Robust Economic Implications of Nonlinear Pricing Kernels

Caio Almeida¹   René Garcia²

¹Graduate School of Economics - FGV
²Edhec Business School

EESP-FGV, Sao Paulo

October 26, 2011
Stochastic Discount Factors (SDFs) and Asset Pricing Models

▶ Asset Pricing Models (APMs) provide approximations to admissible Stochastic Discount Factors (SDFs).

▶ They are usually tested by verifying how well they price a set of primitive assets:

\[ p_t^i = E_t(m_{t+1}x_{i,t+1}^i), \quad i = 1, \ldots, N. \]  \hspace{1cm} (1)

▶ A simple test compares the variance of the APM SDF to the minimum variance obtained by Hansen and Jagannathan (1991).

▶ Alternatively, one may calculate the (quadratic) distance of the APM to the family of admissible SDFs (Hansen and Jagannathan (1997)).
Problem Overview

In this context, some relevant questions are:

1. Does the model correctly price a given set of testing assets?

2. How can we identify sources of model misspecifications?

3. How can we rank a given set of models?

Answers will clearly depend on the choice of testing assets, and on the choice of model misspecification metric.
The central idea of modern finance is that prices are generated by expected discounted payoffs

\[ p_t^i = E_t(m_{t+1}x_{t+1}^i) \]  \hspace{1cm} (2)

Using the definition of covariance and the real risk-free rate \( R_f^t = \frac{1}{E(m)} \), we can write the price as:

\[ p_t^i = \frac{E_t(x_{t+1}^i)}{R_f^t} + cov_t(m_{t+1}, x_{t+1}^i). \]  \hspace{1cm} (3)

For excess returns:

\[ E_t(R^{ei}_{t+1}) = -R_f^t cov_t(R^{ei}_{t+1}, m_{t+1}) \]  \hspace{1cm} (4)

From (3) we derive the following relationship between SDFs and Returns moments:

\[ \frac{\sigma(m)}{E(m)} \geq \frac{|E(R^e)|}{\sigma(R^e)} \]  \hspace{1cm} (5)
Duality with the canonical mean-variance portfolio problem
Our Objective

- We generalize the HJ approach by providing a family of bounds and distances that go beyond the variance.

- As a by product, we obtain a family of strictly positive nonlinear SDFs useful for model diagnostics and pricing purposes.

- Useful for model diagnostics whenever the HJ bounds are not informative about APMs.
  - In such cases there is a need for complementary diagnostic tools.
  - We improve HJ SDF with positivity constraint and Snow’s (1991) higher moment bounds.

- Why useful for pricing purposes? Our SDFs provide a set of possible prices as opposed to a single value (Cochrane and Saa-Requejo (2000))
Testing Asset Pricing Models

- A more stringent test than the minimum variance, is to obtain the minimum variance of SDFs when they are restricted to be strictly positive (HJ 1991).

- Under strictly positive SDFs, sometimes there will be pricing errors in the Euler Equations.
  - This usually happens when there is a large number of primitive assets or a large range of SDF means.
  - In particular, the HJ linear SDF with positivity constraint will present pricing errors in some occasions.
  - In such cases, it is not going to be suitable to test parametric APMs anymore.
Testing Asset Pricing Models

- In cases where the HJ SDF with positivity constraint breaks:

- We use our family of strictly positive SDFs to suggest a sharper variance bound than HJ.
  - We approximate the HJ SDF with positivity constraint via an admissible strictly positive SDF.

- Also whenever Snow's (1991) higher moment bounds are implied by SDFs with pricing errors...

- We offer alternative bounds that consider higher moments based on admissible SDFs.
Our contribution

- How to account for moments other than the variance?

- Our information bounds are based on a family of convex discrepancy functions that capture all the approaches above as particular cases.

- The SDFs implied by our bounds are nonlinear on basis assets and positive: Useful to diagnose nonlinear models / price nonlinear assets.
Outline of Presentation

1. Minimum Discrepancy (MD) SDF Bounds
   - Solving the Problem in the Dual Space
   - A Special Family of Discrepancy Functions

2. Empirical Illustrations of the MD SDF Bounds
   - Diagnosing Consumption-Based Asset pricing Models
   - MD Distance
   - Performance Evaluation of Hedge Funds
Minimum Discrepancy SDF Bounds

- Let $R$ denote the vector of basis assets’ returns whose realizations $\{R_i\}_{i=1,...,T}$ are given in a K-dimensional space.
- Assume that there is a risk-free rate $R_f$ such that all admissible SDFs will have mean equal to $a = \frac{1}{R_f}$.
- The moment conditions for the MD bound will be the Euler equations that any admissible SDF should satisfy to price the basis assets:

$$\frac{1}{T} \sum_{i=1}^{T} m_i \left( R_i - \frac{1}{a} \right) = 0_K$$

(6)

- The Minimum Discrepancy SDF bound is defined by:

$$\hat{m}_{MD} = \arg\min_{\{m_1,...,m_T\}} \frac{1}{T} \sum_{i=1}^{T} \phi(m_i),$$

subject to $\frac{1}{T} \sum_{i=1}^{T} m_i \left( R_i - \frac{1}{a} \right) = 0$, $\frac{1}{T} \sum_{i=1}^{T} m_i = a$, $m_i > 0 \forall i$.

(7)
Choosing the discrepancy: The Cressie Read (CR) Family

- It is simpler to solve the primal discrepancy problem in the dual space (Borwein and Lewis (1991)):

\[
\hat{\lambda} = \arg \sup_{\alpha \in \mathbb{R}, \lambda \in \Lambda} a \ast \alpha - \sum_{i=1}^{T} \frac{1}{T} \phi^*+, \left( \alpha + \lambda' \left( R_i - \frac{1}{a} 1_K \right) \right), \quad (8)
\]

where \( \Lambda \subseteq R^K \) and \( \phi^*+, \) denotes the convex conjugate of \( \phi \) restricted to the positive real line:

\[
\phi^+(z) = \sup_{w > 0} zw - \phi(w) \quad (9)
\]

- The CR discrepancy functions are given by \( \phi(m) = \frac{(m)^{\gamma+1} - a^{\gamma+1}}{\gamma(\gamma+1)} \).

The implied SDF and the Information Frontier

- We can interpret CR dual problems as HARA utility maximizing problems:

\[ \hat{\lambda}_{CR} = \arg \sup_{\lambda \in \Lambda_{CR}} \frac{1}{T} \sum_{i=1}^{T} \left( a^\gamma + \gamma \lambda' \left( R_i - \frac{1}{a} \right) \right) \left( \frac{\gamma + 1}{\gamma} \right) \]  

\hspace{2cm} (10)

- The implied SDF can be recovered via the first order conditions of the problem:

\[ \hat{m}_{MD}^i = T \ast a \ast \frac{(a^\gamma + \gamma \hat{\lambda}_{CR}' \left( R_i - \frac{1}{a} \right) \frac{1}{\gamma})}{\sum_{j=1}^{T} (a^\gamma + \gamma \hat{\lambda}_{CR}' \left( R_j - \frac{1}{a} \right) \frac{1}{\gamma})} \]

- The SDF-related frontier is found by solving (10) for a grid of meaningful values for the SDF mean \( A = \{ a_1, a_2, ..., a_J \} \).

- The SDF-related frontier is given by:

\[ I_{CR}(a_l, \gamma) = \frac{1}{T} \sum_{i=1}^{T} \frac{\hat{m}_{MD}^i(a_l) \frac{\gamma + 1}{\gamma} - a_i^{\gamma + 1}}{\gamma(\gamma + 1)} , l = 1, 2, ..., J \]  

\hspace{2cm} (11)
Skewness and Kurtosis Weights given by Cressie Read Estimators

How do $\gamma$’s weight skewness and kurtosis in HARA-utility problems? $u(W) = -\frac{1}{\gamma+1}(a^\gamma - \gamma W)^{\gamma+1}/\gamma$, $W = \lambda(R - \frac{1}{a})$

$$E[u(v)] \approx u(v0) + \frac{1}{2} u_2(v0) \lambda_{opt}^2 \ast (E(R - E(R))^2 + \frac{1}{6} u_3(v0) \lambda_{opt}^3 \ast (E(R - E(R))^3 + \frac{1}{24} u_4(v0) \lambda_{opt}^4 \ast (E(R - E(R))^4$$

where $v0 = \lambda_{opt} \ast E(R - \frac{1}{a})$ is a scaled expected exc. return.
The Implied SDFs

- Cressie Read SDFs solving the bound problems will be hyperbolic functions of the basis assets returns $R$:

$$\hat{m}_{MD}(R) = \beta \ast \left( 1 + \gamma \hat{\lambda}_{opt}' \left( R - \frac{1}{a} \right) \right)^{\frac{1}{\gamma}}$$  \hspace{1cm} (12)

- They are non-parametric because only depend on the Lagrange Multipliers $\lambda_{opt}$ that re-weight the sample probabilities to minimize the original Cressie Read divergence function.
  - Lagrange Multipliers come from Euler equations for the basis assets returns under the primal problem.

- They are positive because come from solutions to (strictly increasing) utility maximization problems.
  - We impose positivity of the SDFs on the original primal problems...
Some Particular Estimators

- $\gamma = 1$, quadratic portfolio problem, implied SDF is a linear function of returns:

$$\hat{m}_{MD}(\gamma=1) = \arg \min_{\{m_1, \ldots, m_T\}} \frac{1}{T} \sum_{i=1}^{T} \frac{m_i^2 - a^2}{2},$$

subject to \( \frac{1}{T} \sum_{i=1}^{T} m_i (R_i - \frac{1}{a}) = 0, \frac{1}{T} \sum_{i=1}^{T} m_i = a, m_i > 0 \forall i. \quad (13) $$

- $\gamma = 0$, CARA portfolio problem, implied SDF is an exponential function of returns, Stutzer (1995), Kitamura and Stutzer (1997, 2002).

- $\gamma = -1$, logarithmic portfolio problem, implied SDF is a hyperbolic function of returns, Bansal and Lehman (1997).

- $\gamma > 0$, captures the higher moment bounds proposed by Snow (1991).
Empirical Illustrations

1. Diagnosing Asset pricing Models
2. Performance Evaluation of Hedge Funds
Diagnosing the CCAPM with HJ

- Canonical consumption-based asset pricing model

\[ m_t = \beta \left( \frac{C_t}{C_{t-1}} \right)^{-\alpha}. \] (14)

- Data: Annual Equity Return, One year Interest Rate and Consumption Data for the period 1890-2009.

- Equity Return and Interest Rate used to build SDF bound.

- For given values of \( \beta \) and \( \alpha \), we can compute the volatility of the CCAPM SDF.
Further Diagnostics of Consumption Based Models

- Diagnosing the Canonical CCAPM with Cressie Read MD estimators.

- For a large range of MD bounds $\gamma \in [-5, -3, -1, 0, 1, 1.5]$ the CCAPM becomes admissible only with very high $\alpha$’s.

- Members of the CR family with more negative $\gamma$’s demand higher risk-aversion.
  - Snow (1991) showed that the SDF frontier for a moment smaller than 2 becomes more restrictive
  - Julliard and Ghosh (2010) estimate the CCAPM with entropic estimators to capture disaster risk, finding high risk-aversion coefficients.
Analyzing APMs when FF Factors are Primitive Assets

▸ Suppose that we want to diagnose consumption-based APMs when the primitive assets are the three Fama and French factors (Market, SMB, HML).

▸ Factors are available in an annual frequency from 1927 to 2004.

▸ The corresponding average real one year interest rate for this period was 1.49%, giving an SDF mean equal to 0.985.

▸ For this SDF mean, the HJ SDF with positivity constraint presents large pricing errors.
Pricing Errors when FF Factors are Primitive Assets

- The table presents pricing Errors (P.E.) for different implied SDFs from the Cressie Read family.

- We are interested in using those SDFs to help on the diagnostic of APMs.

- The one closest to the HJ SDF with positivity constrain ($\gamma = 1$) with acceptable pricing errors is the $CR(\gamma = 0.3)$.

<table>
<thead>
<tr>
<th>CR $\gamma$</th>
<th>Variance</th>
<th>P.E. Market</th>
<th>P.E. SMB</th>
<th>P.E. HML</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>0.942</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-1.5</td>
<td>0.839</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>0.681</td>
<td>0</td>
<td>-0.1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0.398</td>
<td>0.2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.3</td>
<td>0.360</td>
<td>5.0</td>
<td>0.3</td>
<td>2.1</td>
</tr>
<tr>
<td>0.5</td>
<td>0.344</td>
<td>8.1</td>
<td>0.5</td>
<td>3.3</td>
</tr>
<tr>
<td>1</td>
<td>0.245</td>
<td>154.2</td>
<td>109.2</td>
<td>140.6</td>
</tr>
<tr>
<td>2</td>
<td>0.084</td>
<td>502.3</td>
<td>74.8</td>
<td>271.2</td>
</tr>
</tbody>
</table>
A Variance Bound Sharper than HJ
Hansen and Jagannathan (HJ) Distance

- HJ (1997) measure misspecifications in APMs based on the least-square distance to the family of admissible SDFs.


- Some practical problems with the HJ distance include:
  1. Doesn’t take into account skewness and kurtosis of returns, elements that appear in many nonlinear APMs.
  2. Difficult to distinguish among APMs based on the HJ (1997) distance. (see Lewellen, Nagel, and Shanken (2010))

- There is mixed evidence on the fact that HJ distance with positivity constraint might mitigate the second issue...
Given an APM $y(\theta)$, HJ (1997) measure its degree of misspecification by linearly projecting $y$ into the space of admissible SDFs $M$:

$$\delta_{HJ}(\theta)^2 = \min_{m \in M} \|m - y(\theta)\|^2 = \min_{m \in M} E\{(m - y(\theta))^2\} \quad (15)$$

Standard duality theory for Hilbert spaces (Luemberger (1969)) gives...

$$\delta_{HJ}(\theta)^2 = \max_{\lambda \in \mathbb{R}^n} E\{y^2 - (y - \lambda' x)^2 - 2\lambda' q\} \quad (16)$$

The F.O.C. of problem (4) give the Least Square projection of $y$ into $M$:

$$q = E\{(y - \lambda_{HJ}' \cdot x)x\} \quad (17)$$

$\lambda_{HJ}' \cdot x$ represents the smallest (linear) correction to the asset pricing model $y$ to become an admissible SDF.
Given a proxy APM $y$, and a convex discrepancy function $\phi$, find the admissible SDF closest to $y$ in the $\phi$-discrepancy sense:

$$\delta_{MD}(\theta) = \min_{m \in M} E\{\phi(1 + m - y(\theta))\}$$  \hspace{1cm} (18)

In particular, HJ (1997) is obtained when $\phi(\pi) = \pi^2$.

Under the CR family, $\phi(\pi) = \frac{\pi^{\gamma+1}-1}{\gamma(\gamma+1)}$, (18) becomes:

$$\delta_{CR}(\theta) = \min_{m \in M} E \left\{ \frac{(1 + m - y(\theta))^{\gamma+1} - 1}{\gamma(\gamma + 1)} \right\}$$  \hspace{1cm} (19)
The dual optimization problem is given by (Borwein and Lewis (SJCO, 1991)):

\[ \delta_{CR}(\theta) = \max_{\lambda \in \mathbb{R}^n} \lambda^\prime q - E \left\{ \frac{(1 + \gamma \lambda^\prime \cdot x)^{\frac{\gamma+1}{\gamma}}}{\gamma + 1} + (y(\theta) - 1)\lambda^\prime \cdot x + \frac{1}{\gamma(\gamma + 1)} \right\} \]  

(20)

By the F.O.C., the admissible SDF closest to the APM \( y \) is given by:

\[ m_{CR}(\theta) = y(\theta) - 1 + (1 + \gamma \lambda_{\star}^\prime \cdot x)^{\frac{1}{\gamma}} \]  

(21)

where \( \lambda_{\star} \) is the solution of the optimization problem (20).

- While HJ provide linear corrections (on the basis assets) to the APM \( y(\theta) \)...
- ...the CR problems provide **hyperbolic corrections** to \( y(\theta) \).
Graphical Example: Difference Between HJ and CR
Possible Implications

- Assume that the DGP for returns of industry portfolios is given by the Kraus and Litzemberger (JF, 1976) CAPM with priced co-skewness risk: \( m = a + bR_M + cR_M^2 \)

- We are interested in diagnosing the CAPM model based on the RF asset, the market portfolio \( R_M \), and an industry portfolio \( R_I \). \( (y(\theta) = \theta_1 + \theta_2 \ast R_M) \)

- Will the CAPM capture the risk-return structure of an industry portfolio that has co-skewness priced? No.

- Will we be able to identify the source of misspecification in the CAPM? Based on HJ (1997), the answer is no!

- However, if we search in the family of CR discrepancies, we will identify several nonlinear admissible SDFs.

- Solving for CR(\( \gamma = \frac{1}{2} \)), the dual problem will maximize a cubic utility function, and compensation for risk will be a quadratic polynomial in \( R_M \) and \( R_I \).
Performance Evaluation and Hedge Funds

We adopt monthly data from five relevant market risk factors (January 1996 to March 2004):
- Bond
- Credit Risk
- S&P 500
- Commodities
- Currencies

<table>
<thead>
<tr>
<th>CR Estimators</th>
<th>Bond</th>
<th>Credit</th>
<th>Commodity</th>
<th>S&amp;P</th>
<th>Currency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cressie Read (γ = -3)</td>
<td>-20.0 (47.9%)</td>
<td>-14.4 (34.5%)</td>
<td>-0.7 (1.6%)</td>
<td>1.8 (4.3%)</td>
<td>4.9 (11.7%)</td>
</tr>
<tr>
<td>Cressie Read (γ = -1)</td>
<td>-35.8 (43.7%)</td>
<td>-34.6 (42.3%)</td>
<td>-0.7 (0.9%)</td>
<td>3.7 (4.5%)</td>
<td>7.1 (8.6%)</td>
</tr>
<tr>
<td>Cressie Read (γ = 0)</td>
<td>-39.8 (37.8%)</td>
<td>-56.2 (53.5%)</td>
<td>0.1 (0.1%)</td>
<td>5.7 (5.4%)</td>
<td>3.4 (3.2%)</td>
</tr>
<tr>
<td>Cressie Read (γ = 1)</td>
<td>-23.3 (29.9%)</td>
<td>-44.4 (56.8%)</td>
<td>1.2 (1.6%)</td>
<td>3.8 (4.8%)</td>
<td>-5.4 (6.9%)</td>
</tr>
<tr>
<td>Cressie Read (γ = 3)</td>
<td>-7.9 (28.0%)</td>
<td>-15.1 (53.8%)</td>
<td>0.9 (3.1%)</td>
<td>1.9 (6.6%)</td>
<td>-2.4 (8.5%)</td>
</tr>
</tbody>
</table>

**Table**: Optimal Portfolio Weights for HARA-based Estimators (SDF Mean = 0.9962).
Some examples of Implied SDFs: $\gamma = -1, 0, 3$

Nonparametric SDFs with and without options
Hedge Funds Evaluation

- We use monthly data on 9 categories of Hedge Funds from January 1996 to March 2004.

- The categories include: Convertible arbitrage (C1), Fixed income arbitrage (C2), Event Driven (C3), Equity Market Neutral (C4), Long/Short Equity Hedge (C5), Global Macro (C6), Emerging Markets (C7), Dedicated Short Bias (C8), and Managed Futures (C9).

- We compare performance evaluation of several estimators including the linear factor model, and the Agarwal and Naik method.
Hedge Funds Performance Evaluation: Which CR Criterion to Choose?

Performance for Different HF Classes, and Different Gammas

![Graph showing performance for different HF classes and gammas.](image)
How to Choose a Cressie Read Bound?

- By going after a nice property specific to the empirical problem under analysis.
  - Example: When HJ w.p.c. breaks find CR bound with $\gamma$ as close as possible to 1.

- Could propose a convex combination of CR estimators, since the function will be convex.
  - Particular case: Quadratic (HJ)+ Another CR estimator that weights skewness and kurtosis

- Could use additional assets not among the primitive assets. Example: Option prices.