A credit risk model for large dimensional portfolios with application to economic capital

Kaj Nyström a,*,1, Jimmy Skoglund b,2

a Department of Mathematics, Umeå University, S-90187 Umeå, Sweden
b SAS Institute AB, S-169 70, Sweden

Received 4 May 2004; accepted 19 May 2005
Available online 28 November 2005

Abstract

In this paper we develop a multi-period and multi-state portfolio credit risk model which is applicable to large dimensional portfolios like for example retail and mortgage portfolios. The model includes a methodology for estimation and simulation of systematic transition risk through a model for stochastic migration, a methodology for the modelling of recoveries in the case of stochastic collaterals as well as an approach to dimension reduction of the portfolio. One important application of our model is economic capital (EC) and a concept of EC based on the analogy with classical risk theory is introduced and the questions of allocation as well as risk-adjusted pricing based on the allocation of EC are structured and described. The model is illustrated by an extensive numerical example giving a concretization of the model as well as of several of the concepts introduced.

© 2005 Elsevier B.V. All rights reserved.

JEL classification: G18; G21; G31

Keywords: Economic capital; Value at risk; Conditional value at risk; Credit risk; Credit portfolio model

* Corresponding author.
E-mail addresses: kaj.nystrom@math.umu.se (K. Nyström), jimmy.skoglund@swe.sas.com (J. Skoglund).
1 A longer version of this paper can be requested from the first author.
2 Part of this work was carried out when the second author still was employed at Swedbank.
1. Introduction

In classical risk theory an insurance company is supposed to collect premiums $I$ and suffer losses $X$. Considering a discrete setting and events on a time-horizon of $n\Delta t$, where $n$ is a positive integer and $\Delta t$ is a time-scale of say 0.25 or 0.5 years, we let $I(n)$ be the total of aggregated premiums collected during the interval $[0, n\Delta t]$. Similarly $X(n)$ is the total of losses occurring during $[0, n\Delta t]$. Assuming that the company initially, i.e., at $t = 0$, has an amount of capital equal to $u \in \mathbb{R}_+$ we define $R(u, n\Delta t, \omega) = u + I(n) - X(n)$. $R(u, n\Delta t, \omega)$ is referred to as the risk reserve at $t = n\Delta t$ given an initial amount of $u$ in the reserve. Here we have added $\omega$ in the notation in order to emphasize the random nature of the quantity. For an initial amount of capital $u$ we define $\tau$ to be the first (stochastic) time the risk reserve becomes negative. If $R(u, n\Delta t, \omega) > 0$ for all $n$, then $\tau = \infty$. In risk theory, $P[\tau \leq n\Delta t]$ is usually referred to as the ruin probability. Fixing $\alpha$ to be a very small probability the insurance company can, in theory, solve for the amount of capital needed in order to make sure that the ruin probability is less or equal to $\alpha$. From this argument we realize that the amount of capital, $u$, needed depends on the following. Firstly, $u$ depends on the time horizon $n\Delta t$ and the timing of losses. In particular the amount of capital needed to be protected from ruin during $[0, n\Delta t]$ depends on the probability distribution of the event times of losses. Secondly, $u$ depends on the size of the premiums charged and aggregated into $I(\cdot)$. In particular the amount of capital needed to be protected from ruin is decreasing as the premiums are increasing and vice versa. Thirdly, $u$ depends on the size of the losses suffered and aggregated into $X(\cdot)$. In particular the amount of capital needed to be protected from ruin is increasing as losses become more severe and vice versa.

One of the main ideas explored in this paper is that in order to calculate the amount of capital needed to support a credit, and more generally a portfolio of credits, the analogy with the risk reserve is appropriate and in particular we present a model for credit risk and economic capital (EC) which has the following features. The model is dynamic in the sense that defaults are not considered on one static time horizon. Instead the default time $\tau$ of the counterparty is the quantity we model. The cash flow structure of the credit, i.e., exposures, amortisations and marginals are part of the calculation of capital. In particular excluding revenues would severely overestimate the capital needed. As defaults occur and losses are suffered a dynamic model for the collateral is needed in order to assess the size of loss as the timing of defaults is uncertain and our model contains such a component. Finally, our model is feasible from the point of view of computing. In particular the model contains a scheme to reduce dimensions if the number of credits and counterparties is very large.

In Sections 2 and 3 of this paper we have detailed our concept of EC and our credit portfolio model which has all of the features just described and which is particularly applicable to large dimensional portfolios.

The rest of the paper is organized as follows. In Section 2 we structure and describe our concept of economic capital (EC) based on the language of risk theory. Section 2.1 is devoted to a short recap of a few of the basic ideas of risk theory focusing on the idea of the risk reserve process. In Section 2.2 we define EC and address the questions of allocation of EC and the pricing of credit products based on EC. Our concept of EC, see Definition 1, accounts both for revenues and losses and can be applied to any time horizon. In Section 3 we describe our model for credit risk, which is in particular applicable to large scale portfolios like for example retail and mortgage portfolios but by no means restricted to these areas of application. In Section 3.1 the description is made on an overview level.
and the model put into the context of the existing literature and models. Sections 3.2.1, 3.2.2, 3.2.3 detail the main model pieces on which we base our portfolio model. Section 3.2.1 details a stochastic migration model and as a consequence our model for the probability of default. Section 3.2.2 focuses on our model for recovery processes and Section 3.2.3 is devoted to the analysis of the cash flow structure of the credits. In each section we define concepts of what we call perfect homogeneity. These idealized situations are the basis of the dimension reduction step carried out as the first step in the portfolio modelling. How all of the pieces fit together in a portfolio model and a technical description of the reduction scheme is detailed in Section 3.2.4. Finally in Section 4 we illustrate the model proposed by way of an example.

2. Risk theory and economic capital

In this section we structure and describe our concept of economic capital (EC) which is based on the language of risk theory and the idea of a risk reserve process.

2.1. Risk theory

Classical risk theory is an essential part of the mathematics of insurance and we will in this section give a short recap of a few of the basic ideas of this theory. For reference we refer to Rolski et al. (1999) and the extensive list of references to be found in that book.

In the following presentation we will work in a discrete setting and we will consider events on a time-horizon of \( n \Delta t \). Here \( n \) is a positive integer and \( \Delta t \) is a time-scale of say 0.25 or 0.5 years. Recall that in classical risk theory an insurance company is supposed to collect premiums \( I \) and suffer losses \( X \). Let \( I(n) \) be the total aggregated premiums collected during the interval \([0, n\Delta t]\) and let similarly \( X(n) \) be the total of losses occurring during \([0, n\Delta t]\). Assuming that the company initially, i.e., at \( t = 0 \), has an amount of capital equal to \( u \in \mathbb{R}_+ \) we define

\[
R(u, n\Delta t, \omega) := u + \sum_{k=1}^n C(k).
\]

Here \( C(k) := (I(k) - I(k - 1)) - (X(k) - X(k - 1)) \) is the difference between the premiums collected during \((k\Delta t, (k+1)\Delta t]\) and the losses occurring during the same interval. \( R(u, n\Delta t, \omega) \) is referred to as the risk reserve at \( t = n\Delta t \) given an initial amount of \( u \) in the reserve and \( R(u, \cdot, \omega) \) is a stochastic process defined on an appropriate probability space \((\Omega, \mathbb{F}, P)\). For an initial amount of capital \( u \) we define

\[
\tau = \tau(u, \omega) = \inf\{n\Delta t, R(u, n\Delta t, \omega) \leq 0\}.
\]

Hence \( \tau \) is the first time the risk reserve becomes non-positive. In particular if \( R(u, n\Delta t, \omega) > 0 \) for all \( n \), then \( \tau = \infty \). We note that

\[
S(n\Delta t) = P[\tau > n\Delta t] = P[\inf_{k\leq n} R(u, k\Delta t, \omega) > 0].
\]

In risk theory, \( S(n\Delta t) \) is interpreted as the probability of survival up to time \( n\Delta t \). \( 1 - S(n\Delta t) \) is usually referred to as the ruin probability and it is important to point out that \( \inf_{k\leq n} R(u, k\Delta t, \omega) \) depends on the whole path of the risk reserve process \( R(u, \cdot, \omega) \) up to time \( n\Delta t \). Note that

\[
R_1(u, n\Delta t, \omega) := \inf_{k\leq n} R(u, k\Delta t, \omega)
\]

is the lowest amount in the risk reserve as recorded at any of the points \( k\Delta t \) for \( k \leq n \).
Fixing \( \alpha \) to be a very small probability the insurance company can, in theory, solve for the amount of capital needed in order to make sure that the ruin probability is less or equal to \( \alpha \). The very notion of ruin probability indicates that the insurance company goes bankrupt at \( \tau \) and in order to limit the probability of this event you need a sufficient initial capital reserve. Still in principle, you could base your amount of initial capital on other functionals along the paths \( R(u, \cdot, \omega) \) than the one described by \( R_1(u, n\Delta t, \omega) \). For instance you could consider

\[
R_2(u, n\Delta t, \omega) := u + \sum_{k=1}^{n} \min\{C(k), 0\}.
\]

In this case \( R_2(u, n\Delta t, \omega) \) represents another way of extracting information from the path of the risk reserve process and in this case we add to the initial capital, \( u \), the sum of all the negative cash flows \( C(k) \).

To properly define the amount of initial capital needed in order to limit risk we first have to define the concept of measure of risk. By definition a risk measure is a mapping \( \rho : V \rightarrow \mathbb{R} \cup \{\infty\} \) where \( V \) is a non-empty set of \( \mathbb{F} \)-measurable real-valued random variables defined on \((\Omega, \mathbb{F}, P)\). In this broad definition, the operation of taking expectations also qualifies as a measure of risk. Apart from the expectation operator we will in this paper make use of the following measures of risk. Let \( \alpha \in (0, 1) \) be a small number and let \( Z \) be a random variable defined on \((\Omega, \mathbb{F}, P)\). Then \( \rho_{\text{VaR}}(Z) := \sup\{\lambda \in \mathbb{R} : P[Z \leq \lambda] \geq \alpha\} \) is called the value at risk (VaR) at the confidence level \( \alpha \) of \( Z \). Another and still intuitively clear risk measure is that of conditional value at risk, \( \rho_{\text{CVaR}} \). Using the notation above, \( \rho_{\text{CVaR}}(Z) = E[Z | Z \leq \rho_{\text{VaR}}(Z)] \). That is, \( \rho_{\text{CVaR}}(Z) \) is the expectation of the random variable \( Z \) conditional on the event that \( Z \leq \rho_{\text{VaR}}(Z) \). We refer to Artzner et al. (1999) for more on measures of risk.

If we let \( \rho \) be an arbitrary measure of risks we have, based on our discussion above, two definitions of the amount of initial capital needed in order to limit the risk. In particular, the capital needed under a time frame of \([0, n\Delta t]\) can be defined either as \( u_1(0, n\Delta t) := \min\{u, \rho(R_1(u, n\Delta t, \cdot)) \geq 0\} \) or \( u_2(0, n\Delta t) := \min\{u, \rho(R_2(u, n\Delta t, \cdot)) \geq 0\} \). If we let \( \rho = \rho_{\text{VaR}} \) for \( \alpha \) small, \( u_1(0, n\Delta t) \) can be interpreted as the smallest amount of capital needed in order to make sure that the ruin probability is smaller than \( \alpha \) as asked for in classical ruin theory.

### 2.2. Economic capital

In this section we define economic capital (EC) and address the questions of allocation of EC and the pricing of credit products based on EC. We will, in analogy with the previous section, consider the time interval \([0, n\Delta t]\) where \( n \) is a positive integer and where \( \Delta t \) is a time-scale of say 0.25 or 0.5 years. To outline some of the assumptions we base our analysis and presentation on we start by considering the situation of a customer \( C \) borrowing money from the bank \( B \). We assume without loss of generality that the loan is contracted at time \( t = 0 \) to an amount equal to \( E \) units of currency and that the amount is contracted for the time interval \([0, n\Delta t]\). We assume that the customer agrees to the following terms. At \( t = k\Delta t \), for \( k \in \{0, \ldots, n\} \), \( C \) delivers to \( B \) an amortisation equal to the amount \( A(k\Delta t) \). The remaining exposure of the bank, on the interval \((k\Delta t, (k + 1)\Delta t)\) where \( k \leq n - 1 \), is denoted by \( E(k\Delta t) \). At \( t = k\Delta t \), for \( k \in \{0, \ldots, n - 1\} \), \( C \) delivers to
an interest rate payment equal to \( r_c(k\Delta t)E(k\Delta t)\Delta t \). Here \( r_c(k\Delta t) \) is the interest rate offered to \( C \), at \( t = k\Delta t \), and contracted for the interval \((k\Delta t,(k + 1)\Delta t]\). At \( t = n\Delta t \), \( C \) delivers the remaining exposure \( E(n\Delta t) \). No prepayments are allowed.

To finance this loan we assume that the bank, \( B \), acts as follows. At \( t = 0 \), \( B \) enters the interbank market in the currency in which the loan is denominated and borrows and amount equal to the exposure \( E \) from \( B \). We assume that this amount is due at \( t = n\Delta t \), i.e., at the same time as the loan offered to the customer is supposed to terminate. We also assume that this contract is constructed as a mirror of the contract offered to the customer. That is, at \( t = k\Delta t \), for \( k \in \{0, \ldots, n\} \), \( B \) pays an amortisation equal to the amount \( A(k\Delta t) \) and at \( t = n\Delta t \), \( B \) pays the amount \( E(n\Delta t) \). We also assume that at \( t = k\Delta t \), for \( k \in \{0, \ldots, n - 1\} \), \( B \) delivers an interest rate payment equal to \( r_B(k\Delta t)E(k\Delta t)\Delta t \). Here \( r_B(k\Delta t) \) is the interest rate offered to \( B \), at \( t = k\Delta t \) and contracted for the interval \((k\Delta t,(k + 1)\Delta t]\). That is, \( r_B(k\Delta t) \) is the forward rate at \( t = k\Delta t \) for money due after a period of \( \Delta t \) has elapsed. We denote this forward rate by \( f(k\Delta t, \Delta t) \).

In the following we will consider a time frame of \((0, n\Delta t]\) and we will neglect the specification of the maturity parameter \( n \) as all the expressions involved will have zero contribution from the interval \((n\Delta t, n\Delta t]\) in case \( n > n \).

To continue we consider a portfolio \( \pi \) consisting of the credit just structured and issued to \( C \). Assuming that the customer does not default on the interval \((k\Delta t,(k + 1)\Delta t]\) the bank earns, as the amortisation of the customer is passed through \( B \) to \( B \), an amount equal to \([r_c(k\Delta t) - r_B(k\Delta t)]E(k\Delta t)\Delta t = r_M(k\Delta t)E(k\Delta t)\Delta t \) during this interval. Here we have defined \( r_M(k\Delta t) = r_c(k\Delta t) - r_B(k\Delta t) \) where \( M \) stands for marginal. We let \( \tau \) be the default time of customer \( C \) and we assume that default can only occur, at \( t = k\Delta t \) for \( k \in \mathbb{Z}_+ \). If we assume a time frame of \((0, n\Delta t]\) we get that the PV (from the perspective of the bank \( B \)) of this sequence of gains exposed to default risk equals

\[
PV_M(0, n\Delta t, \pi) := \sum_{k=0}^{\min\{\tau, n-1\}} r_M(k\Delta t)E(k\Delta t)\Delta tD_B(k\Delta t).
\]

Note that the letters PV stand for present value. By construction \( PV_M(0, n\Delta t, \pi) \) is defined by summing the present values of all of the marginals earned by the bank on the outstanding exposure up to the time of default, \( \tau \). \( D_B(k\Delta t) \) denotes the discounting factor, relevant for the bank, of money received at \( k\Delta t \) and we assume, which is reasonable, that the discounting factor is extracted from the interbank market in which the bank interacts when financing the loan.

To account for losses due to default we introduce

\[
PV_L(0, n\Delta t, \pi) := \chi_{[0, n\Delta t]}(\tau)E(\tau)(1 - R(\tau))D_B(\tau),
\]

where \( L \) stands for loss and where \( \chi_{[0, n\Delta t]}(\cdot) \) is the indicator function for the interval \((0, n\Delta t]\). \( E(\tau) \) is the remaining exposure at default and \( R(\tau) \) denotes a recovery factor. That is, \( R(\tau) \in [0, 1] \) and \( E(\tau)(1 - R(\tau)) \) is the loss suffered in the case of default. Note that if \( \tau > n\Delta t \), i.e., the customer does not default during \((0, n\Delta t]\), then \( \chi_{[0, n\Delta t]}(\tau) = 0 \) and \( PV_L(0, n\Delta t, \pi) = 0 \). If instead \( \tau \leq n\Delta t \) then \( \chi_{[0, n\Delta t]}(\tau) = 1 \) and the present value of the loss suffered equals \( PV_L(0, n\Delta t, \pi) = E(\tau)(1 - R(\tau))D_B(\tau) \). The size of this loss depends in particular on the exposure at default, \( E(\tau) \), and the recovery factor at default, \( R(\tau) \).

The (net) present value, to the bank, of having the portfolio \( \pi \) is therefore quantified as

\[
PV(0, n\Delta t, \pi) = PV_M(0, n\Delta t, \pi) - PV_L(0, n\Delta t, \pi).
\]
We write, for $k \in \{0, \ldots, n-1\}$,
\[
\begin{align*}
PV_M(k\Delta t, (k+1)\Delta t, \pi) &= PV_M(0, (k+1)\Delta t, \pi) - PV_M(0, k\Delta t, \pi), \\
PV_L(k\Delta t, (k+1)\Delta t, \pi) &= PV_L(0, (k+1)\Delta t, \pi) - PV_L(0, k\Delta t, \pi).
\end{align*}
\]

Hence
\[
PV(k\Delta t, (k+1)\Delta t, \pi) = PV_M(k\Delta t, (k+1)\Delta t, \pi) - PV_L(k\Delta t, (k+1)\Delta t, \pi)
\]
and
\[
PV(0, n\Delta t, \pi) = \sum_{k=0}^{n-1} PV(k\Delta t, (k+1)\Delta t, \pi).
\]

Most of the pieces $PV(k\Delta t, (k+1)\Delta t, \pi)$, $k \in \{0, \ldots, n-1\}$ will be positive and in particular in case of no defaults all of the pieces are positive representing a solid sequence of marginals for the bank. We emphasize that $PV_M(k\Delta t, (k+1)\Delta t, \pi)$ and $PV_L(k\Delta t, (k+1)\Delta t, \pi)$ are random variables and the randomness enters through interest rates and in particular through the possibility of default.

We now generalize the notation introduced so far to the situation of a portfolio, $\pi$, consisting of credits to $N$ counterparties. We assume that to each counterparty there is exactly one credit and we denote the subportfolio consisting of that credit by $\pi_j$, i.e., $\pi = \pi_1 + \cdots + \pi_N$. Using linearity we define
\[
PV(0, n\Delta t, \pi) := \sum_{k=0}^{n-1} \sum_{j=1}^N PV(k\Delta t, (k+1)\Delta t, \pi_j),
\]
where $PV(k\Delta t, (k+1)\Delta t, \pi_j)$ is defined as above by introducing an index $j$ on all credit specific parameters. That is, $(r_{M,j}, A_j, E_j, R_j, \tau_j, n_j)$ specifies the structure of credit $j$. $PV(k\Delta t, (k+1)\Delta t, \pi)$ is defined by analogy.

In analogy with classical risk theory shortly described in the previous section we associate, to the cash flows on the portfolio $\pi$, the risk reserve process
\[
R(u, n\Delta t, \pi, \omega) = u + \sum_{k=0}^{n-1} PV(k\Delta t, (k+1)\Delta t, \pi) = u + \sum_{k=0}^{n-1} \sum_{j=1}^N PV(k\Delta t, (k+1)\Delta t, \pi_j).
\]

Here we have added $\omega$ in the notation in order to emphasize the random nature of the quantity. We want to quantify, using a risk measure $\rho$, the amount of capital needed to support the portfolio $\pi$. In analogy with the previous section we define
\[
R_1(u, n\Delta t, \pi, \omega) = \min \left\{ \inf_{k \leq n} R(u, k\Delta t, \pi, \omega), 0 \right\},
\]
\[
R_2(u, n\Delta t, \pi, \omega) = u + \sum_{k=0}^{n-1} \min\{PV(k\Delta t, (k+1)\Delta t, \pi), 0\}.
\]

$R_1(u, n\Delta t, \pi, \omega)$ and $R_2(u, n\Delta t, \pi, \omega)$ represent two ways of extracting information from the path of the risk reserve process. In the case of $R_1$ we extract the lowest balance during the period $[0, n\Delta t]$ and in the case of $R_2$ we essentially add to the initial capital, $u$, the aggregated negative contributions to the total PV. Our concept of EC is based on the following definition.
Definition 1 (Economic capital). Let $\pi$ be a portfolio of credits and let $\rho$ be a measure of risk. The economic capital, defined w.r.t $\rho$, needed in order to support portfolio $\pi$ during $[0,n\Delta t]$ is defined as

$$EC(0,n\Delta t,\pi,\rho) := \min\{u, \rho(R_2(u,n\Delta t,\pi,\cdot)) \geq 0\}$$

$$= -\rho\left(\sum_{k=0}^{n-1} \min\{PV(k\Delta t,(k+1)\Delta t,\pi),0\}\right).$$

In general we will write $EC(0,n\Delta t,\pi) = EC(0,n\Delta t,\pi,\rho)$ if the risk measure in use is clear from the context. We furthermore note that $\{EC(0,j\Delta t,\pi)\}_{j \geq 0}$ gives the economic capital needed in order to support the portfolio on arbitrary time-horizons.

By construction the bank will experience periods of positive contribution to the total PV as well as periods of negative contribution to the total PV. The reason we choose the definition of EC given in Definition 1 is based on the following argument. Due to obligations for instance to shareholders it is unreasonable to believe that the bank will be able to save positive contributions from one period in order to use that amount of cash to neutralize negative contributions at subsequent periods. Hence the bank should make sure to be guarded against periods which give a negative contribution to the total PV. In particular we emphasize that our approach to economic capital is based on the simple fact that financial institutions do not raise capital e.g., once a year. In fact, what is sought for is, ceteris paribus, a stable capital base for a pre-defined planning horizon like e.g., 3 years, 5 years or 10 years. The planning horizon determines the choice of $n\Delta t$. Moreover, by construction our concept of economic capital is consistent with classical risk theory.

We will now address the questions of allocation of EC and pricing based on EC but before doing so we define what we mean by a scaling of a portfolio $\pi$ with a factor $\delta \in \mathbb{R}_+$. Let the credit be specified as $(r_M,A,E,R,\tau,\hat{n})$. Then $\delta \pi$ is the credit specified as $(r_M,\delta A,\delta E,R,\tau,\hat{n})$. If $\pi = \pi_1 + \cdots + \pi_N$, then $\delta \pi = \delta \pi_1 + \cdots + \delta \pi_N$.

To start our analysis we again consider the situation of a portfolio $\pi$ consisting of one credit demanding $EC(0,n\Delta t,\pi)$ units of capital. No capital is needed if $EC(0,n\Delta t,\pi) = 0$. We note a number of elementary but important facts from the construction. Firstly, the amount of capital needed is decreasing as the marginal $r_M$ is increasing and vice versa. Secondly, the amount of capital needed is increasing as the severity of loss is increasing and vice versa. In general the amount of capital needed is a non-linear function of $r_M$, i.e., we could write $EC(0,n\Delta t,\pi) = EC(0,n\Delta t,r_M,\pi)$ in order to emphasize this dependence.

Let us now assume that the bank wants a return $\epsilon = \epsilon(\Delta t)$ (where $\epsilon$ could be for instance 0.10) for each unit capital which is held on an interval of length $\Delta t$. That is, the demanded return on the capital used by the portfolio $\pi$ equals $\epsilon EC(0,n\Delta t,\pi)$. This implies that in a calculation of the expected return on the portfolio $\pi$ the demanded return on capital should be posted as a cost. In total

$$P(0,n\Delta t,\pi) = \frac{1}{n\Delta t}[E[PV_M(0,n\Delta t,\pi)] - E[PV_L(0,n\Delta t,\pi)]] - \epsilon EC(0,n\Delta t,\pi)$$

is the average expected return, on an interval of length $\Delta t$, over the horizon $(0,n\Delta t]$ ($E[\cdot]$ is the operation of taking the expectation) on the investment when the cost of capital is accounted for. If $E$ is the original notional issued at $t = 0$ we could, using homogeneity, write the last relation as
\[
\frac{1}{E} P(0, n\Delta t, \pi) = \frac{1}{n\Delta t} \left[ E[\text{PV}_M(0, n\Delta t, \pi/E)] - E[\text{PV}_L(0, n\Delta t, \pi/E)] \right] - \epsilon EC(0, n\Delta t, \pi/E)
\]

and hence the terms in this equation for profitability can be expressed as percentage of the original exposure. As discussed above \( P(0, n\Delta t, \pi) = P(0, n\Delta t, r_M, \pi) \) is a non-linear function of \( r_M \) and

\[
\lim_{r_M \to -\infty} P(0, n\Delta t, r_M, \pi) = \infty, \quad P(0, n\Delta t, 0, \pi) < 0.
\]

Note that \( E[\text{PV}_M(0, n\Delta t, \pi)] \) is a monotonously increasing function of the marginal \( r_M \) and that \( EC(0, n\Delta t, \pi) \) is a monotonously decreasing function of the marginal \( r_M \). Hence \( P(0, n\Delta t, r_M, \pi) \) is monotonously increasing as a function of the marginal \( r_M \). This implies in particular there exists a threshold, \( r_M^* \), such that \( P(0, n\Delta t, r_M^*, \pi) = 0 \) and such that \( P(0, n\Delta t, r_M, \pi) > 0 \) for all \( r_M > r_M^* \). The choice of marginal charged, \( r_M \), is therefore determined by the local competitive pressures where the credit is issued making the silent assumption that the bank always try to get a marginal as high as possible. \( r_M^* \) could therefore be referred to as the point of break-even for the credit from the perspective of the bank.

To continue we now consider a portfolio \( \pi \) consisting of a set of credits to different counterparties, each credit is denoted by \( \pi_j \), i.e., \( \pi = \pi_1 + \cdots + \pi_N \). Again we assume that the bank wants a return \( \epsilon = \epsilon(\Delta t) \) for each unit capital which is held for an interval of length \( \Delta t \). That is, the demanded return on the capital used by the portfolio \( \pi \) equals \( \epsilon EC(0, n\Delta t, \pi) \). In this case \( P(0, n\Delta t, \pi) \) is defined as above using linearity and again this quantity is the average expected return, on an interval of length \( \Delta t \), over the horizon \( (0, n\Delta t] \) on the investment when the cost of capital is accounted for. The question of how to allocate the total amount of capital to each credit is in this case not as trivial as in the case of one credit. In particular we have to split \( EC(0, n\Delta t, \pi) \) into pieces reflecting the contribution of each credit to the overall risk. To do this let \( \delta \in \mathbb{R}^+ \) and define \( h(\delta) = EC(0, n\Delta t, \delta\pi) = EC(0, n\Delta t, \delta\pi_1, \ldots, \delta\pi_N) \). Note that a basic property of any useful measure of risk is that it is homogeneous of degree one (see Artzner et al. (1999)). The measures of risk used in this paper, i.e., value at risk and conditional value at risk, are both homogeneous of degree one. This property implies that if we scale a portfolio \( \pi \) by a factor \( \delta \in \mathbb{R}^+ \), as defined above, then the amount of capital needed is also scaled by \( \delta \). To be precise \( EC(0, n\Delta t, \delta\pi) = \delta EC(0, n\Delta t, \pi) \). Therefore,

\[
h'(1) = EC(0, n\Delta t, \pi) = \sum_{j=1}^N \frac{\delta}{\delta\pi_j} EC(0, n\Delta t, \pi_1, \ldots, \pi_N) := \sum_{j=1}^N \frac{\delta}{\delta\pi_j} EC(0, n\Delta t, \pi).
\]

This decomposition reflects the contribution of each credit \( \pi_j \) to the total risk (or capital) \( EC(0, n\Delta t, \pi) \). Summarizing we have derived the following formula for \( P(0, n\Delta t, \pi) \):

\[
\sum_{j=1}^N \frac{1}{n\Delta t} \left[ E[\text{PV}_M(0, n\Delta t, \pi_j)] - \sum_{j=1}^N E[\text{PV}_L(0, n\Delta t, \pi_j)] \right] - \sum_{j=1}^N \epsilon \frac{\delta}{\delta\pi_j} EC(0, n\Delta t, \pi).
\]

Note that in this case it is not possible to give an as clear-cut discussion, as in the case of one credit, of what marginals imply that the bank break-even on the portfolio. Demanding that \( P(0, n\Delta t, \pi) \geq 0 \) gives an implicit restriction on the marginals \( (r_M^1, \ldots, r_M^N) \). Instead
\(P(0,n\Delta t, \pi)\) can be interpreted as the average expected risk adjusted return on the portfolio, on an interval of length \(\Delta t\) and over the horizon \((0,n\Delta t)\).

In the rest of this section we will proceed as follows. Firstly we will consider an idealized portfolio consisting of a set of credits issued to different counterparties under the assumptions that the credits are what we call perfectly homogeneous. By this we mean that \((r_M, A_j, E_j, R_j, \tau_j, \hat{n}_j) = (r_M, A, E, R, \tau, \hat{n})\) for all credits in the portfolio and for an appropriate set \((r_M, A, E, R, \tau, \hat{n})\). We also assume that the default times \(\{\tau_j\}\) are independent but equally distributed. Secondly we will consider an idealized portfolio consisting of a set of perfectly homogeneous subportfolios, each perfectly homogeneous in the sense just described. That a portfolio is perfectly homogeneous simply means that from a risk perspective all of the credits in the portfolio are equivalent. As our main focus is that of large dimensional portfolios it is natural in a step of dimension reduction to map the original portfolio to a set of idealized and perfectly homogeneous portfolios in a structured fashion. In fact these steps of dimension reduction is described in detail in Section 3.2.4. The reader should also note that a portfolio consisting of simply one credit by necessity also is perfectly homogeneous and hence none of the discussions below exclude the option of treating each credit separately.

Based on the discussion above we firstly consider a perfectly homogeneous portfolio \(\pi\) consisting of \(N\) credits. Let \(\hat{\pi}\) be one such credit. Then

\[
\frac{\partial}{\partial \pi} EC(0, n\Delta t, \pi) = \frac{1}{N} EC(0, n\Delta t, \pi)
\]

and as \(N\) increases the economic capital contribution of each individual credit decreases like \(N^{-1}\). The profitability calculation in this case results in

\[
\frac{1}{EN} P(0, n\Delta t, \pi) = \frac{1}{n\Delta t} \left[ E \left[ PV_M \left( 0, n\Delta t, \frac{\pi}{EN} \right) \right] - E \left[ PV_L \left( 0, n\Delta t, \frac{\pi}{EN} \right) \right] \right] - \epsilon \frac{1}{EN} \frac{\partial}{\partial \pi} EC(0, n\Delta t, \pi).
\]

The right hand side (r.h.s) gives a formula for the return as a fraction of the total exposure \(EN\). Note that in this case, in analogy with the discussion in the case of one credit, the formula just derived can be used to determine lower bound on the marginal \(r_M\) in order for the bank to break-even. We also note that because of diversification the r.h.s in the last formula will be larger than the r.h.s achieved in the case of just one credit. The last relation can also be rewritten as

\[
\frac{1}{EN} P(0, n\Delta t, \pi) = \sum_{k=0}^{n-1} (n\Delta t)^{-1} E \left[ PV_M \left( k\Delta t, (k+1)\Delta t, \frac{\pi}{EN} \right) \right] - \sum_{k=0}^{n-1} (n\Delta t)^{-1} E \left[ PV_L \left( k\Delta t, (k+1)\Delta t, \frac{\pi}{EN} \right) \right] - \sum_{k=0}^{n-1} \epsilon \frac{1}{EN} \frac{\partial}{\partial \pi} EC(k\Delta t, (k+1)\Delta t, \pi),
\]

where, \(EC(k\Delta t, (k+1)\Delta t, \pi) = EC(0, (k+1)\Delta t, \pi) - EC(0, k\Delta t, \pi)\). Finally, the analysis above can easily be generalized to the situation where \(\pi = \pi_1 + \cdots + \pi_m\) and where \(\pi_i\) in itself is a perfectly homogeneous portfolio consisting of \(N_i\) credits. We summarize the key aspects of the discussion above in the following two definitions.
Definition 2 (Term-structures of marginals, expected losses and EC contributions). Let \( \pi = \pi_1 + \cdots + \pi_m \) where each \( \pi_i \) is a perfectly homogeneous portfolio consisting of \( N_i \) credits. Then
\[
\{(n\Delta t)^{-1}E\left[ PV_M\left( k\Delta t, (k+1)\Delta t, \frac{\pi_i}{E_i N_i} \right) \right] \}_{k=0}^{n-1},
\]
\[
\{(n\Delta t)^{-1}E\left[ PV_L\left( k\Delta t, (k+1)\Delta t, -\frac{\pi_i}{E_i N_i} \right) \right] \}_{k=0}^{n-1},
\]
are referred to as the term-structures of average marginal return, average expected loss, on an interval of length \( \Delta t \) and over the horizon \( (0,n\Delta t] \), and contributions to EC expressed as a fraction of exposure for one credit, respectively.

Definition 3 (Risk adjusted return). Let \( \pi = \pi_1 + \cdots + \pi_m \) where each \( \pi_i \) is a perfectly homogeneous portfolio consisting of \( N_i \) credits. Then \( \hat{P}_i(0,n\Delta t, \pi) \), defined as
\[
(n\Delta t)^{-1}E\left[ PV_M\left( 0,n\Delta t, \frac{\pi_i}{E_i N_i} \right) \right] - (n\Delta t)^{-1}E\left[ PV_L\left( 0,n\Delta t, -\frac{\pi_i}{E_i N_i} \right) \right] - \left( \frac{\partial}{\partial \pi_i} EC(0,n\Delta t, \pi) \right)
\]
gives the average risk adjusted return, on an interval of length \( \Delta t \) and over the horizon \( (0,n\Delta t] \), based on the allocation of EC, expressed as a fraction of one unit of exposure to one counterparty, of portfolio \( \pi_i \) assuming a demanded return on capital \( \epsilon = \epsilon(\Delta t) \) for an interval of length \( \Delta t \).

We also introduce the following definition of risk adjusted return on capital (RAROC). For more details on the importance of this and related concepts we refer to Ong (1999) and Saunders and Allen (2002).

Definition 4 (Risk adjusted return on capital (RAROC)). Let \( \pi = \pi_1 + \cdots + \pi_m \) where each \( \pi_i \) is a perfectly homogeneous portfolio consisting of \( N_i \) credits. Then
\[
\frac{(n\Delta t)^{-1}E[PV_M(0,n\Delta t, \pi_i)] - (n\Delta t)^{-1}E[PV_L(0,n\Delta t, \pi_i)] - \frac{\partial}{\partial \pi_i} EC(0,n\Delta t, \pi)}{\frac{\partial}{\partial \pi_i} EC(0,n\Delta t, \pi)}
\]
is the average risk adjusted return on capital (RAROC) of portfolio \( \pi_i \), on an interval of length \( \Delta t \) and over the horizon \( (0,n\Delta t] \), assuming a demanded return on capital \( \epsilon = \epsilon(\Delta t) \) for an interval of length \( \Delta t \).

To summarize assume that the bank at \( t = 0 \) has a credit portfolio \( \pi = \pi_1 + \cdots + \pi_m \) where each \( \pi_i \) is a perfectly homogeneous portfolio consisting of \( N_i \) credits. The bank can use the economic capital framework described in the following way. Firstly, choose a time frame \([0,n\Delta t]\) and calculate the economic capital needed to support the portfolio \( \pi, \ EC(0,n\Delta t, \pi) \), using Definition 1. Secondly, define for each perfectly homogeneous segment or portfolio \( \pi_i \) the term-structures of average marginal return and average expected loss, on an interval of length \( \Delta t \) and over the horizon \( (0,n\Delta t] \), as well as contributions to EC expressed as a fraction of exposure for one credit as in Definition 2. Thirdly, assume that a new credit is issued in segment \( i \). Derive, as above, based on the maturity of the
credit and the present term-structures of average marginal return and average expected
loss as well as contributions to EC, the average risk adjusted return on the credit as in Def-
nitions 3 and 4. Finally, at time $t = 0$ the bank should invest the amount $EC(0,n\Delta t,\pi)$ in
bonds in the interbank market. For each $k \in \{0,\ldots,n-1\}$ an amount equal to
$[EC(0,(k+1)\Delta t,\pi) - EC(0,k\Delta t,\pi)]$ is invested in bonds with maturity $(k+1)\Delta t$.

Finally we note that our treatment of EC is based on a static portfolio in the sense that
we have neglected to include a potential flow of new credits and new counterparties. This is
a deliberate choice in our presentation and again this limitation can be relaxed, at the
expense of more notation, by adding a model for the arrival of new customers and for
the investment strategy of the bank. The latter would result in a model having an addi-
tional dynamic component compared to the framework presented. Clearly then, the con-
cept of economic capital introduced, is most relevant for the portfolio at hand.

3. The credit loss process

The previous section was devoted to the analysis of a credit portfolio $\pi$ without making
any assumptions in particular on the underlying credit risk model. In this section we
intend to describe our model for credit risk and to put our model into the context of
the existing literature and models.

As before we start by splitting the interval $(0,n\Delta t]$ into subintervals $(k\Delta t,(k+1)\Delta t]$ for
$k \in \{0,\ldots,n-1\}$. The ultimate goal for any credit risk model of a non-traded portfolio $\pi$
containing a set of credits is to quantify the loss of notional, due to credit losses, as well as
the recoveries on the portfolio $\pi$ in case of default. We let $\tilde{L}(k\Delta t,(k+1)\Delta t)$ be the total of
all credit losses of notional occurring in $\pi$, i.e., neglecting potential recoveries, during the
interval $(k\Delta t,(k+1)\Delta t]$. $R(k\Delta t,(k+1)\Delta t)$ is the total of all recoveries collected during the
same interval. In this section we will for simplicity discard issues of discounting and just
assume that interest rates are flat and identical to zero. This in particular allows us to focus
on the purely credit risk related components of the loss process. The actual loss on the
portfolio during $(k\Delta t,(k+1)\Delta t]$ is therefore $L(k\Delta t,(k+1)\Delta t) = \tilde{L}(k\Delta t,(k+1)\Delta t) -
R(k\Delta t,(k+1)\Delta t)$ and

$$L(0,n\Delta t) = \sum_{k=0}^{n-1} L(k\Delta t,(k+1)\Delta t)$$

denotes the total of actual losses during $(0,n\Delta t]$. To model the credit loss process a spec-
ification of $L(0,n\Delta t)$ is needed.

Suppose that the portfolio, $\pi$, consists of credits to $N$ counterparties. To each counter-
party there is a default time $\tau_i$ associated. Each counterparty could have several credits and
we will assume in our presentation that if $i$ defaults he/she defaults on all of her credits.
The latter is deviation from what is observed in reality, but the assumption can easily
be relaxed at the expense of introducing a more detailed notation. In any case we let
$E_i(t)$ denote the exposure the bank $B$ has to counterparty $i$ at time $t$, i.e., the non-amortised
notional of the original loan. We have that

$$L(0,n\Delta t) = \sum_{i=1}^{N} \int_{(0,n\Delta t]} (\tau_i) E_i(\tau_i)(1 - R_i(\tau_i)),$$
where $R_i(\tau_i) \in [0, 1]$ is a factor modelling the amount recovered at default. In reality recoveries are not collected instantaneously but this is no loss of generality as we still have flexibility in choosing the function $R_i$. The degrees of freedom in the expression for $L(0, n\Delta t)$ are the main focus in any portfolio model for credit risk. In particular, in the most general situation a model for the joint distribution of $\{\tau_i\}, \{E_i(\tau_i)\}, \{R_i(\tau_i)\}$ is needed. In the following we will not consider situations where the exposures $\{E_i\}$ have any other stochastic features than the one entering through the uncertainty of the default times $\tau_i$, i.e., $E_i = E_i(t)$ is a deterministic function of time. A situation where this is not a correct assumption is in the case of counterparty risk in OTC-derivatives, but we do not think the omission of these cases decreases the value of our presentation. Hence we will focus on default times and the modelling of recovery rates.

The main problems any portfolio model for credit risk has to address are the following. Firstly, a model for the joint distribution of default times $(\tau_1, \ldots, \tau_N)$ is needed. Secondly, models for recovery processes are needed and dependence between the timing of defaults and recovery processes should be possible to include. Thirdly, the model has to be feasible from the point of view of computing. In particular this implies that if $N$ is very large a scheme to reduce the dimensions is needed. In Section 3.2 we have detailed our portfolio model and we claim that our model addresses all of these problems and actually proposes a solution for large dimensional portfolios like for example retail and mortgage portfolios. In particular the issue of dimensionality is resolved by introducing the concept of a perfectly homogeneous portfolio and by creating a tree structure where the original portfolio $\pi$ is mapped to an artificial portfolio $\tilde{\pi}$. The latter portfolio will be seen to consist of a set of perfectly homogeneous subportfolios $\{\tilde{\pi}_j\}$.

3.1. Overview

In this section we intend to describe our model for credit risk on an overview level and put our model into the context of the existing literature and models.

The portfolio models proposed in the literature can roughly be divided into structural models and reduced form or intensity based models.

The basis of the structural approach, which was pioneered in the work of Black and Scholes (1973) and Merton (1974), is the idea that corporate liabilities are contingent claims on the assets of the firm and that the firm defaults when its assets falls below its liabilities. This line of thought has been explored and refined extensively by many researchers over the past fifteen years, e.g., see Longstaff and Schwartz (1995), Leland (1994), Leland and Toft (1996), Leland (1998), Cossin and Pirotte (2001), Zhou (2001a,b), Hilberink and Rogers (2002), Kijima and Suzuki (2001). An attractive feature of the structural approach is that it is based on the microeconomics of the firm and its balance sheet and that it is able to connect this information to a default mechanism. Often equity is used as a proxy for the firm value and models of this type tend to try to make use of information in the equity market for the purpose of calibration. Concerning recovery structural models assume that either the amount recovered is exogenously determined as e.g., a fixed fraction of an equivalent coupon and maturity risk-free bond or endogenously determined as a fraction of the firm value at default as seen in the structural models of Merton (1974), Black and Cox (1976), Leland (1994) and Leland and Toft (1996). In the latter case, dependence between the timing of default and recovery is part of the structure.
In portfolio models based on the structural approach dependence between the default events of firms is introduced on the level of the firm value processes, i.e., the value processed of the firms are assumed dependent in one way or the other resulting in dependence of defaults. Examples of frequently used portfolio models based on the structural approach are CreditMetrics (1997) and Moodys-KMV. A frequent feature of models based on the structural approach is that the models are static and that the default risk is modelled over a fixed time horizon. CreditMetrics (1997) and Moodys-KMV are no exceptions to this. The CreditMetrics and Moodys-KMV models also serve as the foundation for the IRB risk weighted asset formulas proposed in the Basel II accords. In particular, the risk weighted asset formulas are derived assuming an infinitely large homogeneous portfolio where firm values are driven by an equicorrelated one factor model. For further discussions about the foundations of the IRB approach we refer to Gordy (2000).

The reduced form approach differs from the structural in the sense that there is no pre-described default mechanism involved, instead the philosophy is to model defaults as unpredictable realizations from a Poisson type process. Fundamental references for this approach to corporate bond pricing is Litterman and Iben (1991), Artzner and Delbaen (1995), Jarrow and Turnbull (1995), Jarrow et al. (1997), Lando (1998) and Duffie and Singleton (1999). An attractive feature of this approach is that an interest rate theory for corporate bonds containing credit risk can be completely paralleled with the interest rate theory for non-defaultable bonds. As such these models tend to be calibrated using information from the bond market. In Jarrow and Turnbull (1995) and Jarrow et al. (1997) holders of debt receive a fixed fraction of the face amount at the maturity of the defaultable bond. In Duffie and Singleton (1999) holders of debt receive a fraction of the market value of the defaultable bond just prior to default. In Duffie (1998) and Lando (1998) a concept of recovery, based on receiving the same fractional recovery of par at default for bonds of the same issue and seniority, regardless of remaining maturity was introduced into a defaultable debt model. Recent empirical work by Guha (2002) shows that this concept of recovery is supported by data on defaulted bond values, hence providing evidence that this concept of recovery is the appropriate one to describe corporate bonds upon default.

The reduced form approach is also frequently used in the context of non-traded credit risk e.g., for performance measurement and loan pricing using the RAROC methodology, see e.g., Ong (1999), Saunders and Allen (2002) and Section 2 of this paper where we detailed an economic capital approach to loan pricing. Facing the pressure of Basel II many banks are today developing or refining internal rating classifications and portfolio models facilitating a risk based pricing approach.

Duffie and Lando (2001) show the link between intensity and structural based models by assuming that firm asset values are imperfectly observed, and introduce incomplete information credit models. Such models were also introduced in Giesecke (2001) and Cetin et al. (2002). The hybrid models give a common perspective on the structural and intensity based approaches and allow us to see these two types of models as members of a common family. In this context it is relevant to mention that models with incomplete information is also discussed, from a slightly different perspective, in Sobehart et al. (2000). See also Hillegeist et al. (2002) and Tuleda and Young (2003). For more on this type of models we refer to Giesecke (2004a,b).
In reduced form portfolio models default dependence is modelled using correlation between the default intensities of the counterparties but another popular approach to joint defaults is the copula approach. In this approach the marginal default distributions are linked by means of a copula function into a joint default distribution, i.e., essentially the dependence is induced directly on \((\tau_1, \ldots, \tau_N)\) through the copula. One widely used copula is the Gaussian copula introduced, in this setting, by Li (2000). Many other copulas can be constructed, in particular in low dimensions, and we refer to Nelsen (1999) and Rogge and Schönbucher (2003) for more on the topic. The main advantage of copula models is that the marginal default distributions can be calibrated in the same way as in single name intensity models.

Though theoretically appealing, there are few copulas applicable when \(N\) is large, i.e., when there are many default times involved. The standard way of resolving issues of dimensionality in this kind of problems is to reduce dimensionality using factor models. In particular, from an applied perspective, a very important class of dependence models is based on such models and the notion of conditional independence, see for example Yu (2003). This is a general class of models where individual default events depend on a factor structure common to all counterparties. Conditional on this factor structure the defaults are independent.\(^3\) In our model the reduction of dimension is based on similar ideas as in the CreditRisk\(^+\) (1997) and CreditPortfolioView/Wilson (1997) portfolio models. In the CreditRisk\(^+\) model the default threshold is not assumed to be observable. The stochastic default probabilities are driven by a linear reduced form factor model where the factors are independently Gamma distributed. The purpose of the choice of linear form for the factor model and the independent Gamma distribution for factors is to obtain an analytical solution. For this purpose it is also assumed that exposures and recoveries are fixed over some pre-defined horizon. The CreditPortfolioView model is similar to the CreditRisk\(^+\) model albeit without the explicit requirement of analytical solvability the functional form for the factor model is chosen to be the logistic function. Further, the factors are not restricted to the Gamma distribution, nor need they be independent. Our model of the default times of the counterparties in \(\pi, (\tau_1, \ldots, \tau_N)\), is an extension of the CreditPortfolioView model and includes a transition or migration matrix. Important features of this part of our model is that it is multi-period in the sense that defaults are not considered only at some fixed time-horizon, that it is multi-state as the case of only two states, non-default and default, is a special case of the model. In addition our model for recoveries is based on a well defined recovery mechanism and an appropriate modelling of the evolution of collateral values. We believe that our approach of changing the focus from a model of a fixed recovery rate to the modelling of the value processes of collaterals is important. After all it is the value of collateral, potentially adjusted by a haircut factor, as well as the legal aspects of recovery that are the essential ones. The transformation to a recovery factor is, by necessity, deterministic once this information is available. In particular, as the recovery has much impact on the credit loss process, we believe that there are many benefits from using this model compared to using simple portfolio mean recovery rates.

---

\(^3\) Yet another further approach to joint defaults is referred to as credit contagion, see e.g. Davis and Lo (1999, 2001), where defaults incur infectious defaults of other obligators. Some of these models are based on structural components, others on intensity based components. A few more references to papers dealing with models of credit contagion can be found in Jarrow and Yu (2001), Giesecke and Weber (2002, forthcoming) and Egloff et al. (2004).
Furthermore we capture dependence between default events and recoveries by modelling the vector of all variables simultaneously.

3.2. A model proposition

This section is divided into a number of subsections. The first three subsections detail the main model pieces on which we base our portfolio model. The three pieces are a stochastic migration model and as a consequence a model for the probability of default, models for recovery processes and analysis of the cash flow structures of the credits. In the description of these pieces we also define concepts of what we call perfect homogeneity. This idealized situation is the basis of the dimension reduction step carried out as the first step in the portfolio modelling. How all of the pieces fit together in a portfolio model and a technical description of the reduction scheme is detailed in the subsequent subsection.

3.2.1. Transition risk and the probability of default

Suppose that the portfolio, $\pi$, consists of credits to $N$ counterparties. We will in the following assume the existence of a rating scale consisting of $K$ rating classes where belonging to class number $K$ implies default. At $t = 0$ each of the non-defaulted counterparties are located in one of the classes $\{1, 2, \ldots, K-1\}$. To the rating scale we assign a $K \times K$ matrix of transition probabilities, $A = \{A_{ls}\}$, where the last row of $A$ defined as $[0, 0, \ldots, 0, 1]$. In our model the transition matrix will be a matrix valued stochastic process $A(t) = A(t, \omega)$ and $A_{ls}(k\Delta t)$ describes the probability, at $t = k\Delta t$, that a counterparty rated to be in class $l$ will make a transition to class $s$ during the interval $(k\Delta t, (k+1)\Delta t)$. Let for $i = 1, \ldots, N$, $X_i(t) = X_i(k\Delta t)$ be the rating of counterparty $i$ at $t = k\Delta t$. In the following we will assume that, conditional on a sequence of transition matrices $A(t)$, $\{X_i(t)\}_{i=1}^N$ is a set of independent Markov chains defined with respect to these matrices.

Before introducing our model let us recall that in the modelling of default/no default the simple binary logit model has been used quite frequently in models for credit risk and in particular it appears as the specification for systematic default risk in the well-known CreditPortfolioView model, see Wilson (1997). Our basic idea for the modelling of migration risk is to consider the more general multinomial logistic model introduced by Theil (1969). In this context considerations and discussions relevant to ours can also be found in McNulty and Levin (2000), Bahar and Krishan (2000, 2001), Saunders and Allen (2002, pp. 107–120), Keenan and Sobehart (2002), Sobehart and Keenan (2004), Sorge (2004). To set the stage we shall first discuss the logistic model approach to systematic default risk as it appears in Wilson (1997) and then make a smooth transition to the multinomial case.

In a two state model for credit risk, default or not default, $K = 2$, and in this case $p(k\Delta t) = A_{12}(k\Delta t)$ is the only cell in the migration matrix that we have to model. We will

---

4 Best practice in rating classification systems consist of logistic regression score cards for the retail and SME segments and an expert model with score judgements for banks and large corporates. The expert model is often complemented with a benchmark rating (PD) from a structural model of Merton-type. For the logistic score cards a rating classification is generated by a partitioning of the interval $[0, 1]$, i.e., the range of the model. For the expert model score judgements are, due to the scarcity of internal data for calibration, usually mapped to the rating classification of rating agencies.
refer to \( p(k\Delta t) \) as the default rate for the interval \((k\Delta t, (k+1)\Delta t)\). For the portfolio \( \pi \) the approach of Wilson (1997) is encoded as

\[
p(t) = \frac{\exp(\mu + \langle Z(t), \beta \rangle + v(t))}{1 + \exp(\mu + \langle Z(t), \beta \rangle + v(t))}.
\]

Here \( Z(t) = Z(t, \omega) = (Z_1(t, \omega), \ldots, Z_h(t, \omega)) \) is a \( h \)-dimensional vector of random variables and \( \mu \) and \( \beta = (\beta_1, \ldots, \beta_h) \) are constants. The random variables \( Z(t, \omega) \) are referred to as systematic variables and should be thought of as variables describing the evolution of economically relevant variables like interest rates, GDP and unemployment. The innovations \( v(t) \) are assumed to be \( N(0, \sigma) \) variables. The main idea is that all counterparties in \( \pi \) are supposed to satisfy a certain homogeneity with respect to default risk in the sense that all counterparties belong to a specific industry sector, counterparty type or geographical region as well as a specific rating class within. Such a division of counterparties may be obtained through internal credit scoring methods or using the rating agencies grading. In the more general case of a portfolio \( \pi = \pi_1 + \cdots + \pi_m \) where all counterparties in subportfolio \( \pi_j, j \in \{1,2,\ldots,m\}, \) are assumed homogeneous in the sense just described a model for the default rate in each portfolio is specified by introducing an index on all variables and parameters, i.e.,

\[
p_j(t) = \frac{\exp(\mu_j + \langle Z(t), \beta_j \rangle + v_j(t))}{1 + \exp(\mu_j + \langle Z(t), \beta_j \rangle + v_j(t))}.
\]

In the models for \( \pi \) either all \( v_j \in N(0, \sigma_j) \) are assumed independent or \((v_1, \ldots, v_m) \in N(0, \Sigma)\) where \( \Sigma \) is a constant matrix.

A useful property of the logistic model specifying \( p(t) \) is that the logarithm of the ratio of the default rate and the survival rate is a linear model i.e.,

\[
\ln \left( \frac{p(t)}{1 - p(t)} \right) = \mu + \langle Z(t), \beta \rangle + v(t).
\]

Let \( \tilde{p}(t) \) be an empirical time-series of default frequencies. Then we can, assuming that at all times there are sufficiently many defaulted counterparties, apply a frequency interpretation for \( \tilde{p}(t) \). That is, we can assume that there is no measurement error in \( \tilde{p}(t) \) and we can write the equation to be estimated as\(^5\)

\[
\ln \left( \frac{\tilde{p}(t)}{1 - \tilde{p}(t)} \right) = \mu + \langle Z(t), \beta \rangle + v(t).
\]

In the case of a portfolio \( \pi = \pi_1 + \cdots + \pi_m \) we similarly have, for \( j \in \{1,2,\ldots,m\}, \)

\[
\ln \left( \frac{\tilde{p}_j(t)}{1 - \tilde{p}_j(t)} \right) = \mu_j + \langle Z_j(t), \beta_j \rangle + v_j(t).
\]

\(^5\) From an econometric perspective measurement error in the dependent variable need not induce inconsistency of the parameter estimates. Specifically, we can assume that the empirically computed \( \tilde{p}(t) \) measures the true default frequency \( p(t) \) with an measurement error of \( d(t) \sim N(0, \delta) \) independent of \( \tilde{p}(t) \). In the latter case we could subsume the measurement error in the residual term \( v(t) \). Still, for clarity of exposition we stick to our assumption that the \( \tilde{p}(t) \) is the true frequency.
This defines a multivariate linear regression model for which the seemingly unrelated estimate technique, as proposed by Zellner (1962), is efficient. With this estimation technique consistent parameter estimates are obtained by applying least squares equation by equation and in the second step an estimate of the covariance matrix of the least squares residuals is used to improve on the efficiency of the least squares estimator using the generalized least squares estimator. In the special case that the vectors $Z(t)$ are common for all $j$ seemingly unrelated regression reduces to least squares equation by equation, see Amemiya (1985, p. 197). In applications this is quite likely to be satisfied across rating classes within a specific industry sector, counterparty type or geographical region.

We are now ready to specify our model for stochastic migration and again we suppose that the portfolio, $\pi$, consists of credits to $N$ counterparties. In analogy with the model of Wilson described above our model of stochastic migration is based on the following specification:

$$A_{ls}(t, \omega) = f_{ls}(Z(t, \omega)),$$

where we initially have suppressed the specification of the innovations $v$. In particular this implies that we restrict ourselves to matrix valued stochastic processes $A(t)$ which are Markov. Hence the paths of transitions are not allowed to influence the probabilities of transitions in future steps. Recent research, Cantor and Hamilton (2004), shows that in the case of the transition matrices maintained by Moody’s the first order Markov structure is not the right vehicle to use in order to model transitions. Still we think that in our case, as we are mainly focused on large dimensional portfolios like for example retail and mortgage type portfolios, this is less of a serious modelling restriction. The specification of $A$ stated above also implies that all counterparties, which at time $t = 0$ are in rating class $l$, are homogeneous with respect to future rating transitions.

Note that as for each $l \in \{1, \ldots, K\}$, $\sum_r A_{ls}(k \Delta t) = 1$, the choice of functional form for $f_{ls}$ is somewhat limited. The multinomial logit formulation expresses the log of a ratio of probabilities as a function of variables. More specifically for $l \in \{1, \ldots, K - 1\}$, $s \in \{1, \ldots, K - 1\}$,

$$\ln\left(\frac{f_{ls}(Z(t, \omega))}{f_{lk}(Z(t, \omega))}\right) = \tilde{f}_{ls}(Z(t, \omega)).$$

This can also be rephrased as

$$f_{ls}(Z(t, \omega)) = \frac{\exp[\tilde{f}_{ls}(Z(t, \omega))]}{1 + \sum_s \exp[\tilde{f}_{ls}(Z(t, \omega))]} = \frac{1}{1 + \sum_s \exp[\tilde{f}_{ls}(Z(t, \omega))]}.$$

In the following we will by $\tilde{A} = \{\tilde{A}_{ls}\} = \{\tilde{f}_{ls}\}$ denote the $(K - 1) \times (K - 1)$ matrix of log ratios of probabilities. One relevant choice for the function $\tilde{f}_{ls}(Z(t, \omega))$ is $\tilde{f}_{ls}(Z(t, \omega)) = \mu_{ls} + (Z(t), \beta_{ls}) + v_{ls}$ where $\mu_{ls}$ and $\beta_{ls} = (\beta_{ls1}, \ldots, \beta_{lsk})$ are constants and either all $v_{ls} \in N(0, \sigma_{ls})$ are assumed independent or $(v_{l1}, \ldots, v_{K-1K-1}) \in N(0, \Sigma)$ where $\Sigma$ is a constant matrix. Structurally this gives a straightforward generalization of the two state model of Wilson to a multi-state setting. The set of equations can be calibrated to empirical transition frequencies using seemingly unrelated regression in the same manner as the logit model. The historical time series of empirical transition frequencies may come from either internally calculated empirical transition matrices obtained through credit scoring, as would be the case for retail segments as well as small and medium sized corporates, or
in case of large corporates so-called point in time empirical transition matrices published by the rating agencies. We are quite aware that for some segments sufficiently long empirical time series of transition frequencies may not be available. In particular the historical data may not cover a complete business cycle and a combination of empirical data and expertise judgement may be necessary for a conservative calibration of parameters. Although we believe and understand that this is an important issue we do not discuss it further here. A further concern in the multinomial logit case is that the system of equations is of dimension \((K - 1)^2\) as opposed to \(K - 1\) for the simple logit model. As a practical point \(K\) should therefore not be too large.

Note that, conditional on a path \(Z(t)\), the transition of any counterparty is a realization from a multinomial distribution. For a large pool of counterparties that are homogeneous with respect to rating at \(t = 0\) we can make use of the conditional law of large numbers to understand the frequency of transition from rating class \(l\) at \(t = 0\) to rating class \(s\) at \(t = k\Delta t\) by iterating the relation

\[
y(j\Delta t) = A(j\Delta t)y((j - 1)\Delta t).
\]

Here \(y(j\Delta t)\) is a \(K \times 1\) vector and \(y(0)\) contains zeros in all positions except on position \(l\) where the number 1 can be found. This observation will be of particular use for us as it has the potential to reduce the computational burden considerably. We end this section with the following definition.

**Definition 5 (Perfect homogeneity with respect to transition risk).** Two counterparties are said to be perfectly homogeneous with respect to transition risk if both, at \(t = 0\), have the same rating and if their migration in the rating scale is defined by the same transition matrix.

### 3.2.2. Recovery with stochastic collateral

For collateralized credit portfolios the bulk of recovery usually comes from the sale of collateral. Hence, acting as a lender it is crucial, both from a pricing and capital perspective, to have an understanding of the specific risk in collateral value as well as the risk of the joint event of default and large collateral value reduction. The specific risk in collateral value includes both the uncertainty in current collateral value, i.e., what is the market value of collateral, should the counterparty default today, as well as the risk in collateral value evolution over time. In this exposition we shall however assume that the current value of collateral is known with certainty and we focus on the modelling of the risk in collateral value evolution. For this purpose it is typically not feasible to have a specific model for each collateral that describes the evolution of its value. Prime examples include portfolios secured by property such as consumer mortgages as well as object leasing. It is therefore natural to map the evolution of collateral to a set of collateral evolution models. We index these models by \(S_j, j = 1, \ldots, m\) and for each counterparty the future value process of each of its collaterals is modelled by the collateral evolution models. Specifically, denoting by \(\bar{V}(0)\) the current value of a collateral belonging to segment \(j\), we model the value of collateral, at the time \(t = k\Delta t\) as

\[
\bar{V}(t, \omega) = \bar{V}(0)S_j(t, \omega),
\]

where \(S_j(t, \cdot) \in (0, \infty), S_j(0) = 1\). At this level of generality we emphasize that the structure defined is by its nature multiplicative and \(S_j(t, \cdot)\) acts like a stochastic scaling factor.
of the current value of collateral. In case property is used as collateral, one natural way to calibrate the collateral evolution models \( j = 1, \ldots, m \) is to make use of historical data on officially published price indices. The latter are usually segmented on region and type of property. Similarly having historical data from an internal property valuation tool one can use this to construct price indices. However, official price indices have the benefit of objectivity and public scrutiny. As a further example consider object leasing where we may, per object group, estimate value decay curves with support on \((0, 1]\) using samples on historical value decays. The estimated curves may be viewed as deterministic or stochastic as appropriate.

Starting from the current valuation of the collateral of counterparties we model the evolution of the value of collaterals as above. Still an additional component is needed in order to properly model reality. The fact is that in many cases the default value of the collateral is lower than its “going concern” value. A common way of adjusting for this is to introduce an estimate of the fraction of the market value retained. We denote this haircut fraction by \( \gamma \). Using this notation, the (default) value of a collateral, at \( t = k \Delta t \), is given by \( V(t, \omega) = \bar{V}(t, \omega) \gamma \). To be able to estimate recoveries conditional on current collateral value, the collateral evolution model and the assigned haircut it remains to define the exposure at default as well as the legal aspects of recovery. We denote by \( E(t) \) the exposure at time \( t = k \Delta t \). The recovery fraction of exposure at \( t = k \Delta t \) can then be written

\[
R(t, \omega) = \frac{1}{E(t)} \min [E(t), RC(t, \omega) + RB].
\]

Here \( RC(t, \omega) \) is a deterministic function of \( \{S_j(t, \omega)\} \) which captures the legal priority for the financial institution and \( RB \) is the recovery in blanco in excess of what is at most recovered from \( RC \). For loans secured against property this legal priority structure may be somewhat complicated with different tranches of the property value being legally secured by the financial institution. This is the typical case when several mortgage institutions are involved in financing a property. Considering a single property we can express \( RC(t, \omega) \) generally as

\[
RC(t, \omega) = \sum_{d=1}^{D} \max \left[ (V(t, \omega) + \min \left[ (T_u^d - V(t, \omega)), 0 \right] - T_d^u), 0 \right].
\]

(3)

Here \( T_d^u \) and \( T_d^l \) are the upper, respectively, lower threshold of property value tranche \( d = 1, \ldots, D \). These thresholds are expressed in amount secured in the property. As a clarifying example consider a mortgage institution that have first priority up to an amount equal to \( T_1^u \). In that case we obtain

\[
\tilde{R}(t, \omega) = \max \left[ (V(t, \omega) + \min \left[ (T_1^u - V(t, \omega)), 0 \right) \right], 0 \right].
\]

Hence if \( T_1^u \leq V(t, \omega) \) we recover at most \( T_1^u + RB \) and otherwise we recover at most \( V(t, \omega) + RB \). To give yet another and more numerical example assume that the mortgage institution have secured amounts in the interval \([0, 100]\) as well as in the interval \([150, 200]\) with another institution having secured the intervening amount i.e., \([100, 150]\). In terms of the formula (3) \( [T_1^l, T_1^u] = [0, 100], [T_2^l, T_2^u] = [150, 200] \). Applying the formula assuming that the property is sold at an amount equal to 175 i.e., \( V(t, \omega) = 175 \) the institution will, in case of an exposure of 150, recover 100 from the first legal tranche but only 25 from the second legal tranche. Having a null recovery in blanco (i.e., \( RB = 0 \)) the institutions recovery fraction of exposure is therefore 125/150.
Assuming first priority which is the generic case in property financing, object leasing as well as other collateralized lending the formula above can be simplified to a loan to value ratio. We define the loan to value ratio, at time $t = k\Delta t$, as $LTV(t) = E(t)/V(t)$. Below we will make use of the following explicit relation between the current loan to value, $LTV(0)$, and $LTV(t)$, $LTV(t) = LTV(0) \left(1 - \frac{a(t)}{S(t)}\right)$.

Here $a(t) = (E(0) - E(t))/E(0)$ is the non-remaining fraction of current exposure, $E(0)$. We can now write the recovery rate at $t = k\Delta t$ as

$$R(t) = \frac{1}{E(t)} \min \left[ E(t), \frac{E(t)}{V(t)} + R_b \right].$$

In the following we shall assume that the first priority formula (5) for the recovery rate applies. But we also note that the above first priority formula can serve as an approximation of the general case of (2) and (3) if we define today’s loan to value, $LTV(0)$, using $E(0) + \sum_{d=1}^{D-1} \left( T_{d+1}^l - T_d^l \right)$ instead of simply current exposure, $E(0)$. With this in mind we now, in analogy with the concept of perfect homogeneity in transition risk, introduce the concept of perfect homogeneity in recovery risk.

**Definition 6 (Perfect homogeneity with respect to recovery risk).** Two counterparties are said to be perfectly homogeneous with respect to recovery risk if both have the same loan to value at $t = 0$, the value evolution of their collaterals are described by the same index process, they have identical haircuts and recoveries in blanco and their exposure processes are the same. Finally, we want to point out that in analogy with validation requirements of rating classification models we also have to validate the recovery model. In the case of rating systems this is often done using so called cumulative accuracy profiles or accuracy ratio measures, see Keenan et al. (2000). In our model for recoveries the validation must include a validation of the accuracy of the methodology for current (going concern) valuation of collateral, a validation of the accuracy of haircuts employed and a calculation of the accuracy of the employed portions of recovery in blanco. In addition to this one must of course also make sure that a relevant model for the evolution of collateral value is specified.

3.2.3. No default cash flow structures

The no default cash flow structure of a credit is a structure $(r_M, A, E, \hat{n}\Delta t)$ where $r_M$ is the marginal earned by the bank, $A$ represents the amortisation structure, $E$ represents the exposure and $\hat{n}\Delta t$ represents the maturity of the loan. $\hat{n}\Delta t = \infty$ if the credit is a revolving credit line for instance. We have chosen to exclude prepayments in our presentation, hence that component is excluded from the list. The following definition will be of importance.

**Definition 7 (Perfect homogeneity with respect to no default cash flows).** Two counterparties are said to be perfectly homogeneous with respect to no default cash flows if their no default cash flow structures $(r_{M_1}, A_1, E_1, \hat{n}_1\Delta t)$ and $(r_{M_2}, A_2, E_2, \hat{n}_2\Delta t)$ coincide.
The reader notices that there is a bit of overlap between the concept of perfect homogeneity with respect to recovery risk and the concept of perfect homogeneity with respect to no default cash flow structure. In particular two counterparties that are perfectly homogeneous with respect to recovery risk are also perfectly homogeneous with respect to no default cash flows if they have the same marginal, \( r_M \), and maturity, \( \hat{n}\Delta t \).

3.2.4. The portfolio model – combining the pieces

In this section we intend to describe how the pieces of the previous subsections are integrated into a portfolio model. We start with a portfolio \( \pi \) consisting of \( N \) counterparties where \( N \) can be very large (say 2000000–10000000). Logically in any application the first step is therefore to achieve a dimension reduction and in this task our previously introduced definitions of perfect homogeneity with respect to transition risk, recovery risk and no default cash flows will play a vital part. The actual process of reducing the dimension of the portfolio is carried out in two steps. First the portfolio is divided into \( m \) subportfolios, \( \pi = \pi_1 + \cdots + \pi_m \), such that \( \pi_j \) contains \( N_j \) counterparties. In the second step \( \pi_j \) is transformed to a portfolio \( \hat{\pi}_j \) consisting of \( N_j \) counterparties and such that \( \hat{\pi}_j \) is perfectly homogeneous in a sense defined below. \( \hat{\pi}_j \) is an artificial portfolio preserving certain key features of the portfolio \( \pi_j \). The complexity of this portfolio is of the order \( m \) instead of the original order of \( N \) as all credits in a subportfolio \( \hat{\pi}_j \) are identical. In any application calculations are carried out on the portfolio \( \tilde{\pi} = \tilde{\pi}_1 + \cdots + \tilde{\pi}_m \).

We start by explaining the decomposition \( \pi = \pi_1 + \cdots + \pi_m \). To do this we will assume that there are \( K \) rating classes (e.g., the master rating scale for the bank). The counterparties belong to different sectors which we here simplify to private sector, industry sector 1 and industry sector 2. We assume that even though the same rating scale is used for all counterparties the variables driving the stochastic migration model may differ between the three sectors. Hence all counterparties are divided into three subportfolios, i.e., \( \pi = \pi_1 + \pi_2 + \pi_3 \) and we let \( n_1 = 3 \). Secondly each such subportfolio \( \pi_j \) for \( j \in \{1,2,3\} \) is divided into \( (K-1) \) subportfolios (and define \( n_2 = K-1 \)), \( \hat{\pi}_j = \hat{\pi}_{j1} + \cdots + \hat{\pi}_{j(K-1)} \) depending on current position in the rating scale. This implies that all counterparties in each such subportfolio are perfectly homogeneous with respect to transition risk in the sense of Definition 5.

To continue the division we focus on the collateral. We consider a partitioning based on the collateral type and where applicable region where the collateral is located. Combining region and type we get \( n_3 \) possible combinations. For each subportfolio \( \hat{\pi}_{j_2} \) we therefore conduct a further subdivision \( \hat{\pi}_{j_2} = \hat{\pi}_{j_21} + \cdots + \hat{\pi}_{j_2n_1} \). Some of these subportfolios may very well be empty and in reality we will discard those portfolios. Note that by construction the values of all the collaterals of the counterparties in \( \hat{\pi}_{j_2} \) are supposed to evolve according to the same index process.

We continue the division by considering, in the next step, the legal aspects of recovery. For this purpose we define loan to value bands e.g., \((0,0.3],[0.3,0.5], \ldots \) and the counterparties are divided into buckets reflecting the size of the loan to value relative to the bands. We assume that there are \( n_4 \) such bands. For each subportfolio \( \hat{\pi}_{j_3} \) we then obtain \( \hat{\pi}_{j_3} = \hat{\pi}_{j_31} + \cdots + \hat{\pi}_{j_3n_4} \). The \( \gamma \) haircut fraction is so far not part of any division. For simplicity we assume that at this level of granularity the haircut is uniform, something that is reasonable from the very construction. We shall for simplicity also assume that recovery in blanco is uniform at this level of granularity.
The next stage in the division concerns the no default cash flow structure of the credits. We subdivide each leaf in the tree constructed so far with respect to exposure in the sense that current exposures are divided into bands in the same manner as we did for loan to value and the counterparties are divided reflecting the size of the exposure relative to the bands. Assuming $n_J$ buckets we obtain, for each subportfolio $\hat{\pi}_{j_1,j_2,j_3,j_4}$, $\hat{\pi}_{j_1,j_2,j_3,j_4} = \hat{\pi}_{j_1,j_2,j_3,j_4} + \cdots + \hat{\pi}_{j_1,j_2,j_3,j_4,n_J}$. In principle we could now continue this banding and subdivision further for the amortization structure as well as the marginal though we will stop at this level.

This concludes the first part of the dimension reduction and collecting all the leaves in the tree constructed we have the decomposition $\pi = \pi_1 + \cdots + \pi_m$. Each of these subportfolios can be referred to as homogeneous but not perfectly homogeneous and we assume that $N_j$ is large for each $j$. By large we here mean $\geq 100$. This may seem to exclude the possibility of one very big counterparty, but in fact this is not the case because if there is one very large exposure to one counterparty in the portfolio that exposure will in itself represent a subportfolio and not be part of any subdivision scheme.

To carry out the last step we pick a subportfolio $\pi_j$. Now by construction all counterparties in $\pi_j$ are perfectly homogeneous with respect to transition risk in the sense of Definition 5. They are all perfectly homogeneous with respect to collateral value model, haircut and recovery in blando. However they are not perfectly homogeneous with respect to recovery risk in the sense of Definition 6 since the loan to value, current exposure and amortization structure differ between counterparties. Finally $\pi_j$ does not qualify to be homogeneous with respect to no default cash flows in the sense of Definition 7. Before continuing we introduce the concept of perfect homogeneity of a portfolio.

**Definition 8 (Perfect homogeneity of a portfolio).** Let $\hat{\pi}$ be a portfolio containing a set of counterparties. Then $\hat{\pi}$ is said to be perfectly homogeneous if all counterparties are perfectly homogeneous in the sense of Definitions 5–7.

We want to transform $\pi_j$ to a portfolio $\pi_{\hat{j}}$ which is homogeneous in the sense of Definition 8. $\pi_j$ contains $N_j$ credits and the portfolio $\pi_{\hat{j}}$ will contain the same number of credits. To start to produce the representative credit in $\pi_{\hat{j}}$ we will assume that the loan to value equals the average loan to value for all counterparties in $\pi_j$. Recall that the no default cash flows of each credit in the portfolio $\pi_{\hat{j}}$ can be described by a quadruple $(r_{M_l}, A_l, E_l, \hat{n}_l\Delta t)$. Let $\hat{n} = \max\{\hat{n}_l, l \in \{1, \ldots, N_j\}\}$ and consider $k \in \{1, \ldots, \hat{n}\}$. $E(k\Delta t)$ is the exposure of the credit at $t = k\Delta t$ and $r_{M_k} \Delta t E_i(k\Delta t)$ is the marginal earned by the bank during $(k\Delta t, (k + 1)\Delta t]$. $A(k\Delta t)$ is the amortisation received at $t = k\Delta t$. If $k > \hat{n}_j$ we define all quantities involved to be zero. We furthermore define

$$E = \frac{E_1 + \cdots + E_{N_j}}{N_j}, \quad A(k\Delta t) = \frac{A_1(k\Delta t) + \cdots + A_{N_j}(k\Delta t)}{N_j}.$$  

We are therefore left with the task of redeeming the lack of perfect homogeneity in the sense of Definition 7. For this purpose we finally, for $k \in \{1, \ldots, \hat{n}\}$, let

$$r_{M_k}(k\Delta t) = r_{M_1}(k\Delta t) \frac{E_1(k\Delta t)}{E(k\Delta t)} + \cdots + r_{M_{N_j}}(k\Delta t) \frac{E_{N_j}(k\Delta t)}{E(k\Delta t)},$$

i.e., $r_{M_k}(k\Delta t)$ represents an exposure weighted marginal. We are now ready to define the portfolio $\pi_{\hat{j}}$. After the manipulations above this portfolio consists of $N_j$ credits perfectly homogeneous in the sense of Definitions 5 and 6. Furthermore from the no default cash
flow perspective we define each credit to be described by the quadruple \((r_M, A, E, \hat{n}' \Delta t)\). The artificial portfolio is hence constructed in such a way that the cash flows are preserved in an average sense. In particular \(\hat{n}_j\) is perfect homogeneous in the sense of Definition 7 and as a result also perfectly homogeneous in the sense of Definition 8. By construction, if no defaults occur, the cash flows of portfolio \(\pi\) perfectly replicate the cash flows of the original portfolio \(\pi\).

We can conclude that the dimension reduction is complete and that the result is the construction of a portfolio \(\pi = \pi_1 + \cdots + \pi_m\) such that for every \(j\), \(\pi_j\) is a perfectly homogeneous portfolio in the sense of Definition 8. Hence complexity is reduced to the order \(m\) as all credits in a subportfolio \(\pi_j\) are identical. The order of compression is \(N/m\) and assuming 2000000 counterparties and \(m = 10000\) classes the order of compression equals 200.

The next step is now to define models for all of the components. Firstly, to each portfolio \(\pi_j\) there is a model for stochastic migration attached: \(A_j(t, \omega) = f_j(Z_j(t, \omega)).\) Here \(Z_j(t, \omega) = (Z'_j(t, \omega), \ldots, Z_{kj}(t, \omega))\) is a set of explanatory variables. We let \(Z_j(t, \omega)\) be the vector containing all these variables. Secondly, to each portfolio \(\pi_j\) there is an index process \(S_j(t, \omega)\) attached, such that \(S_j(0) = 1\) with probability one. This process models the evolution of the value of collaterals. We let \(S(t, \omega)\) be the vector containing all such processes.

Recall that we, in the discussion of the credit loss process, for simplicity discard issues of discounting by assuming that interest rates are flat and identical to zero. This in particular allows us to focus on the purely credit risk related components of the loss process. With this in mind, combining the models, we obtain that the credit losses of portfolio \(\pi_j\) on the interval \((0, n \Delta t)\) equals

\[
L(0, n \Delta t, \pi_j) = \text{PV}_L(0, n \Delta t, \pi_j) = \sum_{k=0}^{n-1} \sum_{i=1}^{N_j} E(k \Delta t) (1 - R(k \Delta t)) y_{k_1}((k + 1) \Delta t) - y_{k_1}(k \Delta t) N_j.
\]

Here \(E(k \Delta t)\) is the exposure at \(k \Delta t\), \(R(k \Delta t)\) is the recovery rate as defined in (5) and \(\{\tau_{ij}\}_{i=1}^{N_j}\) is the set of stochastic default times for the \(N_j\) members of portfolio \(\pi_j\). As mentioned in Section 3.2.1, the realized default times (and ratings) can be obtained using multinomial sampling conditional on the transition matrix. For \(N_j\) reasonably large \(\sim 100\) we can replace the multinomial sampling scheme with the iteration (1) making use of a conditional law of large numbers approximation

\[
L(0, n \Delta t, \pi_j) = \sum_{k=0}^{n-1} E(k \Delta t) (1 - R(k \Delta t)) [y_{k_1}((k + 1) \Delta t) - y_{k_1}(k \Delta t)] N_j.
\]

Here \(y_{k_1}(k \Delta t)\) is element \((K, 1)\) of the \(K \times 1\) vector vector \(y(k \Delta t)\) defined in Eq. (1). Arguing similarly we have that the income from portfolio \(\pi_j\) on the interval \((0, n \Delta t)\) equals

\[
I(0, n \Delta t, \pi_j) := \text{PV}_I(0, n \Delta t, \pi_j) = \sum_{k=0}^{n-1} r_M(k \Delta t) E(k \Delta t) \Delta t [1 - y_{k_1}(k \Delta t)] N_j.
\]

In total the result from portfolio \(\pi_j\) on the interval \((0, n \Delta t)\) equals \(I(0, n \Delta t, \pi_j) - L(0, n \Delta t, \pi_j)\). Using linearity we can extend all of this notation to a portfolio \(\pi = \pi_1 + \cdots + \pi_m\).

To complete the model we have to specify a model for the vector \(\hat{Z}(t) = (Z(t), S(t))\) and we notice that in this way correlation between variables driving transition risk and the
collateral evolution indices can be captured. One example of such a model is a combined multivariate geometric brownian motion for collateral evolution indices and multivariate gaussian and mean-reverting process for variables driving transition risk. Correlations between the innovations to \( Z(t) \) and the innovations to the multinomial logistic logarithms of the odds ratios may also be taken into account.

4. Application

In this section we illustrate the model proposed by way of an example. Our intention with this section is to give the reader a numerical example and hence a concretization of our model and the theoretical aspects of the previous sections. To do so we will choose a parametrization of the credit risk model and construct and analyze a set of portfolios. We hope to return in a future paper to a case study based on a calibration to real data and the model applied to a real world portfolio. Such a case study would inevitably also raise the important but, in the context of credit risk, difficult question of model validation, see Lopez and Saidenberg (2000).

To start with we will for simplicity and as in the previous section assume that interest rates are flat and zero in order to discard issues of discounting. This in particular allows us to focus on the purely credit risk related components of the loss process and makes the result less difficult to interpret. For the purpose of the application we will also specialize to a time increment \( \Delta t \) equal to 1 year.

Concerning portfolios we will build 16 perfectly homogeneous portfolios. We do this by considering all possible combinations of the following binary choices. The marginal, \( r_M \), for a portfolio is fixed over the analysis horizon at either 10 bp or 50 bp. The start rating for a portfolio is either 1 or 5 on a scale which runs from 1 to 9, 9 being the default class. In particular we assume that there is just one transition matrix involved and hence only one model for stochastic migration. The loan to value (LTV) for a portfolio is either 0.5 or 1. The collateral is modelled by either collateral index model \( S_1 \) or \( S_2 \).

Concerning the composition of the portfolios we furthermore assume that each of the 16 portfolios, denoted \( \pi_j, j = 1, \ldots, 16 \), consists of 10000 counterparties each with an exposure of 1 million SEK. We will also for simplicity assume that there are no amortisations on any of the loans and that recoveries in blanco are zero. This construction yields a total notional for each portfolio \( \pi_j \) of 10 BnSEK and for all of the portfolios, \( \pi = \pi_1 + \cdots + \pi_{16} \), a total notional of 160 BnSEK. The portfolios are displayed in Table 1. The way we have labelled the portfolios we see that the portfolios represent decaying quality with \( \pi_1 \) having the highest quality and \( \pi_{16} \) having the worst quality.

Having defined the portfolios properly we now focus on the model and to specify the underlying model we have to supply the model for stochastic migration and models for the collaterals.

4.1. A model for stochastic migration

For the model of stochastic migration we assume that there is just one systematic or common driver of transition risk and we denote this factor by \( Z(t) \). That is, apart from the idiosyncratic innovations \( Z(t) \) is the only factor entering into the transition matrix \( A \) as described in Section 3.2.1. The evolution of \( Z(t) \) is assumed to follow a discretized Vasicek process,
\[ Z(t + \Delta t) - Z(t) = (1 - \rho)(\lambda - Z(t)) + \sigma_Z(W_Z(t + \Delta t) - W_Z(t)). \]

\( W_Z(t + \Delta t) - W_Z(t) \) specifies the annual increment \((\Delta t = 1)\) of a brownian motion and we define the parameters \(\lambda, \rho\) and \(\sigma\), as follows: \(\lambda = 0, \rho = 0.93, \sigma_Z = 0.1\). This implies that \(Z\) is supposed to follow a mean reverting process with zero mean, mean reverting factor of 0.93 and an annual residual volatility of 10%. Furthermore the process is Gaussian. We set the initial value of \(Z\), \(Z(0)\), to 0. The systematic risk factor can be interpreted as a measure of the deviation of GDP from a trend with the starting point being neither a recession nor a boom.

To obtain scenarios for the transition matrix \(A\) we filter the scenarios on \(Z(t)\) through the multinomial logit model. This requires a specification of the parameters of the multinomial logit model. Denote by \(\mu\) the \(8 \times 8\) constant parameter matrix i.e., \(\mu\) has elements \(\{\mu_{ls}\}, l = 1, \ldots, 8, s = 1, \ldots, 8\). Further, denote by \(\beta\) the corresponding \(8 \times 8\) parameter matrix linking the systematic risk factor to transition probabilities i.e., \(\beta\) has elements \(\{\beta_{ls}\}, l = 1, \ldots, 8, s = 1, \ldots, 8\). The specific parameter matrices for \(\mu\) and \(\beta\) used in this application are given in Tables 2 and 3, respectively. Conditional on that the idiosyncratic innovations \(\{v_{ls}\}, l = 1, \ldots, 8, s = 1, \ldots, 8\) are zero the choice of parameter matrices for \(\mu\) and \(\beta\) implies that negative values of \(Z\) increase the likelihood of transitions to states representing lower credit quality. Table 4 displays the transition matrix conditional on \(Z = 0\) and \(\{v_{ls}\} = 0, l = 1, \ldots, 8, s = 1, \ldots, 8\). We denote this matrix by \(A(Z = 0)\) and it has elements \(A_{ls}(Z = 0), l = 1, \ldots, 9, s = 1, \ldots, 9\). This transition matrix is the starting point of the analysis since we have set \(Z(0) = 0\). Note that the matrix in Table 4 can be seen as an example of an empirical transition matrix and from that perspective the transition matrix may very well lack some of the monotonicity properties you expect from a theoretical transition matrix. Also as the matrix is simply supplied for illustration question concerning its volatility is of less importance.
Table 2
The \( \mu \) parameter matrix for the multinomial logit model

<table>
<thead>
<tr>
<th>( \mu_1 )</th>
<th>( \mu_2 )</th>
<th>( \mu_3 )</th>
<th>( \mu_4 )</th>
<th>( \mu_5 )</th>
<th>( \mu_6 )</th>
<th>( \mu_7 )</th>
<th>( \mu_8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.915</td>
<td>7.390</td>
<td>5.892</td>
<td>3.119</td>
<td>2.879</td>
<td>2.561</td>
<td>2.094</td>
<td>1.182</td>
</tr>
<tr>
<td>4.004</td>
<td>4.893</td>
<td>5.925</td>
<td>4.893</td>
<td>3.400</td>
<td>2.601</td>
<td>1.787</td>
<td>1.072</td>
</tr>
<tr>
<td>2.388</td>
<td>2.702</td>
<td>3.065</td>
<td>3.910</td>
<td>3.066</td>
<td>2.702</td>
<td>1.712</td>
<td>0.768</td>
</tr>
<tr>
<td>9.666</td>
<td>1.256</td>
<td>2.405</td>
<td>1.662</td>
<td>2.729</td>
<td>1.754</td>
<td>0.609</td>
<td>0.431</td>
</tr>
<tr>
<td>-0.666</td>
<td>-0.157</td>
<td>0.053</td>
<td>0.435</td>
<td>0.898</td>
<td>1.879</td>
<td>1.054</td>
<td>0.288</td>
</tr>
<tr>
<td>-2.357</td>
<td>-1.263</td>
<td>-1.006</td>
<td>-0.760</td>
<td>-0.292</td>
<td>0.637</td>
<td>0.839</td>
<td>0.656</td>
</tr>
<tr>
<td>-19.539</td>
<td>-4.544</td>
<td>-0.638</td>
<td>-19.556</td>
<td>-1.736</td>
<td>-0.656</td>
<td>-0.115</td>
<td>0.428</td>
</tr>
</tbody>
</table>

Table 3
The \( \beta \) parameter matrix for the multinomial logit model

<table>
<thead>
<tr>
<th>( \beta_1 )</th>
<th>( \beta_2 )</th>
<th>( \beta_3 )</th>
<th>( \beta_4 )</th>
<th>( \beta_5 )</th>
<th>( \beta_6 )</th>
<th>( \beta_7 )</th>
<th>( \beta_8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.020</td>
<td>-0.020</td>
<td>-0.020</td>
<td>-0.020</td>
<td>-0.020</td>
<td>-0.020</td>
<td>-0.020</td>
<td>-0.020</td>
</tr>
<tr>
<td>0.020</td>
<td>0.020</td>
<td>0.020</td>
<td>-0.020</td>
<td>-0.020</td>
<td>-0.020</td>
<td>-0.020</td>
<td>-0.020</td>
</tr>
<tr>
<td>0.040</td>
<td>0.040</td>
<td>0.040</td>
<td>-0.020</td>
<td>-0.020</td>
<td>-0.020</td>
<td>-0.020</td>
<td>-0.020</td>
</tr>
<tr>
<td>0.060</td>
<td>0.060</td>
<td>0.060</td>
<td>0.060</td>
<td>-0.020</td>
<td>-0.020</td>
<td>-0.020</td>
<td>-0.020</td>
</tr>
<tr>
<td>0.060</td>
<td>0.060</td>
<td>0.060</td>
<td>0.060</td>
<td>0.040</td>
<td>-0.020</td>
<td>-0.020</td>
<td>-0.020</td>
</tr>
<tr>
<td>0.080</td>
<td>0.080</td>
<td>0.080</td>
<td>0.080</td>
<td>0.080</td>
<td>0.10</td>
<td>-0.020</td>
<td>-0.020</td>
</tr>
<tr>
<td>0.100</td>
<td>0.100</td>
<td>0.100</td>
<td>0.100</td>
<td>0.100</td>
<td>0.100</td>
<td>0.100</td>
<td>0.100</td>
</tr>
<tr>
<td>0.100</td>
<td>0.100</td>
<td>0.100</td>
<td>0.100</td>
<td>0.100</td>
<td>0.100</td>
<td>0.120</td>
<td>0.140</td>
</tr>
</tbody>
</table>

Table 4
Transition matrix conditional on \( Z = 0, A(Z = 0) \)

<table>
<thead>
<tr>
<th>( A_1 )</th>
<th>( A_2 )</th>
<th>( A_3 )</th>
<th>( A_4 )</th>
<th>( A_5 )</th>
<th>( A_6 )</th>
<th>( A_7 )</th>
<th>( A_8 )</th>
<th>( A_9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8180</td>
<td>0.0685</td>
<td>0.0485</td>
<td>0.0241</td>
<td>0.0141</td>
<td>0.0115</td>
<td>0.0085</td>
<td>0.0062</td>
<td>0.0001</td>
</tr>
<tr>
<td>0.1530</td>
<td>0.6697</td>
<td>0.1497</td>
<td>0.0093</td>
<td>0.0073</td>
<td>0.0053</td>
<td>0.0033</td>
<td>0.0013</td>
<td>0.0004</td>
</tr>
<tr>
<td>0.0731</td>
<td>0.1779</td>
<td>0.4996</td>
<td>0.1779</td>
<td>0.0399</td>
<td>0.0179</td>
<td>0.0079</td>
<td>0.0038</td>
<td>0.0013</td>
</tr>
<tr>
<td>0.0766</td>
<td>0.1048</td>
<td>0.1507</td>
<td>0.3508</td>
<td>0.1508</td>
<td>0.1048</td>
<td>0.0389</td>
<td>0.015</td>
<td>0.0070</td>
</tr>
<tr>
<td>0.0547</td>
<td>0.0732</td>
<td>0.2309</td>
<td>0.1098</td>
<td>0.3192</td>
<td>0.1205</td>
<td>0.0383</td>
<td>0.0320</td>
<td>0.0208</td>
</tr>
<tr>
<td>0.0282</td>
<td>0.0469</td>
<td>0.0579</td>
<td>0.0850</td>
<td>0.1350</td>
<td>0.3604</td>
<td>0.1578</td>
<td>0.0734</td>
<td>0.0550</td>
</tr>
<tr>
<td>0.01041</td>
<td>0.0311</td>
<td>0.0402</td>
<td>0.0514</td>
<td>0.0821</td>
<td>0.2080</td>
<td>0.2545</td>
<td>0.2120</td>
<td>0.1100</td>
</tr>
<tr>
<td>7.0e-010</td>
<td>0.0022</td>
<td>0.1133</td>
<td>6.8e-010</td>
<td>0.0377</td>
<td>0.1113</td>
<td>0.1912</td>
<td>0.3294</td>
<td>0.2146</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

To continue we let \( A(Z = 3) \) and \( A(Z = -3) \) denote the transition matrices conditional on \( Z = 3, \{v_{ls}\} = 0 \) and \( Z = -3, \{v_{ls}\} = 0 \), respectively, for \( l = 1, \ldots, 8, s = 1, \ldots, 8 \). Tables 5 and 6 display the element by element matrix ratios \( \{A_{ls}(Z = 0)\}/\{A_{ls}(Z = 3)\} \), \( \{A_{ls}(Z = 0)\}/\{A_{ls}(Z = -3)\} \) for \( l = 1, \ldots, 8, s = 1, \ldots, 9 \). Note that the scaling factors in Table 5 display, as expected, a transfer of transition probability mass of the matrix \( A(Z = 0) \) to the lower triangular. Inspecting Table 6 we see that the recession transition matrix, \( A(Z = -3) \), is obtained from the transition matrix \( A(Z = 0) \) through transfer of transition probability mass to the upper triangular part. In our application we have furthermore simplified the structure of the multinomial logit model somewhat assuming that
the non-systematic innovations \( \{v_{ls}\}, l = 1, \ldots, 8, s = 1, \ldots, 8 \) are all zero. Of course, this will not be the case in any application to real data.

4.2. Models for the evolution of collateral values

Considering the evolution of collateral values, the collateral index models \( S_1 \) and \( S_2 \) are modelled using discretized geometric brownian motions

\[
S_1(t + \Delta t) - S_1(t) = \mu_1 S_1(t) \Delta t + \sigma_{S_1} S_1(t) (W_{S_1}(t + \Delta t) - W_{S_1}(t)),
\]

\[
S_2(t + \Delta t) - S_2(t) = \mu_2 S_2(t) \Delta t + \sigma_{S_2} S_2(t) (W_{S_2}(t + \Delta t) - W_{S_2}(t)).
\]

Here \( W_{S_1}(t + \Delta t) - W_{S_1}(t) \) and \( W_{S_2}(t + \Delta t) - W_{S_2}(t) \) represent the increments of two brownian motions. We set the parameters in the model as follows: \( \mu_1 = \mu_2 = 0 \) and \( \sigma_{S_1} = 0.2, \sigma_{S_2} = 0.4 \). The mean annual growth rate for collateral value is hence set to zero in both equations and they differ only with respect to the annual volatility, being 20% for \( S_1 \) and 40% for \( S_2 \). The brownian increments are furthermore assumed to be positively correlated with a correlation of 0.5. Note that by definition (see Section 3.2.2) our model for the evolution of collateral values is multiplicative and the scaling factors are non-negative real numbers. For price indices of property the geometric brownian motions are therefore a natural candidates for a simple model. Other choices are of course feasible but we see no point in adding complexity in terms of processes in this application.

As the final step of the model specification and in order to capture potential covariation between migration to states representing lower credit quality and decreasing values of

---

### Table 5
Element by element transition matrix ratios, \( M = A(Z = 0)/A(Z = 3) \)

<table>
<thead>
<tr>
<th></th>
<th>( M_1 )</th>
<th>( M_2 )</th>
<th>( M_3 )</th>
<th>( M_4 )</th>
<th>( M_5 )</th>
<th>( M_6 )</th>
<th>( M_7 )</th>
<th>( M_8 )</th>
<th>( M_9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_{1} )</td>
<td>0.979</td>
<td>1.104</td>
<td>1.104</td>
<td>1.104</td>
<td>1.104</td>
<td>1.104</td>
<td>1.104</td>
<td>1.039</td>
<td></td>
</tr>
<tr>
<td>( M_{2} )</td>
<td>0.980</td>
<td>0.980</td>
<td>1.104</td>
<td>1.104</td>
<td>1.104</td>
<td>1.104</td>
<td>1.104</td>
<td>1.040</td>
<td></td>
</tr>
<tr>
<td>( M_{3} )</td>
<td>0.959</td>
<td>0.959</td>
<td>0.959</td>
<td>1.148</td>
<td>1.148</td>
<td>1.148</td>
<td>1.148</td>
<td>1.081</td>
<td></td>
</tr>
<tr>
<td>( M_{4} )</td>
<td>0.932</td>
<td>0.932</td>
<td>0.932</td>
<td>0.932</td>
<td>1.185</td>
<td>1.185</td>
<td>1.185</td>
<td>1.116</td>
<td></td>
</tr>
<tr>
<td>( M_{5} )</td>
<td>0.937</td>
<td>0.937</td>
<td>0.937</td>
<td>0.995</td>
<td>1.191</td>
<td>1.191</td>
<td>1.191</td>
<td>1.122</td>
<td></td>
</tr>
<tr>
<td>( M_{6} )</td>
<td>0.950</td>
<td>0.950</td>
<td>0.950</td>
<td>0.950</td>
<td>0.895</td>
<td>1.283</td>
<td>1.283</td>
<td>1.208</td>
<td></td>
</tr>
<tr>
<td>( M_{7} )</td>
<td>0.907</td>
<td>0.907</td>
<td>0.907</td>
<td>0.907</td>
<td>0.907</td>
<td>1.023</td>
<td>1.086</td>
<td>1.225</td>
<td></td>
</tr>
<tr>
<td>( M_{8} )</td>
<td>0.998</td>
<td>0.998</td>
<td>0.998</td>
<td>0.998</td>
<td>0.998</td>
<td>0.940</td>
<td>0.885</td>
<td>1.347</td>
<td></td>
</tr>
</tbody>
</table>

### Table 6
Element by element transition matrix ratios, \( M = A(Z = 0)/A(Z = -3) \)

<table>
<thead>
<tr>
<th></th>
<th>( M_1 )</th>
<th>( M_2 )</th>
<th>( M_3 )</th>
<th>( M_4 )</th>
<th>( M_5 )</th>
<th>( M_6 )</th>
<th>( M_7 )</th>
<th>( M_8 )</th>
<th>( M_9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_{1} )</td>
<td>1.023</td>
<td>0.907</td>
<td>0.907</td>
<td>0.907</td>
<td>0.907</td>
<td>0.907</td>
<td>0.907</td>
<td>0.963</td>
<td></td>
</tr>
<tr>
<td>( M_{2} )</td>
<td>1.022</td>
<td>1.022</td>
<td>0.906</td>
<td>0.906</td>
<td>0.906</td>
<td>0.906</td>
<td>0.906</td>
<td>0.962</td>
<td></td>
</tr>
<tr>
<td>( M_{3} )</td>
<td>1.049</td>
<td>1.049</td>
<td>1.049</td>
<td>0.876</td>
<td>0.876</td>
<td>0.876</td>
<td>0.876</td>
<td>0.930</td>
<td></td>
</tr>
<tr>
<td>( M_{4} )</td>
<td>1.085</td>
<td>1.085</td>
<td>1.085</td>
<td>0.853</td>
<td>0.853</td>
<td>0.853</td>
<td>0.853</td>
<td>0.906</td>
<td></td>
</tr>
<tr>
<td>( M_{5} )</td>
<td>1.075</td>
<td>1.075</td>
<td>1.075</td>
<td>1.013</td>
<td>0.846</td>
<td>0.846</td>
<td>0.846</td>
<td>0.898</td>
<td></td>
</tr>
<tr>
<td>( M_{6} )</td>
<td>1.074</td>
<td>1.074</td>
<td>1.074</td>
<td>1.074</td>
<td>0.796</td>
<td>0.796</td>
<td>0.796</td>
<td>0.845</td>
<td></td>
</tr>
<tr>
<td>( M_{7} )</td>
<td>1.112</td>
<td>1.112</td>
<td>1.112</td>
<td>1.112</td>
<td>1.112</td>
<td>0.986</td>
<td>0.929</td>
<td>0.824</td>
<td></td>
</tr>
<tr>
<td>( M_{8} )</td>
<td>1.026</td>
<td>1.026</td>
<td>1.026</td>
<td>1.026</td>
<td>1.026</td>
<td>1.090</td>
<td>1.157</td>
<td>0.760</td>
<td></td>
</tr>
</tbody>
</table>
collaterals we assume that the increments of $W_Z, W_{S_1}$, and $W_{S_2}$ to be described by an equi-correlation matrix with correlation 0.5.

Through the previous steps we have described all of the pieces in the portfolio model as demanded by 1–4. For any time horizon $n\Delta t$ we can now simulate paths of transition (and default) frequencies and collateral values and analyze this information using appropriate measures of risk as described in Section 2.1. We have made the choice to present the results for the horizons of 1 year, 5 years and 10 years (i.e., for $n = 1, 5$ and 10). Note however that path dependence is a feature of the present model and that in order to obtain results at a horizon of e.g., 10 years we need to compute the complete path.

4.3. The credit loss process

In Table 7 we consider the credit loss process generated by our model and applied to the portfolios constructed. Since the level of marginals are not influencing the credit loss process we only have to present the calculation for a subset of our portfolios. Therefore the reader only finds the results for the portfolios ($\pi_1, \pi_2, \pi_3, \pi_4$) and ($\pi_9, \pi_{10}, \pi_{11,} \pi_{12}$). The table consists of ten columns. Columns two to four give the expected losses as percentage of the original notional on the portfolios and is by necessity a non-decreasing function of the time horizon $n\Delta t$ ($n = 1, 5$ and 10). Considering for instance portfolio $\pi_1$ we see that that during the first year the expected losses are essentially zero but expected to accumulate to $0.0062 \times 10$ BnSEK in 10 years. Similarly columns five through seven give the value at risk of the losses at the 99.9% confidence level as percentage of the original notional on the portfolios. The construction of these numbers are as follows. For each time horizon $n\Delta t$ ($n = 1, 5$ and 10) we construct the accumulated loss distribution and calculate the value at risk by calculating the appropriate quantiles. Again the numbers calculated are by necessity a non-decreasing function of the time horizon $n\Delta t$. Finally the last three columns give the conditional value at risk of the losses at the 99.9% confidence level as percentage of the original notional on the portfolios. The understanding of the numbers constructed follows from the same discussion as in the case of value at risk. We also note that, by construction, the numbers for the conditional value at risk are always larger than the numbers for the value at risk at the same confidence level.

Table 7
The credit loss process

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Cumulative expected loss %</th>
<th>Cumulative loss VaR (99.9) %</th>
<th>Cumulative loss CVaR (99.9) %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 year</td>
<td>5 years</td>
<td>10 years</td>
</tr>
<tr>
<td>$\pi_1$</td>
<td>0.00001</td>
<td>0.10026</td>
<td>0.62558</td>
</tr>
<tr>
<td>$\pi_2$</td>
<td>0.00094</td>
<td>0.41091</td>
<td>1.61792</td>
</tr>
<tr>
<td>$\pi_3$</td>
<td>0.00048</td>
<td>0.45114</td>
<td>1.84846</td>
</tr>
<tr>
<td>$\pi_4$</td>
<td>0.00198</td>
<td>0.72770</td>
<td>2.57942</td>
</tr>
<tr>
<td>$\pi_9$</td>
<td>0.00122</td>
<td>0.27940</td>
<td>1.00697</td>
</tr>
<tr>
<td>$\pi_{10}$</td>
<td>0.155</td>
<td>1.47093</td>
<td>3.09288</td>
</tr>
<tr>
<td>$\pi_{11}$</td>
<td>0.07853</td>
<td>1.50092</td>
<td>3.37282</td>
</tr>
<tr>
<td>$\pi_{12}$</td>
<td>0.32593</td>
<td>2.72464</td>
<td>5.22319</td>
</tr>
</tbody>
</table>

Expected losses and risk, defined using value at risk and conditional value at risk, expressed as percentage of the original exposure.
To analyze the numbers and to make comparisons between the portfolios we note that the numbers produced are consistent with the following intuition. The expected losses and the risk, measured using either value at risk or conditional value at risk are increasing functions of loan to value and of the volatility in the model for collateral values. This can be examplified by first comparing the result for $\tilde{\pi}_1$ and $\tilde{\pi}_2$ as these two portfolios only differ with respect to loan to value and secondly by comparing the result for $\tilde{\pi}_1$ and $\tilde{\pi}_3$ as these two portfolios only differ with respect to the model for the evolution of collateral. However if we compare portfolio $\tilde{\pi}_3$ and portfolio $\tilde{\pi}_4$, which both are modelled by the more volatile collateral index model $S_2$ and which differ only with respect to loan to value, then there is no actual difference in risk as measured by value at risk and conditional value at risk at the 99.9% confidence level. The effects of comparably lower loan to value is hence essentially wiped out by the volatile collateral values.

Focusing on the portfolios $(\tilde{\pi}_9, \tilde{\pi}_{10}, \tilde{\pi}_{11}, \tilde{\pi}_{12})$ which all have an initial rating of 5, to be compared to the portfolios $(\tilde{\pi}_1, \tilde{\pi}_2, \tilde{\pi}_3, \tilde{\pi}_4)$ which all have initial rating 1, we see that the expected loss and risk is, due to the lower credit quality, higher than for the otherwise comparable portfolios $(\tilde{\pi}_1, \tilde{\pi}_2, \tilde{\pi}_3, \tilde{\pi}_4)$. Furthermore the expected loss and risk when going from low to high collateral volatility i.e., from collateral index model $S_1$ to $S_2$ increase relatively more for the portfolios $(\tilde{\pi}_9, \tilde{\pi}_{10}, \tilde{\pi}_{11}, \tilde{\pi}_{12})$ than for the portfolios $(\tilde{\pi}_1, \tilde{\pi}_2, \tilde{\pi}_3, \tilde{\pi}_4)$. Therefore portfolios with lower credit quality in terms of initial rating seem to be more sensitive to collateral volatility. The explanation for this is the choice we made for a positive correlation between all of the variables driving the underlying model. Still for the portfolios $(\tilde{\pi}_9, \tilde{\pi}_{10}, \tilde{\pi}_{11}, \tilde{\pi}_{12})$ we observe, as we did for the set of portfolios $(\tilde{\pi}_1, \tilde{\pi}_2, \tilde{\pi}_3, \tilde{\pi}_4)$, that a high/low loan to value does seem to have less impact on risk, as measured by value at risk or conditional value at risk at the 99.9% confidence level, if the collateral value volatility is high as manifested by the collateral index model $S_2$.

4.4. The return process

In the previous discussion we considered the credit loss process, hence ignoring the level of marginals. In Table 8 we also account for the income generated from all the credits in the form of marginals. That is, in this case the stochastic variable we consider is $I(0, n\Delta t, \pi_j) - L(0, n\Delta t, \pi_j)$ where the notation $I(0, n\Delta t, \pi_j)$ and $L(0, n\Delta t, \pi_j)$ was introduced at the end of Section 3.2.4. Therefore Table 8 is referred to as risk measures applied to the return process. Again the table consists of ten columns and columns two to four give the expected returns. Columns five to seven give the value at risk at the 99.9% confidence level and the last three columns give the conditional value at risk at the 99.9% confidence level. Note that the only difference between the set $(\tilde{\pi}_1, \tilde{\pi}_2, \tilde{\pi}_3, \tilde{\pi}_4)$ of portfolios and the set $(\tilde{\pi}_5, \tilde{\pi}_6, \tilde{\pi}_7, \tilde{\pi}_8)$ of portfolios is that in case of the latter set the bank only earns a marginal of 10 bp to be compared with the marginal of 50 bp earned on the first set. Inspecting Table 8 for these portfolios we see considerable differences in both the expected return and in the risk of the return. In particular for the set $(\tilde{\pi}_6, \tilde{\pi}_7, \tilde{\pi}_8)$ the expected return is negative where as it is not for any portfolio in the comparable set $(\tilde{\pi}_2, \tilde{\pi}_3, \tilde{\pi}_4)$. The buffer provided by the larger marginals for the set of portfolios $(\tilde{\pi}_1, \tilde{\pi}_2, \tilde{\pi}_3, \tilde{\pi}_4)$ is therefore important for both the expected return and the risk in return. The same effects are also seen when comparing the set of portfolios $(\tilde{\pi}_9, \tilde{\pi}_{10}, \tilde{\pi}_{11}, \tilde{\pi}_{12})$ with the set of portfolios $(\tilde{\pi}_{13}, \tilde{\pi}_{14}, \tilde{\pi}_{15}, \tilde{\pi}_{16})$. From the results in Tables 7 and 8 we can conclude that a comparably low loan to value may or may not be a buffer for risk. Wether it is depends on the volatility.
of the collateral. Another natural conclusion is that comparably high marginals provide a significant buffer.

4.5. Economic capital

We now consider the calculation of economic capital (EC) for \( \bar{\pi} = \bar{\pi}_1 + \cdots + \bar{\pi}_{16} \) as well as the calculation of the contributions to EC. Recall that by Definition 1

\[
EC(0, n\Delta t, \bar{\pi}) = -\rho \left( \sum_{k=0}^{n-1} \min \{ I(k\Delta t, (k+1)\Delta t, \bar{\pi}) - L(k\Delta t, (k+1)\Delta t, \bar{\pi}), 0 \} \right),
\]

where \( \rho \) is a measure of risk and \( I(k\Delta t, (k+1)\Delta t, \bar{\pi}) = I(0, (k+1)\Delta t, \bar{\pi}) - I(0, k\Delta t, \bar{\pi}) \), \( L(k\Delta t, (k+1)\Delta t, \bar{\pi}) = L(0, (k+1)\Delta t, \bar{\pi}) - L(0, k\Delta t, \bar{\pi}) \). Furthermore \( \{EC(0, j\Delta t, \pi)\}_{j \geq 0} \) gives the economic capital needed to support the portfolio on arbitrary time-horizons. In Table 9 we have computed, based on this definition, the economic capital required at a horizon of 10 years. The economic capital is computed using the risk measures value at risk and conditional value at risk at the 99.9% confidence level. Hence we have in the application, from the term-structure \( \{EC(0, j\Delta t, \pi)\}_{j \geq 0} \), chosen the single horizon of 10 years as the basis for economic capital. The rational for the introduction of the term-structures in Section 2.2 is that the term-structure is non-decreasing. As a practical point we may therefore want to avoid situations of a steeply rising term-structure as this would potentially require the acquisition of capital along the path. Compared to the more common horizon of one-year we have, by the choice of a 10-year horizon, achieved further stability in the capital base.

From Table 9 we can conclude that using conditional value at risk at the 99.9% confidence level as our measure of risk, the economic capital for the total portfolio is 9.726% of
160 BnSEK. This yields a capital reserve of approximately 15.552 BnSEK. Using value at risk at the same confidence level results in a smaller capital buffer.

In the same table the reader can also find the total of economic capital allocated to the portfolios \( \tilde{\pi}_j \), \( j = 1, \ldots, 16 \) in accordance with the discussion in Section 2. In the simulation based case an interesting question is how to compute the risk contributions. In the case of value at risk, given a set of scenarios and a confidence level, there will be a scenario which realize the VaR-number. Following Mausser and Rosen (1998) we take that scenario as our estimator of the derivative of VaR. In the case of conditional value at risk we estimate the contributions by averaging over the estimated VaR derivatives. The allocations of the required capital reserve rank the portfolios \( \tilde{\pi}_j \) in the order we expect. The allocation has therefore succeeded in attributing a higher capital reserve for ceteris paribus less creditworthy portfolios in terms of rating, portfolios with more volatile collateral, portfolios with less marginals and in case of comparably low collateral volatility high loan to value portfolios. In general conditional value at risk, seen as an average of value at risk numbers, enjoys greater stability than value at risk. There are also theoretical reasons to prefer the use of conditional value at risk, see Artzner et al. (1999).

### 4.6. Risk adjusted return and risk adjusted return on capital

Above we calculated economic capital for the total portfolio \( \tilde{\pi} \) and considered its allocation to the subportfolios \( \tilde{\pi}_j \) in accordance with the discussion in Section 2. In the following discussion we let economic capital be defined using conditional value at risk as the risk measure. In Table 10 we have computed the 10-year horizon average risk adjusted return for each of the portfolios \( \tilde{\pi}_1, \ldots, \tilde{\pi}_{16} \) assuming a demanded return on capital of 10%. For portfolio \( \tilde{\pi}_j \) this is calculated as

#### Table 9
Economic capital for the total portfolio expressed as percentage of the original exposure

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Economic capital CVaR(99.9) (%)</th>
<th>Economic capital VaR(99.9) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{\pi} )</td>
<td>9.726</td>
<td>9.162</td>
</tr>
<tr>
<td>( \tilde{\pi}_1 )</td>
<td>2.008</td>
<td>2.264</td>
</tr>
<tr>
<td>( \tilde{\pi}_2 )</td>
<td>2.310</td>
<td>2.523</td>
</tr>
<tr>
<td>( \tilde{\pi}_3 )</td>
<td>2.518</td>
<td>2.271</td>
</tr>
<tr>
<td>( \tilde{\pi}_4 )</td>
<td>2.575</td>
<td>2.526</td>
</tr>
<tr>
<td>( \tilde{\pi}_5 )</td>
<td>4.445</td>
<td>4.696</td>
</tr>
<tr>
<td>( \tilde{\pi}_6 )</td>
<td>4.739</td>
<td>4.955</td>
</tr>
<tr>
<td>( \tilde{\pi}_7 )</td>
<td>4.955</td>
<td>4.703</td>
</tr>
<tr>
<td>( \tilde{\pi}_8 )</td>
<td>5.013</td>
<td>4.959</td>
</tr>
<tr>
<td>( \tilde{\pi}_9 )</td>
<td>6.009</td>
<td>7.614</td>
</tr>
<tr>
<td>( \tilde{\pi}_{10} )</td>
<td>7.418</td>
<td>8.190</td>
</tr>
<tr>
<td>( \tilde{\pi}_{11} )</td>
<td>8.561</td>
<td>7.031</td>
</tr>
<tr>
<td>( \tilde{\pi}_{12} )</td>
<td>8.913</td>
<td>7.898</td>
</tr>
<tr>
<td>( \tilde{\pi}_{13} )</td>
<td>8.420</td>
<td>10.023</td>
</tr>
<tr>
<td>( \tilde{\pi}_{14} )</td>
<td>9.829</td>
<td>10.599</td>
</tr>
<tr>
<td>( \tilde{\pi}_{15} )</td>
<td>10.973</td>
<td>9.440</td>
</tr>
<tr>
<td>( \tilde{\pi}_{16} )</td>
<td>11.324</td>
<td>10.307</td>
</tr>
</tbody>
</table>

The capital allocation is expressed as percentage of the total amount of economic capital. Hence the allocations by necessity sum to 100%.
where \( n\Delta t = 10 \) and where we are using the notation of Section 2. Dividing this number with the economic capital allocated to portfolio \( \tilde{\pi}_j \) we get an average risk adjusted return on capital. This type of performance measure is also commonly abbreviated RAROC. Hence in Table 10, column two gives the average risk adjusted return expressed as percentage of the original notional while column three gives the average risk adjusted return on capital expressed as percentage of the capital allocated. The risk adjusted return on capital is presented only in case it is positive. It is interesting to compare the results of Table 10 with the results in Table 8 on the cumulative expected return. In particular using column 4 in Table 8 we can compute the 10-year horizon average expected return for the total portfolio \( \tilde{\pi} \) as approximately 0.534% and from Table 10, using column 2, we can compute the 10-year horizon average risk adjusted return for the total portfolio \( \tilde{\pi} \), i.e., taking the cost (or demanded return) into account, to be approximately \(-0.579\%\). Hence taking the cost of holding capital into account the total portfolio \( \tilde{\pi} \) is no longer performing on average. In fact the average risk adjusted return and average risk adjusted return on capital is seen to be negative for most of the portfolios \( \tilde{\pi}_j, j = 1, \ldots, 16 \).

5. Summary

In this paper we have proposed a portfolio credit risk model which, though general, is in particular applicable to large dimensional portfolios like for example large retail and mortgage portfolios. Key features of the model are as follows. Firstly, the model is dynamic in the sense that defaults are based on default times \( \{\tau_j\} \) instead of being restricted to a static time horizon. Secondly, our focus on dynamic models for the value of collaterals allow us to adequately assess the size of loss at the time of default. Thirdly, the cash flow structures of the credits, i.e., exposures, amortisations are properly.
accounted for in the credit loss process. Finally, the model proposed is feasible from the point of view of computing. In particular a key idea is the concept of perfectly homogeneous portfolio and to create a tree structure using which the original portfolio is mapped to a set of artificial perfectly homogeneous subportfolios in a structured manner.

As a first application of our model we have defined economic capital (EC) and addressed the questions of allocation of EC and the pricing of credit products based on EC. Our concept of EC is inspired by classical risk theory and the concept of risk reserve process and accounts for both the cash flow structure of the credits, i.e., the exposures, amortisations and marginals as well as losses and can be applied to any time horizon. The inclusion of revenues allows us to avoid a potential overestimation of the capital needed.

In the final section of the paper the model proposed and its application to EC are illustrated by way of an example. One key focus of future papers will be to produce case studies based on the calibration to real data and to apply the model to real world portfolios.

References