In other words, since we can get different mean values of the fiscal reaction ρ for different points of time, we can do the same with respect to interest and GDP growth. This may imply different conclusions about sustainability, depending on the period considered. We do not need to focus only on the mean values of these variables associated with the entire analysis period.

We will use below, as a criterion of sustainability, the approximation given by (2.13). We will consider the average value of the fiscal reaction coefficient estimated over the period considered, comparing it with the difference between the average values of the nominal (log) interest rate (i) and the (logarithmic) growth rate of nominal GDP (θ).

Let us take, for the beginning of the analysis, the entire period of estimation. Accordingly, consider the estimated average fiscal reaction between January 2003 and June 2016, which was 5.67%. Taking the Selic as an interest rate, we have a mean $(i-\theta)$, for the same period, of $12.81 - 10.80 = 1.51\%$, what indicates (considering a fiscal reaction value of 5.67) a sustainable trajectory throughout the study period.

If, on the other hand, the analysis is restricted to a more recent period of the Brazilian economy (say, as of January 2012), the average of the estimated fiscal reaction coefficients should be recalculated considering only the values in this sub-sample, and the result is now 4.75%, from a mean $(i-\theta)$, for the same period, of $10.11 - 8.16 = 1.95\%$. Again, we have a sustainable behavior, despite the strong reduction in the mean fiscal reaction.

However, analyzing Figure 5.1, the three estimation methods provide indications of a more pronounced decline in the fiscal reaction coefficient from 2014. In fact, if sustainability analysis is redone for the period between January 2014 and June 2016, the conclusion is reversed.

Although a substantial drop in the average of the coefficient of fiscal reaction estimates (from 4.75 to 4.27%) is not observed, the combination of the increase of the interest rate in the period with the fall in GDP results in a mean difference between them of 6.10% ($= 12.79\% - 6.69\%$) in the period, well below the estimated fiscal reaction coefficient. In this case, the indication is of an unsustainable trajectory.

Although the Selic rate is generally higher than the effective rate on consolidated government gross debt, the same is not true of net debt as we are doing here. The Central Bank calculates an implicit interest rate on this type of debt. As the quality of liabilities tends to be higher than the quality of public assets (hence many publications are based on gross debt, rather than net debt), the implicit interest rate on net debt tends to be higher than the Selic rate.

If we use this rate instead of the Selic rate, the conclusions about sustainability are similar except for the period between January 2012 and June 2016. In this period, in this new analysis, debt is unsustainable. Tables 6.1 and 6.2 summarize the results considering each tax, respectively.

Table 6.1 - Public Debt Sustainability per Sub-Period (interest rate = Selic)

	SINCE JAN 2003	SINCE JAN 2012	SINCE JAN 2014
SELIC (ln)	12,31	10,11	12,79
% GDP (ln)	10,80	8,16	6,69
FISCAL REACTION	5,67	4,75	4,27
SELIC- PIB	1,51	1,95	6,10
SUSTAINABILITY (SELIC-PIB < FISCAL REACTION)	SUSTAINABLE	SUSTAINABLE	UNSUSTAINABLE

Table 6.2 - Public Debt Sustainability per Sub-Period (implicit interest rate)

	SINCE JAN 2003	SINCE JAN 2012	SINCE JAN 2014
IMPLICIT (ln)	15.68	17.86	20.39
% GDP (ln)	10.80	8.16	6.69
FISCAL REACTION	5.67	4.75	4.27
SELIC- GDP	4.88	9.70	13.70
SUSTAINABILITY (SELIC-GDB < FISCAL REACTION)	SUSTAINABLE	UNSUSTAINABLE	UNSUSTAINABLE

We conclude that, regardless of the interest rate (and the estimation method) adopted, the public debt reaches an unsustainable trajectory at the end of the sample. However, it should be noted that, despite the notable decline in the fiscal reaction observed in Figure 5.1, cyclical factors such as the fall in GDP and the increase in the interest rate also contribute strongly to this result.

Particularly in the case of the Selic-based analysis, the transition to unsustainability as of January 2014 (in relation to the period beginning in January 2012) is due to changes in GDP and interest rates (with $i-\theta$ rising from 1.95 to 6.10%) rather than to fiscal reaction coefficient variation. Actually, the latter changes much less, from 4.75 to 4.27.

7. Conclusions

This paper presents some contributions. First, it exemplifies a case where estimation with constant coefficients over time can lead to wrong answers. In fact, the conclusions about sustainability are greatly enriched when the estimation method allows comparisons in different sub periods of the sample. Second, the paper compares, in a specific case, three distinct estimation methods with variable coefficients, showing that all of them lead to similar and statistically close conclusions. Third, it provides an important quantitative analysis of the fiscal reaction to the Brazilian economy in the recent period. Coupled with the recent substantial increase in the debt-to-GDP ratio, both gross and net, the analysis suggests a strong reversal of the fiscal reaction process, from a drop over time to a strong increase in the period after mid 2016. Similarly, the analysis of the data shows the importance, to reverse the difficult fiscal scenario, of lower interest rates and slowing growth. It would be interesting in future work, from the point of view of evaluating the robustness of the results, to use different definitions of the public sector, and a data sample allowing greater temporal amplitude.

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Annex I - Unit Root Tests

	Critical value 5%	$h_t = \text{output gap}$		$\pi_t = \text{inflation}$		$RD_t = \text{RISCO}$		$b_t = \text{Debt/GDP}$	
		Test statistics	p-value	Test statistics	p-value	Test statistics	p-value	Test statistics	p-value
ADF	-2,93	-3,79*	0,03	-2,32	0,17	-2.39**	0,19	-1,39	0,57
PP	-2,93	-4,75**	0,008	-1,1	0,70	-2.01**	0,28	-2,84	0,06
KPSS	0,46	0.17*	-	0,58	-	0,74	-	0.51	-

	Critical value 5%	$s_t = \text{Surplus/PIB}$		$se_t = \text{foreign savings}$		$i_t = \text{interest (SELIC)}$		$i_t^* = \text{Interest (Implied)}$	
		Test statistics	p-value	Test statistics	p-value	Test statistics	p-value	Test statistics	p-value
ADF	-2,93	-1,81	0,39	-2,02	0,27	-1,98	0,29	-1,83	0,37
PP	-2,93	-1,61	0,46	-2.99*	0,04	-1,43	0,55	-1,92	0,32
KPSS	0,46	0.7	-	0.6	-	0.17	-	0.17	-

** - significant at level 0.05, * - significant at level 0.1. ADF tests with the best specification found.

Annex II - Maximum Likelihood Estimates of the Hyperparameters - coefficients of transition matrix T_t in equation (4.2) – for the State Space Model

Variable	Estimate
intercept (μ_t)	$\phi_1 = -0,0239$
debt/GDP (b_t)	$\phi_2 = 0.9537$
surplus/GDP at t-1 (s_{t-1})	$\phi_{31} = 0.9956$
foreign savings (se_t)	$\phi_{32} = 0.8347$
output gap (h_t)	$\phi_{33} = -0.9741$
debt risk (r_t)	$\phi_{34} = 0.7782$