INVESTMENT INCENTIVES, DEBT, AND WARRANTS*

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This paper models and characterizes investment incentive problems associated with debt financing. The decision problem of residual claimants is explicitly formulated and their investment policies are characterized. The paper also analyzes the use of conversion features and warrants to control distortionary incentives. These claims reverse the convex shape of levered equity over the upper range of the firm's earnings, and this mitigates the incentive to take risk. It is shown that, under certain conditions, such claims can be constructed to restore net present value maximizing incentives and simultaneously meet the financing requirements of the firm.

1. Introduction

A number of authors have recognized that maximizing the value of the equity claim and maximizing the value of the firm can, with risky debt outstanding, lead to different investment policies. Fama and Miller (1972), Stiglitz (1972), and Myers (1977) all provide interesting examples of such situations. More recently, the analogy between levered equity and a call option, first noted by Black and Scholes (1973), helped to focus attention on 'risk' as a source of these adverse incentives. The Black-Scholes model gives the value of an option as an increasing function of the instantaneous variance of the underlying asset. Hence, it was argued, shareholders have an incentive to substitute 'risky' for 'less risky' operating and investment policies, even when this may lower the net present value (NPV) of the firm as a whole.

Such appeals to option pricing results rely on models which price claims by taking as exogenous the value of the underlying asset. Yet the same appeals use comparative statics from these models to determine endogenously the value of the firm and argue for its dependence on the structure of the claims which are issued. This circularity leads Smith and Warner (1979, p. 118) to conclude that 'the implications drawn from the option pricing model are only suggestive'.

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Jensen and Meckling (1976, p. 336), in their discussion of stockholder–bondholder conflicts, also caution against careless application of the Black–Scholes model.

The approach taken here avoids these problems by explicitly modeling the decision problem faced by ‘insiders’ who wish to maximize the value of their residual claim while financing operations with debt. The use of value maximization as the decisionmakers’ objective ignores distinctions between equity holders and management, and abstracts from the important issues of risk sharing which motivate much the literature on bilateral principal–agent problems. It does allow us, however, to focus directly on the incentive effects of the limited liability of corporate equity. In the absence of monitoring problems, a ‘first-best’ solution leads to $NPV$ maximizing investment choices. This solution is then compared to investment policies which prevail when insiders cannot guarantee their choices at the time of financing. A ‘second-best’ problem is formulated which, as in the standard principal–agent setting, involves an additional constraint on stockholders’ actions. The resulting allocation is shown to involve overinvestment in ‘risky’ projects.

One benefit of the model is that it provides a means of addressing formally the suggestions of Jensen and Meckling (1976) and Smith and Warner (1979), that option claims issued with debt may be rationalized as attempts to mitigate distortionary incentives. Although our single-period discrete time framework limits the generality of the analysis, it also generates insights into how the structure of these claims works to control incentives to take risk. In a continuous time environment, a warrant can be duplicated as a combination of riskless debt and equity. This would suggest that a bond–warrant combination would lead to the same incentive effects as debt and outside equity. In discrete time, however, warrants issued with debt have very different incentive consequences. The continuous time spanning portfolio must be revised at every instant to eliminate the put component in the put–call parity relation. In discrete time, the put component added to the firm’s capital structure turns out to be a key factor in the ability of warrants to control risk incentives. Within the very limited context of our model, in fact, these claims can be structured to neutralize adverse risk incentives.

A number of other recent papers have addressed issues similar to those studied here. Stiglitz and Weiss (1981) and Dothan and Williams (1981) describe circumstances which lead equity holders to shift to projects with higher risk in risk neutral and continuous time environments, respectively. Yasuhara (1981) and Mikkelsen (1980) show that conversion privileges help to restore $NPV$ maximizing incentives in a two-state setting. This paper extends these results in several ways. First, we provide a more precise and general definition of the relative risk of projects. Second, by using continuous production functions rather than mutually exclusive alternatives we can employ first-order conditions to more fully characterize investment incentives associ-
ated with different contracts. Most importantly, we address the financing and incentive problems simultaneously, showing that the correct incentives can be induced by a convertible bond or debt–warrant combination which will indeed meet the firm’s financing requirements. The resulting contract involves debt with a risky promised payment, so we have not simply introduced riskless debt and outside equity in disguise.

Section 2 describes the investment opportunities and the asset pricing regime under which the firm operates. In Section 3 the firm’s investment and financing behaviors are characterized with and without moral hazard. Section 4 describes the effects of warrants and convertibles on the structure of the residual claim, and the incentives this structure induces. A brief conclusion is provided in Section 5.

2. Technology and valuation

We consider a firm which consists of a group of entrepreneurs with monopoly access to two investment projects. Each of these generates rents as measured by net present value (NPV). The entrepreneurs wish to capture these rents in the form of a high market value for their residual claim. Monitoring costs, however, prevent them from guaranteeing the allocation of funds across the two projects at the time they issue debt in exchange for these funds. To focus on a limited set of issues, we study a single period case. Each project consists of a scale function with a random shock, and the firm’s end of period cash flow, $\tilde{X}$, aggregates the returns to these projects,

$$\tilde{X}(I_1, I_2) \equiv k(I_1)(1 + \tilde{R}_1) + k(I_2)(1 + \tilde{R}_2).$$

$I_1$ and $I_2$ represent the amounts invested in each project. Their relative magnitudes are assumed unobservable to outsiders. Let $k'(.)$ denote the derivative of the scale function. We assume:

A.1. $k(\cdot)$ is strictly concave with $k'(\cdot) > 0$, $k(0) = 0$,

$$\lim_{I \to \infty} k(I) = A < \infty, \quad \lim_{I \to 0} k'(I) = \infty.$$

A.2. $R_1$ and $R_2$ have bounded support on $(0, \overline{R})$.

Assumptions A.1 and A.2 will help to bound the value of the firm and its claims. The infinite marginal product and strict positivity of the shock also ensure that some investment is undertaken in each project, regardless of financial structure. Limited liability for the firm requires that $\tilde{X}(I_1, I_2) \geq 0$, for any $(I_1, I_2)$. This is also implied by A.1. and A.2. Thus, investment incentives
are not distorted by limited liability of the firm as a whole, and we can focus on the incentive effects of the debt instruments it employs.

In order to study the conflict of interest between stockholders and bondholders, it is important to be able to value the firm's cash flows unambiguously. Otherwise, even in the absence of monitoring problems, capital structure may not be a matter of indifference. The simplest way of avoiding these complications is to assume:

A.3. The capital markets are complete.

Let $V(\tilde{\gamma})$ represent the value of some random flow $\tilde{\gamma}$. Assumption A.3 and 'zero-arbitrage' assumptions ensure the existence of a unique vector of state-claim prices, $\{p(s)\}$, and a positive random variable $\tilde{W}$, which satisfy

$$V(\tilde{\gamma}) = E(\tilde{\gamma}\tilde{W}) = \int_S p(s)\gamma(s)\,ds.$$ (2)

Here $S$ is the universal set of 'states of the world', and $s$ is a realization from it. $\tilde{W}$ can be thought of as the source of 'systematic' risk in the economy. It is also the payoff on a portfolio with a bounded second moment.\(^1\)

Let $(1 + r_f)$ be the gross return per dollar on the riskless asset. We can normalize the state-prices in (2) so that $p(s)(1 + r_f) = p(s)/\int_S p(s)\,ds$ integrates to one and so provides a probability measure. Let $E(.)$ be the expectation operator over this measure. We will refer to it as a certainty-equivalence operator since $V(\tilde{\gamma})(1 + r_f) = E(\tilde{\gamma})$.

We will now put more structure on the relationship between $\tilde{R}_1$ and $\tilde{R}_2$, so that we can capture formally the notion that equity holders have incentives to take extra risk. What constitutes extra 'risk' depends on the asset pricing regime which prevails. Accordingly, the following seems a natural way of extending the Rothschild–Stiglitz (1970) notion of increasing risk to a setting where portfolio opportunities are important:

A.4. $1 + \tilde{R}_1 = 1 + \tilde{R}$ and $1 + \tilde{R}_2 = 1 + \tilde{R} + \tilde{z}$, where $E(\tilde{z}|R, W) = E(\tilde{z}) = 0$.

By conditioning on $\tilde{W}$ as well as $\tilde{R}$, we ensure that $\tilde{R}_2$ is a 'price-preserving' as

\(^1\)The existence of $\tilde{W}$ and the boundedness of its second moment is implied by the Riesz representation theorem applied to the Hilbert space generated by all portfolios of marketed assets and the inner product norm $(X, Y) = E(XY)$. For a rigorous discussion of these issues, see Harrison and Kreps (1979).
well as a ‘mean-preserving’ spread of $\tilde{R}_1$. To see this, note that

$$\int_S p(s) \tilde{z}(s) \, ds = E(\tilde{z}\tilde{W})$$

$$= E\left[ E(\tilde{z}\tilde{W}|W, R) \right]$$

$$= E\left[ \tilde{W}E(\tilde{z}|W, R) \right] = 0.$$  \hspace{1cm} (3)

Thus, we can loosely interpret $\tilde{z}$ as ‘non-systematic’ or ‘residual’ risk.\(^2\)

Finally, debt and equity claims are specified as $\max\{\ldots\}$ and $\min\{\ldots\}$ functions. These have points of non-differentiability where the arguments are equal. To characterize optimal policies with first-order conditions, we must differentiate the value of these claims, which are integrals over them. This can be done if the kinks in the functions are of zero measure or if:

A.5. $\tilde{R}$, $\tilde{z}$, and $\tilde{W}$ are continuously distributed with joint density $f(R, z, W)$.

3. Investment choice

We first consider the investment decisions the firm would undertake if contracts were costlessly enforceable. The policies which result maximize the net present value of the firm. With this as a standard of comparison, we then characterize the distortions that monitoring problems induce.

We will assume that the firm finances its operations with debt. Under costless enforcement, capital structure is a matter of indifference, so the assumption is innocuous. A more careful discussion is therefore postponed. Let $m$ represent the promised payment on the single-period debt issued by the firm, and let $I$ denote the vector $(I_1, I_2)$. We can write the problem faced by the holders of the residual equity claim as:

$$\max_{I_1, I_2, m} \int_S p(s) \max\{X(I, s) - m, 0\} \, ds,$$

subject to $I_1 + I_2 \leq E_0 + \int_S p(s) \min\{X(I, s), m\} \, ds$,

where $E_0$ is a positive initial amount of funds contributed by the firm's

\(^2\)Jagannathan (1982), working independently, has shown that a condition similar to A.4 is sufficient to ensure that a call option on one asset is more valuable than a call on an otherwise similar but less ‘risky’ asset. Contrary to Merton (1973), Rothschild–Stiglitz risk is insufficient.
This problem is algebraically equivalent to maximizing the NPV of the firm. The NPV is by definition \( \int_{s} p(s) X(I, s) \, ds - (I_1 + I_2) \). Substitution for \((I_1 + I_2)\) from the budget constraint in P.1 yields for the NPV the value of the equity less \( E_0 \), a constant which will not affect the choice of \( I \). Letting \( \hat{I} \) denote the optimal choice, the first-order conditions for NPV maximization are

\[
k'\left(\hat{I}_i\right) \int_{S} p(s)(1 + R_i(s)) \, ds = 1, \quad i = 1, 2.
\]

(4)

The firm invests until the marginal value product is equal to the cost of funds in current dollars.

Since \( \epsilon \) has a zero price over the entire state space, (4) implies equal investment in each project. To see this, note

\[
\int_{S} p(s)(1 + R_1(s)) \, ds = \int_{S} p(s)(1 + R(s)) \, ds + \int_{S} p(s) z(s) \, ds
\]

\[= \int_{S} p(s)(1 + R_2(s)) \, ds.\]

(5)

Denote as \( \hat{m} \) the solution to the budget constraint evaluated at \( \hat{I}_1 + \hat{I}_2 \). To motivate the problems discussed below, we will assume that \( \hat{I}_1 + \hat{I}_2 \) is sufficiently large and \( E_0 \) sufficiently small to make \( \hat{m} \) risky. Letting \( \bar{m}(I) = \inf_{s} X(I, s) \), we require:

A.6. \( \hat{I}_1 + \hat{I}_2 > E_0 + \int_{S} p(s) \bar{m}(\hat{I}) \, ds = E_0 + \bar{m}(\hat{I})/(1 + r_f) \).

Without A.6 we would have no problem to study. If the firm can be fully financed with riskless debt, none of the incentive problems associated with risky debt obtain.

If insiders cannot guarantee their choice of \( \{I_1, I_2\} \) at the time of financing, bondholders will attribute to them those actions which are in the insiders' own interests. Bondholders will pay into the firm only in accord with these (rational) forecasts. This further constrains insiders' actions, for they can control the decisions bondholders attribute to them only indirectly. We model this by formulating the insiders' decision problem in two stages: To determine bondholders' forecasts, we ask what shareholders will do, given a quantity of

\[\text{The assumption that } E_0 > 0 \text{ proves convenient later in ruling out perverse cases such as bankruptcy in all states. It simply ensures that the initial owners could generate some value through the firm even without access to external sources of funds.}\]
funds to invest, $F$, and a promised payment on the debt, $m$. We solve:

$$\text{P.2. } \max_{I_1, I_2, s} \int_S p(s) \max\{ X(I, s) - m, 0 \} \, ds$$

subject to $I_1 + I_2 \leq F$.

Let $G(m, F)$ denote the set of solution vectors to P.2. It depends on the promised payment on the debt and the total capitalization of the firm, both of which are observable at the time of financing. The insider’s investment-financing problem can now be written

$$\text{P.3. } \max_{m, F, I} \int_S p(s) \max\{ X(I, s) - m, 0 \} \, ds,$$

subject to $F \leq \int_S p(s) \min\{ m, X(I, s) \} \, ds + E_0,$

$I \in G(m, F)$.

Two aspects of P.2 and P.3 deserve comment. The first is why the firm would issue risky debt at all, given the existence of outside equity as an alternative. A number of motives could be advanced to motivate the use of risky debt. Jensen and Meckling (1976) have argued that managerial incentives to allocate the firm’s resources for their private benefit are likely to be more severe for firms which are largely equity financed. Within the context of the model presented here, Green (1982) has demonstrated that the introduction of a simple corporate tax regime can motivate the issuance of risky debt. Proof of this involves arguments analogous to the proof of Lemma 1 below. Further, while taxes will alter in a non-proportional manner the claim held by the firm as a whole on its assets, they will generally not change the shape of the residual claim, since it is last in priority to both the government and the debt. An exception to this would be a case where other deductions, such as those from depreciation, exceeded the total promised payment on the debt. A tax regime such as that employed by Kraus and Litzenberger (1973), however, would simply involve multiplying the objective functions in P.2 and P.3 by unity less the tax rate. Since this is merely an affine transformation, it cannot alter the solutions to P.2, which determine the investment allocation. Thus formally incorporating these considerations into the model, while it involves considerable notational inconvenience, provides no added insight into the nature of the problem. We have accordingly simply constrained the firm to issue risky debt.

Perhaps a more fundamental question is why insiders would agree to value maximization as an appropriate objective function. A sufficient condition for such unanimity is unimpeded access to the financial markets on private
account, but this in turn raises the question of why insiders would choose to issue corporate securities with their attendant limited liability. By financing the firm on private account insiders could substitute their own personal liability for the limited liability of the corporation, and perhaps avoid the sorts of agency problems discussed here. We ignore this issue, with apologies for the partial equilibrium nature of the analysis, in the belief that rationalizing the corporate form itself takes us outside the scope of this inquiry, namely, how the structure of corporate liabilities influences corporate investment decisions.\(^4\)

Problem P.3 has the structure of a standard principal–agent problem. Were the second constraint to be replaced by the simple feasibility condition \(I_1 + I_2 \leq F\), P.3 would reproduce the ‘first-best’ problem P.1. Any \(\{I_1, I_2\}\) combination in \(G(m, F)\) meets this feasibility requirement, but not all feasible choices at \(F\) are in \(G(m, F)\). Since P.1 produces the highest \(NPV\) for the firm, solutions to P.3 will generally fall short of this maximum. By construction, bondholders who provide \(F - E_0\) receive a claim with a present value of \(F - E_0\). Thus the deviation from \(NPV\) maximization must be reflected in a loss of value for the shareholders’ claim. This loss represents the ‘agency cost of debt’ in this model.

Due to considerations discussed in Mirrlees (1975), the first-order conditions for P.3 will not in general be necessary. By construction, however, the optimal choice of \(I\), denoted \(I^*\), must also be a solution to P.2, for which the first-order conditions are necessary.\(^5\) These are

\[
k'(I_i^*) \int_{\hat{s}}^s p(s)(1 + R(s)) \, ds = \lambda^*, \quad i = 1, 2, \tag{6}
\]

where \(\lambda^*\) is the multiplier for the budget constraint and \(\hat{s}\) is the set of non-bankrupt states, \(\{s : X(I^*, s) > m\}\). For the comparison between \(I^*\) and the \(NPV\) maximizing allocation, \(\hat{l}\), to be at all interesting, we must rule out the trivial cases where \(I^* = \hat{l}\) or \(I^* = (0, 0)\). The following lemma accomplishes this:

**Lemma 1.** At the optimal pair \(\{m^*, F^*\}\), the debt is risky (\(\hat{s} \neq S\)), the budget constraint is binding in P.2 (\(\lambda^* > 0\)), and \(\hat{s} \neq \emptyset\).

**Proof.** See appendix A.

The intuition behind the proof of this lemma is quite simple. As long as the debt is riskless, insiders choose the allocation which maximizes the firm’s

\(^4\)See Fama and Jensen (1983) for a detailed discussion of questions regarding the determination of organizational form.

\(^5\)The \(\max\{,\}\) function has left and right-hand derivatives which both exist. These may differ, however, on the set of states \(\{s : X(I, s) = m\}\). Assumption A.5 guarantees that this set has zero measure, so that the right and left-hand derivatives of the integral are equal.
present value. Short of the \( NPV \) maximizing scale, there is an incentive to raise more capital and expand until either the debt becomes risky and agency costs arise, or \( \hat{I} \) is attained. We have ruled out the latter by assumption A.6.

To get a more precise understanding of the nature of the distortions which moral hazard induces, we can multiply the state prices in (4) and (6) by \((1 + r_f)\) as described at the end of section 2. This gives a characterization of the optima in terms of certainty equivalents. For the \( NPV \) maximizing allocation, we will have

\[
k'(\hat{I}_r)/k'(\hat{I}) = \frac{\bar{E}(1 + \hat{R}_r)}{\bar{E}(1 + \hat{R})}
\]

(7)

Let \( \hat{x}_m \) be the indicator function for the event \( X(I^*) > m \). Eq. (6) yields

\[
k'_i(I_i^*)/k'_j(I_j^*) = \frac{\bar{E}[\hat{x}_m(1 + \hat{R}_j)]}{\bar{E}[\hat{x}_m(1 + \hat{R})]} = \frac{\bar{E}(1 + \hat{R}_j) + \text{cov}(\hat{x}_m, 1 + \hat{R}_j)/\bar{E}(\hat{x}_m)}{\bar{E}(1 + \hat{R}) + \text{cov}(\hat{x}_m, 1 + \hat{R})/\bar{E}(\hat{x}_m)}.
\]

(8)

The first terms in the numerator and denominator of this last expression are the numerator and denominator of (7), which characterizes the \( NPV \) maximizing choices. In the allocation described by (8), the certainty equivalents from (7) are each adjusted by a premium which depends on the covariability with the event of bankruptcy. Since \( \hat{x}_m \) is a deterministic increasing function of \( \hat{X}(I) \), projects are rewarded at the margin for positive covariability with the firm as a whole on the ‘risk adjusted’ probability space \( (\mathcal{S}, \mathcal{F}, P(1 + r_f)) \). This suggests an ‘anti-diversification’ effect at work in the firm’s investment choices.

For the case at hand, let \( i = 1 \) and \( j = 2 \), and write the shock terms as defined in A.4. Then (7) yields immediately \( k'(\hat{I}_1)/k'(\hat{I}_2) = 1 \), which implies equal investment in both projects. Let

\[
\bar{V} = \bar{E}(\hat{x}_m)\bar{E}(1 + \hat{R}) + \text{cov}(\hat{x}_m, 1 + \hat{R}).
\]

(9)

From (8), we obtain

\[
k'(I_1)/k'(I_2) = \left[ \frac{\bar{V} + \text{cov}(\hat{x}_m, z)}{\bar{V}} \right] = 1 + \text{cov}(\hat{x}_m, z)/\bar{V}.
\]

(10)

Eq. (10) makes clear the nature of the ‘risk incentive’ problem. \( \bar{V} \) is positive, for it is equal to \((1 + r_f)\int_S p(s)\hat{x}_m(s)(1 + R(s))ds\), and \( \hat{R} \) is positive while \( \hat{x}_m \) is 1 for some states by Lemma 1 and zero otherwise. Therefore the right-hand side of (10) is bigger than one if \( \text{cov}(\hat{x}_m, z) > 0 \). Since \( z \) is a part of \( \hat{X}(I) \), and since it is mean independent of the other component, \( \hat{R}, z \) will be positively correlated with \( \hat{X}(I) \) and hence \( \hat{x}_m \). This positive covariability is preserved by the valuation or ‘risk adjustment’ because of the mean independence of \( z \) from
If the right-hand side of (10) is larger than one, this must imply a smaller marginal product for $I_1^*$ and/or a larger marginal product for $I_1^*$. Eq. (10) also provides some insight into when the problem is likely to be more severe. As the debt becomes less risky, $\bar{x}_m$ approaches the constant unity, and the covariance term in (10) goes to zero while the denominator does not. The following proposition merely establishes formally the conditions for (10) to be greater than one.

**Proposition 1.** The firm overinvests in the risky project relative to the less risky project.

**Proof.** See appendix B.

**4. The role of warrants**

The discussion of the previous section demonstrated formally that the firm issues risky debt and that a risk incentive problem exists. Further, we were able to characterize the sources of the distortionary incentives. This, of itself, simply elaborates and reinforces results that have been available in the literature for some time. More importantly, the analysis provides a framework within which to examine the effects of conversion features and warrants issued with debt on the risk incentives described above. These claims impose a certain structure or ‘shape’ on the residual claim which alters the incentives of the equity holders to take risk. By varying the parameters of the contract, insiders control this shape and hence their own incentives. Within the simple, single-period, two-project setting described above, these parameters can be chosen so as to enforce NPV maximizing behavior and meet the financing requirements of the firm simultaneously. This can be done within the context of limited liability and with a risky promised payment on the debt portion of the claim. Thus, we are not simply introducing riskless debt ‘by the back door’.

The conditions required to achieve this result are very restrictive. The single-period assumption allows us to abstract from the underinvestment problem discussed by Myers (1977), and the dividend payout and claim dilution problems examined by Smith and Warner (1979). It also enforces identity between the life of the project and the life of the bonds, and gives a very simple structure to the conversion decision. All options are exercised at the same point. The assumption of only two projects allows us to characterize deviations from NPV maximizing behavior solely in terms of the difference between the two, their relative risk. Moreover, we have abstracted from a myriad of other incentive problems which may exist in corporations, and will generally influence their financing behavior. Most important of these, surely, are potential conflicts of interest between management and equity holders. An explicit role for management in the model would complicate the analysis in at least two respects. It would introduce the problem of perquisite consumption
described in Jensen and Meckling (1976). It would also raise issues of optimal risk sharing within the firm studied in the literature on bilateral principal-agent contracts, and this in turn would confuse the role of market values in the determination of the firm’s policies.

Given these limitations, our results do not imply that all agency costs are eliminated in equilibrium. Tradeoffs between these costs and other considerations may very well shape observed financial practices at the margin. However, by ‘solving’ the risk incentive problem with convertibles in this abstract setting we can better understand (a) how their characteristics make convertible bonds an especially effective tool for controlling this particular agency problem, (b) the factors which would determine how the parameters of the contract might be chosen to minimize costs in a more complex setting, and (c) by the very restrictiveness of the assumptions, what the limitations of these contracts are in practice.

That convertibles and detachable warrants may serve such a role has been suggested by Jensen and Meckling (1976, p. 354). They state:

‘It seems that the incentive effects of warrants would tend to offset to some extent the incentive effects of the existence of risky debt because the owner manager would be sharing part of the proceeds associated with a shift in the distribution of returns with the warrant holders.’

This conjecture fails to make clear just how this ‘sharing’ would reduce initial shareholders’ incentives to increase the risk of the firm. For instance, if insiders were to simply issue outside equity with the risky debt, they would be sharing any wealth expropriated from bondholders. Yet, it is not clear that this would reduce their incentives to engage in this expropriation. Their claim remains a residual one, even if there are other residual claims outstanding. While equity holders may have to share the payoffs in the upper tail of the firm’s distribution, they still derive no benefit at all from its lower tail. One would therefore expect them to do whatever increases the value of that upper tail on which their claim entirely depends; and if investing in risky projects affects this goal, they will so invest. We show that what is crucial about the ‘sharing’ induced by warrants is that equity holders share only ‘the upper tail of the upper tail’. There must be a set of states over which the fixed claim on the debt is paid, while the option claim is not exercised, if the desired effect on incentives is to be produced. On these states, outsiders ‘put’ their fraction of the outside equity to insiders in exchange for the exercise price.6

6Haugen and Senbet (1981), in their study of the agency costs of equity, use put options to counteract the incentives to take the risk induced by executive call options. They argue that the effect is analogous to that which convertible bonds might induce. Their solution, however, relies on the unlimited liability of the manager and takes the scale of the firm as given. In the present context, it is precisely the limited liability of the equity claim which creates the adverse incentive effects we describe.
To understand the effects of warrants on the investment allocation we must consider their effect on the ‘shape’ of the residual claim. With straight debt, this claim is convex, as shown by the curve $O-A-B-C$ in fig. 1. The existence of warrants alters this shape. Assume that $h$ warrants are issued. Each gives the holder the right to purchase $h$ shares for an exercise price of $e$ per share. $n_0$ shares are outstanding initially. Each warrant holder takes as given the exercise behavior of others and believes that $\bar{n}$ warrants will be exercised. A holder of $n^*$ warrants will exercise if

$$\left[ \frac{n^*h}{n_0 + \bar{n}h + n^*h} \right] \left[ X(I, s) - m + \bar{n}he + n^*he \right] > n^*he.$$

(11)

This simplifies to

$$X(I, s) > \bar{e}n_0 + m,$$

(12)

which is independent of $n^*$ and $\bar{n}$. Constantinides (1981) has shown that in a multiperiod setting optimal exercise need not result in this simple ‘block exercise’ structure. The single-period assumption employed here abstracts from effects of dividend payouts and differences between American and European warrants.

Fig. 1. Straight debt with promised payment $m$ results in a residual claim described by $O-A-B-C$. Adding warrants for a fraction $r$ of the shares, with a total exercise price of $e$, makes $O-A-B-D$ the residual claim.
Thus, each agent has a dominant exercise strategy, and in a Nash equilibrium the warrants will be exercised in a block. Let $c = \hat{h}h\tilde{e}$ be the total amount paid into the firm on exercise in exchange for the fraction of equity claims $r = \hat{h}h/(n_0 + \hat{h}h)$. Condition (12) is, then, equivalent to

$$X(I, s) > e \left( \frac{1 - r}{r} \right) + m. \quad (13)$$

This leaves initial equity holders with the following claim:

$$\min\left\{ (1 - r) \left[ X(I, s) - m + e \right], \max\left[ X(I, s) - m, 0 \right] \right\}. \quad (14)$$

A convertible bond would produce the same flows as the above with $m = e$. The convertible holders surrender their fixed claim of $m$, thus exercising the option.

The claim described by (14) is a concave transformation of the shareholders' claim under straight debt, $\max\{ \hat{X}(I) - m, 0 \}$, which is convex. Referring again to fig. 1, we see that the residual claim with warrants, $O-A-B-D$, is convex over $O-A-B$, but concave over $A-B-D$. The question we wish to address is to what extent this concavity mitigates the distortionary incentives caused by the convexity of the equity claim with straight debt.\footnote{Haugen and Senbet's (1981) put option solution, referred to in footnote 5, would induce a very different pattern of flows. With unlimited liability for the holders of the put, the convex region of $O-A-B-D$ would be eliminated.} Note the importance of the
intermediate states where the warrants are not exercised while the fixed claim is paid. These states constitute the difference between issuing warrants and simply issuing outside equity. As illustrated in fig. 2, outside equity, or warrants with a zero exercise price, leave the convexity of the residual claim intact.

These arguments suggest that by appropriately choosing the parameters of the contract \( r, m, \) and \( e \), we can make the residual payoff either convex or concave. If we choose \( m \) so that it is risky and \( e = 0 \), we have the convex claim depicted in fig. 2. This will lead to overinvestment in the riskier project. If we choose \( m \) so that it is riskless and issue warrants with \( e > 0, 0 < r < 1 \), we create a concave claim. The fixed claim is paid in all states, but the warrant is exercised only on a subset. This puts point \( A \), in fig. 1, at the origin. By thus reversing the shape of the payoff, one ought to be able to reverse the incentives. The concave payoff ought to induce underinvestment in the riskier project. Given these two extremes, some contract between them with \( m \) risky, \( e > 0 \), and \( 0 < r < 1 \) ought to enforce the correct incentives. This logic is exploited to prove Proposition 2 below.

Enforcing the correct incentives is not very helpful if the resulting contract cannot finance the firm simultaneously. That is, given that bondholders anticipate \( NPV \) maximizing behavior by shareholders, the value of the contract must equal the funds required. For debt with warrants, the contract is defined by the promised payment on the debt, \( m \), the exercise price, \( e \), and the fraction of the shares, \( r \). Its payoff is \( \max\{ r[\tilde{X}(I) - m + e] + m - e, \min[m, \tilde{X}(I)] \} \).

The financing requirement is that \( \{r, m, e\} \) be chosen so that

\[
\hat{F} = \hat{I}_1 + \hat{I}_2 = E_0 + \int_S p(s) \max\{ r[\tilde{X}(I, s) - m + e] + m - e, \min[m, \tilde{X}(I, s)] \} \, ds.
\]

Let \( \hat{F} \) denote the optimal capitalization, i.e., \( \hat{F} = \hat{I}_1 + \hat{I}_2 \). To control the risk incentive problem we require that \( \{r, m, e\} \) be chosen so that \( \hat{I} \) solves

\[
P.4. \quad \max_{I_1} \int_S p(s) \min\{(1 - r)[X(I, s) - m + e], \max[X(I, s) - m, 0] \} \, ds,
\]

subject to \( I_1 + I_2 \leq \hat{F} \).

In section 3 we found that \( NPV \) maximization implied equal investment in the two projects. Denote as \( I^* \) the solution to P.4, and let \( s_1 = \{s: X(I, s) > \} \)
\[ m + e \left( 1 - \frac{r}{r} \right) \}

The first-order conditions for \( P.4 \) yield

\[
\frac{k'(I_1^m)}{k'(I_2^m)} = 1 + \frac{\int_{s_1} p(s)z(s) \, ds - (1 - r)\int_{s_1} p(s)z(s) \, ds}{\int_{s_1} p(s)(1 + R(s)) \, ds - (1 - r)\int_{s_1} p(s)(1 + R(s)) \, ds}.
\]

(16)

Note that with \( e > 0 \), and \( 0 < r < 1 \), \( s_1 \) is a proper subset of \( \hat{s} \). With \( 1 + \hat{R} > 0 \), this implies the denominator in the second term of (16) is positive. If our contract is to enforce \( \hat{I} \), we must be able to choose \( e^c > 0, 0 < r^c < 1 \), and \( m^c > \bar{m}(\hat{I}) \) so that \( [k'(I_1^m)/k'(I_2^m)] = 1 \), or so that

\[
\int_{s_1} p(s)z(s) \, ds - (1 - r)\int_{s_1} p(s)z(s) \, ds = 0.
\]

(17)

The expression on the left-hand side of (17) depends on the bankruptcy and exercise points, and hence on \( I, m, r, \) and \( e \). Call this function, which we must equate to zero at \( \hat{I}, g(I; m, r, e) \), and denote the value of the outsiders' payoff, the integral in (15), as \( V_c(I; m, r, e) \). Proposition 2 shows that a solution to the simultaneous equations \( V_c(I; m, r, e) = \hat{I} + 1 - E_0 \) and \( g(I; m, r, e) = 0 \) exists and has the desired characteristics. The proposition is stated for convertible debt, where \( e = m \). Since this is just a special case of bonds with warrants, the result must hold for warrants as well.

**Proposition 2.** There exists a contract \( \{m^c, r^c, m^c\} \) (i.e., a convertible bond) which controls the risk incentive problem and finances the firm simultaneously, and which has the following characteristics:

(a) The promised payment on the debt, \( m^c \), is risky.
(b) Exercise occurs only on a proper subset of the non-bankrupt states.

A proof is contained in appendix C. Lemmas establish the continuity of the \( V_c \) and \( g \) functions in the parameters of the contract, that the risk incentive can be 'reversed' by issuing convertible debt with a riskless promised payment, making \( g[I; \bar{m}(\hat{I}), r, \bar{m}(\hat{I})] < 0 \), and that either straight debt or riskless convertible debt can support the NPV maximizing allocation. It is then shown that this implies an intersection of the level sets of \( V_c \) and \( g \) functions at a point where \( m > \bar{m}(\hat{I}) \) and \( r > 0 \).

5. Conclusion

We believe that the results of the last section shed light on why convertibles and warrants are particularly well suited to the problem of controlling risk
incentives. This in turn helps rationalize the widespread coupling of debt and option claims. The analysis also makes apparent some limitations of these instruments. First, the option privileges issued with debt have market value, and therefore the promised payment on the debt, while still risky, will be less than it would be under straight debt financing if there were no monitoring or enforcement problems. This may reduce the firm's ability to exploit some of the benefits of debt financing. In short, there are ways in which option features with debt limit the firm's freedom as well as ways in which they increase it. Secondly, we have focused on the two-project firm for a reason — our ability to characterize the deviation from optimality by means of a single ratio of marginal scale products. The result of section 3, that there is overinvestment in the risky relative to the less risky project, will hold in a multiproject firm for any two projects that can be ordered according to this definition of relative risk. A single specification of \( \{ m^c, r^c, e^c \} \), however, may not restore the NPV maximizing choice of the ratio \( k'(I_j)/k'(I_i) \) for all \( (i, j) \) pairs simultaneously. Nevertheless, the formalization presented here seems to capture much of the intuition behind previous discussions of the role of complex financial instruments, while it adds insight into just just how the structure of these claims works to control distortionary incentives engendered by risky debt.

### Appendix A. Proof of Lemma 1

The last claim, that \( \delta \neq 0 \), follows immediately from the fact that this would imply a zero value for the equity claim. Such a solution is dominated by setting \( F = E_0 \), which generates some value, and seeking no outside financing. So \( \delta \neq 0 \). The left-hand side of (6) is the derivative of the value of the equity claim with respect to \( I_i \). Its positivity, so long as \( \delta \neq 0 \), ensures that \( \lambda^* > 0 \), so the budget constraint is binding [and also that (6) holds with equality, as written].

It remains to show that \( m^* \) is risky. Suppose not. Then, \( \delta = 0 \) and the \( \max \{ X(I^*, s) - m^*, 0 \} \) is \( X(I^*, s) - m^* \) for all \( s \). This implies that the objective function for P.2 is (locally) just the present value of the firm less a constant \( m^*/1 + r_f \). The solution will not depend on \( m^* \). \( \hat{I}(F^*) \) will be differentiable by the differentiability of the constraint and objective function in P.2 and the implicit function theorem. These conditions also ensure that if, in fact, this solution is optimal, the first-order conditions for P.3 are necessary. Substituting for \( I \) with \( \hat{I}(F^*) \) to eliminate the second constraint, the first-order condition with respect to \( F \) is

\[
\int_S p(s) \sum_i k_i \left[ \hat{I}_i(F^*) \right] (1 + R_i(s)) \left( \frac{\partial \hat{I}_i(F^*)}{\partial F} \right) ds - \lambda = 0. \quad (A.1)
\]

Equality holds because \( F = E_0 \) is always attainable at zero cost, so \( F^* \) must be
positive. Using (6) with $\hat{s} = S$, (A.1) can be written

$$\lambda^* \sum_i \partial \hat{I}_i(F^*)/\partial F = \lambda.$$  

(A.2)

Differentiation of the budget constraint in P.2 yields

$$\sum_i \partial \hat{I}_i(F^*)/\partial F = 1.$$  

(A.3)

The first-order condition for $m$, from P.3, is

$$-\int_0^1 p(s) \, ds + \lambda \int_0^1 p(s) \, ds = 1.$$  

(A.4)

But (A.4) and (A.3) imply $\lambda^* = 1$ in (A.2). If $\lambda^* = 1$ and $\hat{s} = S$, then (6) and (4) give $I^* = \hat{I}$. This contradicts assumption A.6 that $\hat{I}$ cannot be financed with riskless debt. Q.E.D.

Appendix B. Proof of Proposition 1

In hopes of improved readability, tildes are omitted from this proof. It should be understood that $R$, $W$, $z$, and $\chi_m$ are all random variables.

Using eq. (6), we need to show that

$$k'(I_1^*)/k'(I_2^*) = 1 + \left[ \int_0^1 p(s) z(s) \, ds / \int_0^1 p(s)(1 + R(s)) \, ds \right] > 1,$$  

(B.1)

or that

$$0 < \int_0^1 p(s) z(s) \, ds = E[Wz\chi_m] = E[W \chi_m z | R, W] = E[W h(R, W)],$$

(B.2)

where

$$h(R, W) = E(\chi_m z | R, W).$$  

(B.3)

If, for a particular $R$, the conditional probability of no bankruptcy is zero [i.e., $E(\chi_m | R, W) = 0$], then $h(R, W) = 0$ also. If it is zero for all $R$, then the shareholders receive nothing, which cannot be an optimal solution to the
problem. For any \( R \) such that \( \mathbb{E}(X_m | R, W) > 0 \), we have

\[
h(R, W) = \mathbb{E}(X_m | R, W) \left[ \frac{\mathbb{E}(X_m z | R, W)}{\mathbb{E}(X_m | R, W)} \right]
= \mathbb{E}(X_m | R, W) \mathbb{E}(z | z > a_z(R), R, W)
\geq \mathbb{E}(X_m | R, W) \mathbb{E}(z | R, W) = 0,
\]

where

\[
a_z(R) = (m - (1 + R) \left| k(I_1^*) + k(I_2^*) \right| )/k(I_2^*).
\]

Strict inequality will hold in (B.4) for some \( R \) unless the probability of bankruptcy is zero, and Lemma 1 rules out this possibility. Thus, \( h(R, W) \) is positive for some \( R \), and non-negative for all \( R \). Since \( W \) is positive, \( \mathbb{E}[W h(R, W)] > 0 \) and (B.2) holds. Q.E.D.

**Appendix C. Proof of Proposition 2**

We first require some preliminary results.

**Lemma 2.** The functions \( V_c(I; m, r, e) \) and \( g(I; m, r, e) \) are both continuous in the parameters of the contract \( \{ m, r, e \} \).

**Proof.** Assumption A.5 allows us to write \( g \) as an iterated integral,

\[
g(I; m, r, e) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W \left[ \int_{a_z}^{\infty} zf(z, R, W) \, dz \right. \right.
\left. - (1 - r) \int_{a_z^*}^{\infty} zf(z, R, W) \, dz \right] \, dW \, dR,
\]

where \( a_z \) is as defined in eq. (B.5) and \( a_z^* = a_z + [e(1 - r)]/[rk(I_2)] \). The interior integrals are continuous in their lower limits. These in turn are continuous in \( m, e \) and \( r \). The fact that the integral of a continuous function is continuous ensures the result. Similarly, we can write the value of the payoffs to outsiders as

\[
V_c(I; m, e, r) = \iint W \max\{ r [X(I) - m + e] + m - e, \min[X(I), m] \} f(z, R, W) \, dR \, dz \, dW.
\]
The \( \max(\ldots) \) and \( \min(\ldots) \) functions are continuous in their arguments. Integration over a continuous density preserves this continuity. Q.E.D.

We now show the risk incentive problem can be reversed.

**Lemma 3.** With riskless debt and a warrant outstanding the exercise of which is uncertain, \( g(I; m, r, e) < 0 \).

**Proof.** With \( m \) riskless, \( \hat{s} = S \) and \( \hat{z} \) has a zero price over \( S \). Therefore,

\[
g(I; m, e, r) = \int_S p(s) z(s) \, ds - (1 - r) \int_{\hat{S}_1} p(s) z(s) \, ds
\]

\[= - (1 - r) \int_{\hat{S}_1} p(s) z(s) \, ds.\]

(C.3)

Exercise is assumed uncertain so \( \hat{s}_1 \) is a proper subset of \( S \). But the proof of Proposition 1 shows that \( \int_{\hat{S}_1} p(s) z(s) \, ds > 0 \). This proves the result for the assumed case, because uncertain exercise implies \( 0 < r < 1 \). Q.E.D.

We now demonstrate the existence of two contracts which support the \( NPV \) maximizing allocation.

**Lemma 4.** For any finite \( \varepsilon \), there exists an \( \bar{r} \), \( 0 < \bar{r} < 1 \), which supports investment plan \( \hat{I} \) in combination with the riskless promised payment \( \bar{m}(\hat{I}) \).

**Proof.** Let \( G(r) \equiv V_e(\hat{I}; \bar{m}(\hat{I}), r, e) \). If \( r = 0 \), the warrant is worthless. The value of the outside claim is just that of the riskless debt and assumption A.6 states this is less than the cost of inputs. If \( r = 1 \), the option is always exercised and outsiders own the entire firm. Since \( \hat{I} \) maximizes \( NPV \), we have

\[ G(0) = \int_S p(s) \bar{m}(\hat{I}) \, ds < \hat{I}_1 + \hat{I}_2 - E_0 < \int_S p(s) X(\hat{I}, s) \, ds = G(1). \]

(C.4)

By Lemma 2, \( G(.) \) is continuous. Therefore, (C.4) and the intermediate value theorem imply the existence of \( r \in (0, 1) \) such that \( G(r) = \hat{I}_1 + \hat{I}_2 - E_0 \). Q.E.D.

**Lemma 5.** At \( r = 0 \) and \( e \) arbitrary, there exists a finite risky fixed payment \( \bar{m} \) which supports the allocation \( \hat{I} \).

**Proof.** With \( r = 0 \), the warrant is worthless so the contract is straight debt. Choose \( m' = \sup_s X(\hat{I}, s) \). That \( m' < \infty \) follows from A.2. Clearly with \( m = m' \),
outsiders own the entire firm, and the firm has a positive NPV at $\hat{i}$. This, with A.6, gives us

$$V_c(\hat{i}; \bar{m}(\hat{i}), 0, e) < \hat{i}_1 + \hat{i}_2 - E_0 < V_c(\hat{i}; m', 0, e).$$ (C.5)

By Lemma 2, $V_c$ is continuous and the intermediate value theorem implies there exists $\bar{m} \in (\bar{m}(\hat{i}), m')$ such that $V_c(\hat{i}; \bar{m}, 0, e) = \hat{i}_1 + \hat{i}_2 - E_0$. Q.E.D.

We now proceed to the main result. Our task is to show that the system

$$h(m, r) = g(\hat{i}; m, r, m) = 0, \tag{C.6}$$

$$v(m, r) = V_c(\hat{i}; m, r, m) - (\hat{i}_1 + \hat{i}_2 - E_0) = 0, \tag{C.7}$$

has at least one solution $\{m^c, r^c\}$ with $m^c > \bar{m}(\hat{i})$ and $0 < r^c < 1$. Let $\bar{m} = \bar{m}(\hat{i})$ and let $\hat{r}$ be a point where $\{\bar{m}, \hat{r}\}$ supports $\hat{i}$. That $\hat{r}$ exists and that $0 < \hat{r} < 1$ is ensured by Lemma 4. Thus we have, at point $B$ in fig. 3 that $v(\bar{m}, \hat{r}) = 0$.

Fig. 3. Any continuous curve connecting $D$ and $B$ where the financing constraint is met ($v = 0$) must intersect the curve on which risk incentives are neutralized ($h = 0$) at least once. Convertible bonds parameterized by $\{\bar{m}, \hat{r}\}$ and $\{m^c, 0\}$ support the optimal scale. $m^T$ is the promised payment which ensures $h > 0$ for any $r$. 

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Lemma 5 proves that a finite $\hat{m}$ exists for which $v(\hat{m},0) = 0$. This is point $D$ in the figure. By assumption A.5, we can differentiate (C.7) to obtain

$$\partial v(m, r) / \partial m = \int_{S_1} p(s) \, ds - \int_{S} p(s) \, ds > 0,$$

(C.8)

$$\partial v(m, r) / \partial r = \int_{S_1} p(s) X(\hat{i}, s) \, ds > 0.$$  

(C.9)

Consider the boundaries of the rectangle $A - B - C - D$. (C.8) and (C.9) imply $v$ is strictly increasing in $m$ and $r$. Thus, on the top and right-hand edges of the rectangle, $D - C - B$ (excluding the endpoints $D$ and $B$), $v(m, r) > 0$. On $D - A - B$, excluding the endpoints, $v(m, r) < 0$. But $v(m, r)$ is a continuous surface by Lemma 2, so that it must pass through the $r, m$ plane at least once on the interior of the $A - B - C - D$ rectangle. This intersection will describe a continuous path connecting the points $D$ and $B$.

Now we establish a similar result concerning the function $h$. Recall

$$h(m, r) = \int_{S_1} p(s) z(s) \, ds - (1 - r) \int_{S} p(s) z(s) \, ds,$$  

(C.10)

where, with $m = e$, $s_1 = \{s: X(\hat{i}, s) > m/r \}$. At the point $A$, $r = 0$ implying $s_1 = \emptyset$, and $m = \bar{m}$ implying $\hat{s} = S$. Since $\hat{z}$ has a zero price over $S$, we have $h(\bar{m}, 0) = 0$. At point $E$, $r = 1$ so the second term in $h$, with the factor $(1 - r)$, is zero. $\hat{s}$ is still the entire set $S$ and again $h(\bar{m}, 1) = 0$. For $r \in (0, 1)$ and $m = \bar{m}$, $s_1$ is a proper subset of $\hat{s} = S$, and

$$h(\bar{m}, r) = - (1 - r) \int_{s_1} p(s) z(s) \, ds < 0.$$  

(C.11)

The last inequality is, with $\hat{s} = s_1$, what is demonstrated in Proposition 1. Thus, on $\bar{m} \times (0, 1)$, $h(\bar{m}, r) < 0$. Define $m^T = \inf \{ m: z(s) > 0 \text{ for } s \in \hat{s} \}$. That $m^T < \infty$ follows from our assumption that $\hat{R}$ has bounded support, A.2. Recall that $\bar{m}$ is the maximum riskless promised payment on the debt. Therefore, for $r = 0$, $m > \bar{m}$, $h(m, 0) = \int_{S_1} p(s) z(s) \, ds > 0$. At $r = 1$, $(1 - r) = 0$ and with $m > \bar{m}$, $h(m, 1) - \int_{S} p(s) z(s) \, ds > 0$. Both inequalities are proved in Proposition 1. With $m = m^T$, $z(s)$ is positive for any $s \in \hat{s} \supseteq s_1$. So $h(m^T, r) > 0$ for all $r \in (0, 1)$. Thus we have established that on the locus $A - G - F - E$ (excluding points $A$ and $E$) $h(m, r) > 0$, while on $A - E$ (excluding $A$ and $E$), $h(m, r) < 0$. Since the surface is continuous, it must pass through the plane at least once on the interior of $[\bar{m}, m^T] \times [0, 1]$. The continuous path representing the level set $\{ m, r: h(m, r) = 0 \}$ must connect the points $A$ and $E$. It follows that the
two sets \( \{m, r : v(m, r) = 0\} \) and \( \{m, r : h(m, r) = 0\} \) have at least one point of intersection \( \{m^*, r^*\} \). Since these sets lie entirely within the interior of \([\bar{m}, m^T] \times [0, 1]\), except for the points \(A\) and \(E\), and \(B\) and \(D\), any such point of intersection must have the characteristics \(m^* > \bar{m}, 0 < r^* < 1\). Q.E.D.

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