Value-at-risk versus expected shortfall: A practical perspective[☆]

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Abstract

Value-at-Risk (VaR) has become a standard risk measure for financial risk management. However, many authors claim that there are several conceptual problems with VaR. Among these problems, an important one is that VaR disregards any loss beyond the VaR level. We call this problem the "tail risk". In this paper, we illustrate how the tail risk of VaR can cause serious problems in certain cases, cases in which expected shortfall can serve more aptly in its place. We discuss two cases: concentrated credit portfolio and foreign exchange rates under market stress. We show that expected shortfall requires a larger sample size than VaR to provide the same level of accuracy.
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1. Introduction

Value-at-Risk (VaR) has become a standard risk measure for financial risk management due to its conceptual simplicity, ease of computation, and ready applicability. Nevertheless, VaR has been charged as having several conceptual problems. Artzner et al. (1997, 1999), among others, have cited the following shortcomings: (i) VaR measures only percentiles of profit–loss distributions and disregards any loss beyond the VaR level (we term this problem the "tail risk" [1]); and (ii) VaR is not coherent, since it is not subadditive. [2]

To remedy the problems inherent in VaR, Artzner et al. (1997) have proposed the use of expected shortfall. Expected shortfall is defined as the conditional expectation of loss for losses beyond the VaR level. By its very definition, expected shortfall takes into account losses beyond the VaR level. Expected shortfall is also demonstrated to be subadditive, which assures its coherence as a risk measure.

In this paper, we compare VaR and expected shortfall by summarizing the authors' four papers (Yamai and Yoshiba, 2002a; Yamai and Yoshiba, 2002b; Yamai and Yoshiba, 2002c; Yamai and Yoshiba, 2002d). In particular, focusing on tail risk, we illustrate how it can result in serious problems in certain real-world cases. [3]

Our main points are summarized below.

 (i) Rational investors who maximize their expected utility may be misled by the use of VaR as a risk measure. They are likely to construct positions with unintended weaknesses that result in greater losses under conditions beyond the VaR level. [4]

 (ii) VaR is unreliable under market stress. Under extreme asset price fluctuations or an extreme dependence structure of assets, VaR may underestimate risk.

(iii) Investors or risk managers can solve such problems by adopting expected shortfall, which by definition takes into account losses beyond the VaR level.

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[1] We have followed the terminology of the BIS (Bank for International Settlements) Committee on the Global Financial System (2000).

[2] A risk measure ρ is subadditive when the risk of the total position is less than or equal to the sum of the risk of individual portfolios. Let X and Y be random variables denoting the losses of two individual positions. A risk measure ρ is subadditive if the following equation is satisfied:

$$\rho(X + Y) \leqslant \rho(X) + \rho(Y).$$

Intuitively, subadditivity requires that "risk measures should take into account risk reduction through portfolio diversification effects".

[3] Recently, various studies on VaR and expected shortfall have been reported. As in our studies, Consigli (2004) evaluates tail risk of VaR and expected shortfall by applying extreme value theory. Other than tail risk, Acerbi (2004) generalizes the concept of expected shortfall to propose "spectral measures of risk". Rau-Bredow (2004) evaluates convexity of VaR and expected shortfall by calculating first and second derivatives of each risk measure.

[4] This result is shown by Basak and Shapiro (2001) in dynamic portfolio optimization framework. Yamai and Yoshiba (2002a) illustrate this problem in simple examples of far-out-of-the-money option and concentrated credit portfolio.

(iv) The effectiveness of expected shortfall, however, depends on the accuracy of estimation.

The rest of the paper is organized as follows. Section 2 defines the concepts of VaR and expected shortfall. Section 3 discusses some examples of the tail risk of VaR. Section 4 illustrates estimation errors for VaR and expected shortfall. Section 5 concludes the paper.

2. Value-at-risk and expected shortfall

In this section, we introduce the concepts of VaR and expected shortfall, pointing out that VaR and expected shortfall give essentially the same information under normal distributions.

2.1. Definition of value-at-risk and expected shortfall

VaR is defined as the "possible maximum loss over a given holding period within a fixed confidence level". That is, mathematically, VaR at the $100(1 - \alpha)\%$ confidence level is defined as the upper 100α percentile of the loss distribution. Suppose X is a random variable denoting the loss of a given portfolio. Following Artzner et al. (1999), we define VaR at the $100(1 - \alpha)\%$ confidence level ($\mathrm{VaR}_\alpha(X)$) as

$$\mathrm{VaR}_\alpha(X) = \sup\{x \mid P[X \geqslant x] > \alpha\}, \tag{1}$$

where $\sup\{x|A\}$ is the upper limit of x given event A, and $\sup\{x|P[X \geqslant x] > \alpha\}$ indicates the upper 100α percentile of loss distribution. This definition can be applied to both discrete and continuous loss distributions.

Artzner et al. (1997) proposed expected shortfall (also called "conditional VaR", "mean excess loss", "beyond VaR", or "tail VaR") to alleviate the problems inherent in VaR. Expected shortfall is the conditional expectation of loss given that the loss is beyond the VaR level; that is, the expected shortfall is defined as follows:

$$\mathrm{ES}_\alpha(X) = E[X \mid X \geqslant \mathrm{VaR}_\alpha(X)]. \tag{2}$$

The expected shortfall indicates the average loss when the loss exceeds the VaR level.

2.2. VaR and expected shortfall under normal distribution

When the profit–loss distribution is normal, VaR and expected shortfall give essentially the same information.[5] Both VaR and expected shortfall are scalar

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[5] More precisely, if the profit–loss distribution belongs to the elliptical distribution family, either VaR or expected shortfall suffice for information about loss distribution, as both would be redundant. Normal distribution belongs to this family. See Embrechts et al. (2002), for example.

multiples of the standard deviation. [6] Therefore, VaR provides the same information on tail loss as does expected shortfall. For example, VaR at the 99% confidence level is 2.33 times the standard deviation, while expected shortfall at the same confidence level is 2.67 times the standard deviation.

In the following, we compare VaR and expected shortfall in cases where the profit–loss distribution is not normal.

3. Tail risk of VaR: A practitioner approach

In this paper, we say that VaR has tail risk when VaR fails to summarize the relative risk of available portfolios due to its underestimation of the risk of portfolios with fat-tailed properties and high potential for large losses. [7] The tail risk of VaR arises because it measures only a single quantile of the profit–loss distributions, disregarding any loss beyond the VaR level. [8] This may lead one to perceive securities with higher potential for large losses as less risky than securities with lower potential for large losses.

Yamai and Yoshiba (2002c) show that VaR and expected shortfall are free from tail risk when the underlying profit–loss distribution is normal. [9] On the other hand, VaR may have tail risk if the profit–loss distribution is not normal. Non-normality of the profit–loss distribution is caused by non-linearity of the portfolio position or non-normality of the underlying asset prices.

We illustrate the problem of tail risk with two examples [10]: concentrated credit portfolio and currency portfolio under market stress. In these examples, asset returns have fat-tailed properties and high potential for large losses. For the first example, we show that utility-maximizing investors with VaR constraints choose to invest in securities with a high potential for large losses beyond the VaR level. In the second example, we show that VaR entails tail risk when asset returns are described by the extreme value distribution.

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[6] When the loss distribution is normal, expected shortfall is calculated as follows:

$$\mathrm{ES}_\alpha(X) = E[X \mid X \geqslant \mathrm{VaR}_\alpha(X)] = \frac{1}{\alpha \sigma_X \sqrt{2\pi}} \int_{\mathrm{VaR}_\alpha(X)}^{\infty} t \cdot \mathrm{e}^{-t^2/2\sigma_X^2} \mathrm{d}t = \frac{\mathrm{e}^{-q_\alpha^2/2}}{\alpha \sqrt{2\pi}} \sigma_X,$$

where q_α is the upper 100α percentile of standard normal distribution. For example, from this equation, expected shortfall at the 99% confidence level is the standard deviation multiplied by 2.67, which is the same level as VaR at the 99.6% confidence level.

[7] See Yamai and Yoshiba (2002c) for the detail of authors' definition of tail risk.

[8] A risk measure free from tail risk is not always subadditive. For example, when the underlying distribution is generalized Pareto with large tail index ($\xi \geqslant 1$), VaR is not subadditive and has no tail risk. See footnote 51 of Yamai and Yoshiba (2002d) for details.

[9] More precisely, if the profit–loss distribution belongs to the elliptical distribution family, VaR and expected shortfall are free from tail risk. See Theorems 14 and 15 of Yamai and Yoshiba (2002c).

[10] See Yamai and Yoshiba (2002a) and Yamai and Yoshiba (2002d) for more examples of tail risk of VaR.

3.1. Risk control for expected utility-maximizing investors: A simple illustration with credit portfolio

Basak and Shapiro (2001) show that utility-maximizing investors with VaR constraint optimally choose to construct vulnerable positions that can result in large losses exceeding the VaR level. They demonstrate this using a dynamic portfolio optimization framework. [11] Yamai and Yoshiba (2002a) illustrate this problem using simple examples of far-out-of-the-money option and a concentrated credit portfolio.

This section provides a simple illustration of how the tail risk of VaR may result in serious practical problems in credit portfolios. [12] The case discussed here was introduced in Yamai and Yoshiba (2002a).

Suppose that an investor invests 100 million yen in the following four mutual funds: (1) concentrated portfolio A, consisting of only one defaultable bond with a 4% default rate; (2) concentrated portfolio B, consisting of only one defaultable bond with a 0.5% default rate; (3) a diversified portfolio that consists of 100 defaultable bonds with a 5% default rate; and (4) a risk-free asset. To simplify, we assume that the profiles of all bonds in these funds are as follows: the maturity is one year, occurrences of default events are mutually independent, the recovery rate is 10%, and yield to maturity is equal to the coupon rate. We further assume that the yield to maturity, default rate, and recovery rate are fixed until maturity. Table 1 gives the specific profiles of bonds included in these mutual funds.

Assuming logarithmic utility, the expected utility of the investor is given below. [13]

$$
\begin{aligned}
E[u(W)] = & \sum_{n=0}^{100} 0.96 \cdot 0.995 \cdot 0.05^n \cdot 0.95^{100-n} \cdot {}_{100}C_n \cdot \ln \tilde{w}(1,1) \\
& + \sum_{n=0}^{100} 0.04 \cdot 0.995 \cdot 0.05^n \cdot 0.95^{100-n} \cdot {}_{100}C_n \cdot \ln \tilde{w}(0.1,1) \\
& + \sum_{n=0}^{100} 0.96 \cdot 0.005 \cdot 0.05^n \cdot 0.95^{100-n} \cdot {}_{100}C_n \cdot \ln \tilde{w}(1,0.1) \\
& + \sum_{n=0}^{100} 0.04 \cdot 0.005 \cdot 0.05^n \cdot 0.95^{100-n} \cdot {}_{100}C_n \cdot \ln \tilde{w}(0.1,0.1),
\end{aligned}
\tag{3}
$$

where

$$
\begin{aligned}
\tilde{w}(a,b) = & \, 1.0475 a X_1 + 1.0075 b X_2 + 1.055 X_3 \frac{100 - 0.9n}{100} \\
& + 1.0025(W_0 - X_1 - X_2 - X_3),
\end{aligned}
$$

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[11] For general issues in optimization of VaR and expected shortfall, see Rockafellar and Uryasev (2002), and Alexander et al. (2004).

[12] For VaR and expected shortfall in credit portfolio, see also Frey and McNeil (2002), for example.

[13] For example, the probability that both of the concentrated portfolios A and B do not default and that n bonds of the diversified portfolio default is $0.96 \cdot 0.995 \cdot 0.05^n \cdot 0.95^{100-n} \cdot {}_{100}C_n$ where ${}_mC_n$ is the number of combinations choosing n out of m.

Table 1
Profiles of bonds included in the mutual funds

	Number of bonds included	Coupon (%)	Default rate (%)	Recovery rate (%)
Concentrated portfolio A	1	4.75	4.00	10
Concentrated portfolio B	1	0.75	0.50	10
Diversified portfolio	100	5.50	5.00	10
Risk-free asset	1	0.25	0.00	–

Note. The occurrences of defaults are mutually independent.

W final wealth,
W_0 initial wealth,
X_1 amount invested in concentrated portfolio A,
X_2 amount invested in concentrated portfolio B,
X_3 amount invested in diversified portfolio.

We analyze the impact of risk management with VaR and expected shortfall on the rational investor's decisions by solving the following five optimization problems, where the holding period is one year. [14]

(1) No constraint

$$\max_{\{X_1, X_2, X_3\}} E[u(W)].$$

(2) Constraint with VaR at the 95% confidence level

$$\max_{\{X_1, X_2, X_3\}} E[u(W)]$$

subject to   VaR(95% confidence level) $\leqslant 3$.

(3) Constraint with expected shortfall at the 95% confidence level

$$\max_{\{X_1, X_2, X_3\}} E[u(W)]$$

subject to   expected shortfall(95% confidence level) $\leqslant 3.5$.

(4) Constraint with VaR at the 99% confidence level

$$\max_{\{X_1, X_2, X_3\}} E[u(W)]$$

subject to   VaR(99% confidence level) $\leqslant 3$.

(5) Constraint with expected shortfall at the 99% confidence level

$$\max_{\{X_1, X_2, X_3\}} E[u(W)]$$

subject to   expected shortfall(99% confidence level) $\leqslant 3.5$.

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[14] We solved the optimization problems using a quasi Newton method in Microsoft Excel solver, and checked the optimality by plotting investor's utility in the space of portfolio weight. Here, we focus on the analyses of solutions rather than the numerical method to get them.

Table 2
Portfolio profiles (95% confidence level)

	No constraint (1)	VaR constraint[a] (2)	Expected shortfall constraint[b] (3)
Portfolio (%)			
Concentrated portfolio A (default rate: 4%)	7.4	20.1	2.9
Concentrated portfolio B (default rate: 0.5%)	0.0	0.0	2.0
Diversified portfolio	92.6	79.9	95.1
Risk-free asset	0.0	0.0	0.0
Risk measure (million yen)			
VaR	3.35	3.00	2.75
Expected shortfall	5.26	14.35	3.50

[a] Optimize with the constraint that VaR at the 95% confidence level is less than or equal to 3.
[b] Optimize with the constraint that expected shortfall at the 95% confidence level is less than or equal to 3.5.

We analyze the effect of risk management with VaR and expected shortfall by comparing solutions (2)–(5) with solution (1).

Table 2 shows the results of the optimization problem with a 95% VaR or expected shortfall constraint. The solution of the optimization problem with a 95% VaR constraint ((2) in Table 2) shows that the amount invested in concentrated portfolio A is greater than that of solution (1); that is, the portfolio concentration is enhanced by risk management with VaR. Fig. 1 depicts the tails of the cumulative probability distributions of the profit–loss of the portfolios. It shows how risk management with VaR brings about this undesirable result.

Fig. 1. Cumulative probability of profit–loss: the left tail (95% confidence level).

When constrained by VaR, the investor must reduce his/her investment in the diversified portfolio to reduce maximum losses with a 95% confidence level. Instead, he/she should increase investments either in concentrated portfolios or in a risk-free asset. Concentrated portfolio A has little effect on VaR, since the probability of default lies beyond the 95% confidence interval. Concentrated portfolio A also yields a higher return than other assets, except diversified portfolio. Thus, the investor chooses to invest in concentrated portfolio A. Although VaR is reduced, the optimal portfolio is vulnerable due to its concentration and larger losses under conditions beyond the VaR level.

On the other hand, when constrained by expected shortfall ((3) in Table 2), the investor optimally reallocates his investment to a risk-free asset, significantly reducing the portfolio risk. The investor cannot increase his investment in the concentrated portfolio without affecting expected shortfall, which takes into account the losses beyond the VaR level. Unlike risk management with VaR, risk management with expected shortfall does not enhance credit concentration.

Next, we examine whether raising the confidence level of VaR solves the problem. Table 3 gives the results of the optimization problem with a 99% VaR or expected shortfall constraint. It shows that when constrained by VaR at the 99% confidence level, the investor optimally chooses to increase his/her investment in concentrated portfolio B because the default rate of concentrated portfolio B is 0.5%, outside the confidence level of VaR. On the other hand, risk management with expected shortfall reduces the potential loss beyond the VaR level by reducing credit concentration.

VaR may enhance credit concentration because it disregards losses beyond the VaR level, even at high confidence levels. On the other hand, expected shortfall reduces credit concentration because it takes into account losses beyond the VaR level as a conditional expectation.

Table 3
Portfolio profiles (99% confidence level)

	No constraint (1)	VaR constraint[a] (4)	Expected shortfall constraint[b] (5)
Portfolio (%)			
Concentrated portfolio A (default rate: 4%)	7.4	0.7	0.7
Concentrated portfolio B (default rate: 0.5%)	0.0	18.8	0.5
Diversified portfolio	92.6	64.9	65.6
Risk-free asset	0.0	15.6	33.2
Risk measure (million yen)			
VaR	6.77	3.00	3.13
Expected shortfall	7.83	7.33	3.50

[a] Optimize with the constraint that VaR at the 99% confidence level is less than or equal to 3.
[b] Optimize with the constraint that expected shortfall at the 99% confidence level is less than or equal to 3.5.

Fig. 2. Cumulative distribution of profit–loss when tail risk of VaR occurs.

This illustration suggests that if investors can invest in assets whose loss is infrequent but large (such as concentrated credit portfolios), the problem of tail risk can be serious. Furthermore, investors can manipulate the profit–loss distribution using those assets, so that VaR becomes small while the tail becomes fat (see Fig. 2).

In general, expected shortfall is more consistent with expected utility maximization under less stringent conditions than VaR. Yamai and Yoshiba (2002c) show that VaR is consistent with expected utility maximization when portfolios are ranked by first-order stochastic dominance, while expected shortfall is consistent with expected utility maximization when portfolios are ranked by second-order stochastic dominance. Thus, VaR is more likely to have unanticipated effect on utility maximization than expected shortfall.

3.2. Risk measurement under market stress

In this section, we show that VaR may entail tail risk if the underlying asset price fluctuations are extreme; in other words, if the market is under stress.

A typical case of market stress can be seen in the financial market crisis of fall 1998. Concerning this crisis, the BIS Committee on the Global Financial System (1999) notes that "a large majority of interviewees admitted that last autumn's events were in the 'tails' of distributions and that VaR models were useless for measuring and monitoring market risk". In this section, we focus on this particular case.

We assume that the multivariate extreme value distributions represent asset returns under market stress. Under this assumption, Yamai and Yoshiba (2002d) investigate the conditions of the tail risk of VaR and expected shortfall employing asset return simulation with those distributions.

We introduce a case study from Yamai and Yoshiba (2002d), with a brief explanation of multivariate extreme value theory.

We analyze the daily logarithmic changes in currency exchange rate of three industrialized countries and 18 emerging economies. The raw historical data are the exchange rates per one US dollar from November 1, 1993 to October 29, 2001. We examine the exchange rates for the 21 currencies and among those the dependence structures of five currencies in Southeast Asian countries.

3.2.1. Extreme value theory and copulas in financial risk estimation

Before introducing our analyses, we briefly describe extreme value theory and copulas. First, we examine one asset Z, in our case, daily logarithmic changes in each of the exchange rates for the 21 currencies. Let F be the distribution function of Z. The distribution function of $(Z - \theta)$ given that Z exceeds θ is

$$F_\theta(x) = \Pr\{Z - \theta \leqslant x \mid Z > \theta\} = \frac{F(x) - F(\theta)}{1 - F(\theta)}, \quad \theta \leqslant x. \tag{4}$$

Univariate extreme value theory says that the distribution function F_θ converges to a generalized Pareto distribution $G_{\xi,\sigma}(x)$ when the value of θ is sufficiently large (see Embrechts et al., 1997 for example).

$$G_{\xi,\sigma}(x) = 1 - \left(1 + \xi \cdot \frac{x}{\sigma}\right)^{-1/\xi}, \quad x \geqslant 0. \tag{5}$$

With Eqs. (4) and (5), when the value of θ is sufficiently large, the distribution function of exceedances $\max(Z, \theta)$, denoted by $F_m(x)$, is approximated as follows:

$$F_m(x) \approx (1 - F(\theta))G_{\xi,\sigma}(x - \theta) + F(\theta) = 1 - p\left(1 + \xi \cdot \frac{x - \theta}{\sigma}\right)^{-1/\xi}, \quad x \geqslant \theta, \tag{6}$$

where $p = 1 - F(\theta)$ is the tail probability. The distribution is described by three parameters: the tail index ξ, the scale parameter σ, and the tail probability p. The tail index ξ represents how fat the tail of the distribution is; when ξ is large, the tail is fat. The scale parameter σ represents how dispersed the distribution is; when σ is large, the distribution is highly dispersed. Assuming the confidence level of VaR and expected shortfall is less than p, we use this distribution of exceedances to calculate VaR and expected shortfall.

Next, we notice a pair of two assets (Z_1, Z_2). To identify joint distribution of the two random variables, we need to specify the dependence structure of the two variables other than marginal distribution functions. A copula is useful for describing the dependence structure. A copula C is a function that satisfies the relationship

$$F(x_1, x_2) = C(F_1(x_1), F_2(x_2)), \tag{7}$$

where $F(x_1, x_2)$ is the joint distribution function $P[Z_1 \leqslant x_1, Z_2 \leqslant x_2]$, and $(F_1(x_1), F_2(x_2))$ the marginal distribution functions $(P[Z_1 \leqslant x_1], P[Z_2 \leqslant x_2])$. The above Eq. (7) shows that the copula represents the dependence structure in the joint

distribution. The copula is the part not described by the marginals and is invariant under the transformation of the marginals. [15]

Here, we note the distribution of the bivariate exceedances $m_{(\theta_1,\theta_2)}(Z_1, Z_2) = (\max(Z_1, \theta_1), (\max(Z_2, \theta_2))$ with some threshold $\theta = (\theta_1, \theta_2)$. Multivariate extreme value theory says that the marginal distributions of $m_{(\theta_1,\theta_2)}(Z_1, Z_2)$ converge to the distribution of Eq. (6) and that the copulas of $m_{(\theta_1,\theta_2)}(Z_1, Z_2)$ converge to a class of copulas, as the threshold $\theta = (\theta_1, \theta_2)$ becomes large. Ledford and Tawn (1996) show that copulas in the class satisfy the following equation:

$$C(u_1, u_2) = \exp\left\{-V\left(-\frac{1}{\ln u_1}, -\frac{1}{\ln u_2}\right)\right\}, \tag{8}$$

where

$$V(z_1, z_2) = \int_0^1 \max\{sz_1^{-1}, (1-s)z_2^{-1}\}\mathrm{d}H(s) \tag{9}$$

and H is a non-negative measure on $[0, 1]$ satisfying

$$\int_0^1 s\mathrm{d}H(s) = \int_0^1 (1-s)\mathrm{d}H(s) = 1. \tag{10}$$

Following Hefferman (2000), we call this type of copula the *bivariate extreme value copula*.

One bivariate extreme value copula is the Gumbel copula. The Gumbel copula is expressed by

$$C(u_1, u_2) = \exp\{-[(-\ln u_1)^\alpha + (-\ln u_2)^\alpha]^{1/\alpha}\}, \tag{11}$$

for a parameter $\alpha \in [1, \infty]$. [16] The dependence parameter α controls the level of dependence between random variables. When $\alpha = 1$, it corresponds to full dependence; while $\alpha = \infty$ corresponds to independence.

3.2.2. Tail risk of VaR for each currency (univariate analyses)

Let us compare two different losses denoted by random variables Z_1 and Z_2. As in Eq. (6), the distribution functions of exceedances $m(Z_1) = \max(Z_1, \theta)$ and $m(Z_2) = \max(Z_2, \theta)$ are approximated by Eqs. (12) and (13) if θ is sufficiently large.

$$F_{m(Z_1)}(x) = 1 - p\left(1 + \xi_1 \cdot \frac{x - \theta}{\sigma_1}\right)^{-1/\xi_1}, \tag{12}$$

$$F_{m(Z_2)}(x) = 1 - p\left(1 + \xi_2 \cdot \frac{x - \theta}{\sigma_2}\right)^{-1/\xi_2}. \tag{13}$$

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[15] See Embrechts et al. (2002), Joe (1997), Nelsen (1999), Frees and Valdez (1998), etc. for details. See Jouanin et al. (2004) and Geman and Kharoubi (2004) for financial application of copula.

[16] Eq. (11) is obtained by defining V of Eq. (9) as $V(z_1, z_2) = (z_1^{-\alpha} + z_2^{-\alpha})^{1/\alpha}$.

Fig. 3. Example of the distribution of exceedances.

Given $\xi_2 > \xi_1$, Z_2 has a fatter tail than Z_1. In this case, if $\rho(\xi_1)$ is larger than $\rho(\xi_2)$ for some risk measure $\rho(\cdot)$, the risk measure $\rho(\cdot)$ has tail risk since Z_2 has a higher potential for larger loss than Z_1.

To illustrate an example of the tail risk of VaR, Fig. 3 plots Eqs. (12) and (13) for the following parameter values: tail probability $p = 0.1$; threshold $\theta = 0.05$; tail indices $\xi_1 = 0.1$ and $\xi_2 = 0.5$; and scale parameters $\sigma_1 = 0.05$ and $\sigma_2 = 0.035$. In this example, $\xi_2 > \xi_1$ and the VaR at the 95% confidence level is higher for Z_1 than for Z_2. VaR at the 95% confidence level has tail risk as the distribution functions intersect beyond the VaR confidence level.

We next estimate the parameters of the distribution shown in Eq. (6) using the daily logarithmic changes in each of the exchange rates for the 21 currencies. We employ the maximum likelihood method described in Embrechts et al. (1997) and McNeil (2000). [17] We vary the tail probability p as 1%, 2%, ..., 10%, [18] and estimate the parameters ξ, σ, and θ for each. We then calculate the VaR and expected shortfall at confidence levels of 95% and 99% using the estimated parameter values (see Yamai and Yoshiba (2002d) for details). Table 4 shows a part of the estimation

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[17] For estimation of parameters by the maximum likelihood method, we use the S-Plus function libraries of McNeil's Evis. With a limited sample of excess data, some authors apply the Bayesian Markov chain Monte Carlo to estimate parameters (see Coles, 2001; Medova and Kyriacou, 2002, for example).

[18] The choice of threshold is crucial in applying extreme value distribution to real data. The issue of the choice implies a balance between bias and variance. If a threshold is too low, it is likely to violate the asymptotic properties, leading to bias; if it is too high, a threshold will generate few excesses, leading to high variance (see Coles, 2001). We followed the approach described in Embrechts et al. (1997). We plot empirical mean excess functions, estimated tail indexes for varied thresholds, and checked whether our choice of threshold is valid. The method to choose thresholds are proposed by various authors. Daníelsson et al. (2001) propose a two step subsample bootstrap method to determine the optimal threshold that minimizes the asymptotic mean squared error.

Table 4
Tail indices and VaR of some currencies

	Japanese Yen	Malaysian Ringgit	South Korean Won	Thai Baht	Chilean Peso	Mexican New Peso	Venezuelan Bolivar
ξ	0.080	0.737	0.685	0.430	0.177	0.646	0.862
VaR(95%) (%)	1.00	0.43	0.73	0.77	0.52	0.84	0.34

results of the tail indices ξ and VaR at a 95% confidence level for the Japanese yen and six emerging currencies, given that p is 10%.

Table 4 shows that VaR at the 95% confidence level has tail risk. First, the tail is fatter for emerging economies than for Japan. The tail indices ξ are substantially larger for emerging economies than for Japan. The currencies of emerging economies pose a higher potential for large losses than the Japanese yen. Second, the VaR at the 95% level for the Japanese yen is larger than that for emerging economies. Thus, VaR at the 95% level has tail risk.

For detailed results, including analyses of expected shortfall, see Yamai and Yoshiba (2002d).

3.2.3. Tail risk of VaR for selected pairs of currencies (bivariate analyses)

Below, we provide an example in which VaR has tail risk in certain pairs of exchange rate data, selecting five currencies in Southeast Asian countries: the Indonesian rupiah, the Malaysian ringgit, the Philippine peso, the Singapore dollar, and the Thai baht.

Following the method of Longin and Solnik (2001), we assume that the marginal distributions of bivariate exceedances are approximated by (6) and that their copula is approximated by the Gumbel copula. Given tail probabilities p_1 and p_2, we estimate the following parameters: the tail indices of the marginals (ξ_1 and ξ_2), the scale parameters of the marginals (σ_1 and σ_2), the thresholds (θ_1 and θ_2), and the dependence parameter of the Gumbel copula (α).

We estimate those parameters on the right tails of the logarithmic changes of each pair of Southeast Asian currencies by the maximum likelihood method for a tail probability of 10% ($p_1 = p_2 = 0.1$). Table 5 shows the results of the estimation.

In the bivariate analyses, we focus on the dependence structure rather than the tail-fatness of the marginals. We assume that the dependence structure is represented by the copula. We adopt the Gumbel, Gaussian, and Frank copulas to represent different tail dependencies. Among the Gumbel, Gaussian and Frank copulas, the Gumbel copula has the strongest tail dependence of the two random variables, and the Frank copula the weakest. Changing from the Gumbel to the Gaussian and Frank copulas weakens tail dependence. [19]

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[19] The Gumbel copula corresponds to asymptotic dependence and the Gaussian and Frank copulas correspond to asymptotic independence (Ledford and Tawn, 1996).

Table 5
Estimates of the bivariate extreme value distribution of daily logarithmic changes in southeast Asian exchange rates (Gumbel copula)

Currencies		α	ξ_1	σ_1	θ_1	ξ_2	σ_2	θ_2
Indonesia (Rupiah)	Malaysia (Ringgit)	1.2658	0.4088	0.0128	0.0084	0.7371	0.0030	0.0016
Indonesia (Rupiah)	Philippines (Peso)	1.3056	0.4088	0.0128	0.0084	0.4156	0.0046	0.0035
Indonesia (Rupiah)	Singapore (Dollar)	1.3316	0.4088	0.0128	0.0084	0.3256	0.0020	0.0028
Indonesia (Rupiah)	Thailand (Baht)	1.3855	0.4088	0.0128	0.0084	0.4298	0.0051	0.0035
Malaysia (Ringgit)	Philippines (Peso)	1.2578	0.7371	0.0030	0.0016	0.4156	0.0046	0.0035
Malaysia (Ringgit)	Singapore (Dollar)	1.5288	0.7371	0.0030	0.0016	0.3256	0.0020	0.0028
Malaysia (Ringgit)	Thailand (Baht)	1.3186	0.7371	0.0030	0.0016	0.4298	0.0051	0.0035
Philippines (Peso)	Singapore (Dollar)	1.3120	0.4156	0.0046	0.0035	0.3256	0.0020	0.0028
Philippines (Peso)	Thailand (Baht)	1.4267	0.4156	0.0046	0.0035	0.4298	0.0051	0.0035
Singapore (Dollar)	Thailand (Baht)	1.4364	0.3256	0.0020	0.0028	0.4298	0.0051	0.0035

Note. The foreign exchange rate data is sourced from Bloomberg. The estimation period is from November 1, 1993 to October 29, 2001.

The Gaussian copula is

$$C(u,v) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v)), \tag{14}$$

where Φ_ρ is the distribution function of a bivariate standard normal distribution with a correlation coefficient ρ, and Φ^{-1} is the inverse function of the distribution function for the univariate standard normal distribution.

The Frank copula [20] is

$$C(u,v) = -\frac{1}{\delta} \ln \left(\frac{1 - e^{-\delta} - (1 - e^{-\delta u})(1 - e^{-\delta v})}{1 - e^{-\delta}} \right). \tag{15}$$

The dependence parameters ρ and δ control the level of dependence between random variables. For the Gaussian copula, $\rho = \pm 1$ corresponds to full dependence while $\rho = 0$ corresponds to independence. For the Frank copula, $\delta = \pm \infty$ corresponds to full dependence while $\delta = 0$ corresponds to independence.

In comparing the results using the three copulas, we set the values of the dependence parameters of those copulas (Gumbel: α, Gaussian: ρ, and Frank: δ) so that Spearman's rho (ρ_S) is equal across those copulas. [21] By setting Spearman's rho as equivalent for all the three copulas, we can eliminate the effect of global dependence and examine the pure effect of tail dependence, since Spearman's rho is a measure of global dependence.

After the estimation of the tail indices (ξ_1 and ξ_2), the scale parameters (σ_1 and σ_2), the thresholds (θ_1 and θ_2), and the dependence parameter of the Gumbel copula

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[20] This definition of the Frank copula follows Joe (1997).

[21] Spearman's rho $\rho_S(Z_1, Z_2)$ is a rank correlation defined by the linear correlation of the marginals, as $\rho_S(Z_1, Z_2) \equiv \frac{\text{Cov}(F_{Z_1}(Z_1), F_{Z_2}(Z_2))}{\sqrt{V[F_{Z_1}(Z_1)]V[F_{Z_2}(Z_2)]}}$. Here, we note $F_1(Z_1)$ and $F_2(Z_2)$ are uniform random variables on [0, 1], and $V[F_1(Z_1)] = V[F_2(Z_2)] = 1/12$. Using copula $C(u,v)$, Spearman's rho is expressed as $\rho_S(Z_1, Z_2) = 12 \iint \{C(u,v) - uv\} \, du \, dv$, where $U \equiv F_1(Z_1)$ and $V \equiv F_2(Z_2)$. See Frees and Valdez (1998).

Table 6
VaR of the simulated sums of the foreign exchange rates

Currencies		VaR (95%)			VaR (99%)		
		Frank (%)	Gaussian (%)	Gumbel (%)	Frank (%)	Gaussian (%)	Gumbel (%)
Indonesia	Malaysia	2.337	2.331	2.257	6.852	6.958	7.041
Indonesia	Philippines	2.464	2.464	2.408	6.573	6.746	7.002
Indonesia	Singapore	2.118	2.133	2.132	5.993	6.094	6.270
Indonesia	Thailand	2.562	2.551	2.482	6.788	7.015	7.298
Malaysia	Philippines	1.161	1.154	1.111	3.427	3.504	3.570
Malaysia	Singapore	0.844	0.834	0.811	2.442	2.558	2.677
Malaysia	Thailand	1.232	1.220	1.166	3.692	3.778	3.850
Philippines	Singapore	1.043	1.047	1.035	2.497	2.588	2.720
Philippines	Thailand	1.455	1.440	1.395	3.650	3.802	3.992
Singapore	Thailand	1.114	1.114	1.102	2.754	2.882	3.037

(α) as in Table 5, we calculate Spearman's rho (ρ_S) using a numerical integration. And we determine the dependence parameter of the Gaussian copula (ρ) and that of the Frank copula (δ) so that Spearman's rho (ρ_S) is equal to the Gumbel copula. [22] We then simulate logarithmic changes in two exchange rates $X_{i,1}$ and $X_{i,2}$ with distributions whose marginals are Eq. (6), and whose copulas are the Gumbel, Gaussian, and Frank copula. Finally, we calculate the VaR and expected shortfall of the sums of the logarithmic changes in two exchange rates, $X_{i,1} + X_{i,2}$, running ten million simulations for each case.

Table 6 shows a part of the results from those simulations. We find that for most pairs the VaR at the 95% confidence level has tail risk, since the VaRs are larger for the Frank copula (the weakest tail dependence) than for the Gumbel copula (the strongest tail dependence). This means that VaR fails to take into account tail dependencies among exchange rates in emerging economies. On the other hand, the VaRs at the 99% confidence level in this example have no tail risk.

For detailed results, see Yamai and Yoshiba (2002d).

4. Estimation error of VaR and expected shortfall

Expected shortfall has better properties than VaR with respect to tail risk. However, expected shortfall does not always yield better results than VaR. In this chapter, we argue that expected shortfall is likely to result in worse estimates than VaR if we adopt simulation methods for estimation.

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[22] To find the parameters with the same Spearman's rho, we use FindRoot function of Mathematica with three defined functions to calculate Spearman's rho for each parameter of the Gaussian, Frank, and Gumbel copulas. Spearman's rho for Gaussian copula is $\rho_S(Z_1, Z_2) = \frac{6}{\pi}\arcsin(\rho)$. Spearman's rho for the Frank copula is represented by "Debye" functions (see Frees and Valdez, 1998). Spearman's rho for the Gumbel copula is not a closed-form function of α, and we calculate it by numerical integration using a general expression $\rho_S(Z_1, Z_2) = 12\iint\{C(u,v) - uv\}\,\mathrm{d}u\,\mathrm{d}v$.

Estimates of VaR and expected shortfall are affected by estimation error, such as limited sample size results in the sampling fluctuation. Suppose that we estimate the VaR of a given portfolio by Monte Carlo simulations. The VaR estimates vary according to the realizations of random numbers. To reduce estimation error, we must increase the sample size of the simulations, which is a highly time-intensive task. This chapter compares the estimation errors of VaR and those of expected shortfall.

We assume that underlying asset prices have generalized Pareto distribution, as introduced in the former chapter,

$$F_m(x) \approx (1 - F(\theta))G_{\xi,\sigma}(x - \theta) + F(\theta) = 1 - p\left(1 + \xi \cdot \frac{x - \theta}{\sigma}\right)^{-1/\xi}, \quad x \geqslant \theta.$$
(16)

The distribution is described by three parameters: the tail index ξ, the scale parameter σ, and the tail probability p. Since tail index ξ represents how fat the tail of the distribution is, the tail is fat when ξ is large.

We evaluate the estimation errors of VaR and expected shortfall by obtaining 10,000 estimates of those risk measures. To obtain each estimate, we run Monte Carlo simulations with a sample size of 10,000, assuming that the underlying loss have generalized Pareto distributions with $\xi = 0.1, 0.3, 0.5, 0.7, 0.9$, [23] and obtain VaR and expected shortfall [24] at the 99% confidence level with each tail index ξ. [25] We iterate this procedure 10,000 times, and obtain 10,000 estimates of VaR and expected shortfall. Then we calculate the average value, the standard deviation, and the 95% confidence level of those estimates. The estimation errors of VaR and expected shortfall are compared by relative standard deviation (the standard deviation divided by the average). Table 7 summarizes the results. Fig. 4 depicts the relative standard deviations.

The estimation error of expected shortfall is larger than for VaR when the underlying loss distribution is fat-tailed. As ξ approaches one (i.e., as the underlying loss distribution becomes fat-tailed), the relative standard deviation of the expected shortfall estimate becomes much larger than that of the VaR estimate. For $\xi = 0.9$, the relative standard deviation of the expected shortfall estimate is more than 60 times that of the VaR estimate. On the other hand, when ξ is close to zero, the relative standard deviation of VaR and expected shortfall estimates are both small and nearly equivalent.

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[23] For simplicity, we set other parameters as $\sigma = 1$ and $p = 1$ ($\theta = 0$).

[24] Under generalized Pareto distribution, VaR and expected shortfall are analytically solved. However, we use a generalized Pareto distribution to obtain the simulation estimates of VaR and expected shortfall in order to illustrate how the tail fatness of an underlying distribution affects estimation errors. Yamai and Yoshiba (2002b) derive the same result under stable distribution, where VaR and expected shortfall are not solved analytically.

[25] The estimate of VaR at the 99% confidence interval is in the upper one percentile of the empirical loss distribution. For the sample size 10,000, we take the VaR estimate as the 100th largest sample of loss; That is, we take $X_{(100)}$ as the VaR estimate where the sequence $X_{(1)}, \ldots, X_{(100)}, \ldots, X_{(10,000)}$ is the loss sample rearranged in decreasing order. We take the first 100 loss averages of the rearranged sample as the expected shortfall estimate.

Table 7
Estimates of VaR and expected shortfall under generalized Pareto distributions

ξ	Risk measures	Average (a)	S.D. (b)	Relative S.D. (c) = (b)/(a)	Confidence interval (95%)
0.1	VaR	5.84	0.16	0.027	[5.59–6.11]
	Expected Shortfall	7.58	0.26	0.034	[7.16–8.03]
0.3	VaR	9.92	0.40	0.040	[9.29–10.60]
	Expected Shortfall	15.54	1.04	0.067	[13.99–17.35]
0.5	VaR	17.97	1.00	0.056	[16.40–19.68]
	Expected Shortfall	37.68	6.78	0.180	[30.81–47.10]
0.7	VaR	34.40	2.51	0.073	[30.51–38.76]
	Expected Shortfall	115.55	111.64	0.966	[76.04–178.41]
0.9	VaR	68.92	6.32	0.092	[59.25–79.97]
	Expected Shortfall	535.15	3027.57	5.657	[208.17–998.51]

Fig. 4. Relative standard deviation of estimates (solid line: expected shortfall, dotted line: VaR).

This result can be explained as follows: when the underlying distribution is fat-tailed, the probability of infrequent and large loss is high. The expected shortfall estimates are affected by whether large and infrequent loss is realized in the obtained sample, since expected shortfall considers the right tail of the loss distribution. On the other hand, the VaR estimates are less affected by large and infrequent loss than the expected shortfall estimates, since the VaR method does not take into regard loss beyond the VaR level. Therefore, when the underlying loss distribution becomes more fat-tailed, the expected shortfall estimates become more varied due to infrequent and large losses, and their estimation error grows larger than the estimation error of VaR.

We also investigate whether the increase in sample size reduces the estimation error of expected shortfall. We run 10,000 sets of Monte Carlo simulations with sample sizes of 10,000, 100,000, and 1,000,000 for $\xi = 0.5, 0.7, 0.9$. Table 8 shows the

Table 8
Convergence of expected shortfall estimates

Sample size	$\xi = 0.5$		$\xi = 0.7$		$\xi = 0.9$	
	Relative S.D.	Confidence interval (95%)	Relative S.D.	Confidence interval (95%)	Relative S.D.	Confidence interval (95%)
10,000	0.180	[30.81–47.10]	0.966	[76.04–178.41]	5.657	[208.17–998.51]
100,000	0.058	[35.24–41.42]	0.328	[96.07–150.29]	2.168	[307.63–939.23]
1,000,000	0.020	[36.97–39.19]	0.159	[106.73–134.55]	1.598	[387.66–905.40]

average, the standard deviation, and the 95% confidence interval of those 10,000 estimates.

The increase in sample size from 10,000 to 1,000,000 reduces the relative standard deviations (the standard deviation divided by the average) of the expected shortfall estimates. [26]

5. Concluding remarks

We have compared VaR with expected shortfall, emphasizing the problem of tail risk, or the problem whereby VaR disregards losses beyond the VaR level. This problem can cause serious real-world problems, since information provided by VaR may mislead investors. Investors can safeguard against this problem by adopting expected shortfall, since this method also considers losses beyond the VaR level. Expected shortfall is a better risk measure than VaR in terms of tail risk.

The advantages of expected shortfall do not come without certain disadvantages. When the underlying distribution is fat-tailed, the estimation errors of expected shortfall are much greater than those of VaR. To reduce estimation error, we need to increase the sample size of the simulation. Thus, expected shortfall is most costly when it most needs to be free from tail risk under the fat-tailed distribution.

These findings imply that the use of a single risk measure should not dominate financial risk management. Each risk measure offers its own advantages and disadvantages. Complementing VaR with expected shortfall represents an effective way to provide more comprehensive risk monitoring.

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[26] Table 8 shows that, when the underlying loss is a generalized Pareto distribution with $\xi = 0.7$, we must have a sample size of several million in order to ensure the same level of relative standard deviation as when we estimate VaR with a sample size of 10,000 (0.073). Even with a sample size of one million, the relative standard deviation of expected shortfall estimate is given as 0.159.

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