Dynamic Asset Pricing Theory

Darrell Duffie
Graduate School of Business
Stanford University

Author’s Manuscript
Copyright ©2002 Princeton University Press

September 16, 2006
Contents

1 Introduction to State Pricing  
   1A Arbitrage and State Prices .................................. 11  
   1B Risk-Neutral Probabilities .................................. 13  
   1C Optimality and Asset Pricing .................................. 13  
   1D Efficiency and Complete Markets .............................. 16  
   1E Optimality and Representative Agents .......................... 17  
   1F State-Price Beta Models ...................................... 19  

2 The Basic Multiperiod Model  
   2A Uncertainty .................................................. 29  
   2B Security Markets .............................................. 30  
   2C Arbitrage and State Pricing .................................. 30  
   2D Individual Agent Optimality .................................. 33  
   2E Equilibrium and Pareto Optimality .............................. 35  
   2F Equilibrium Asset Pricing .................................... 35  
   2G Arbitrage and Martingale Measures ............................. 36  
   2H Valuation of Redundant Securities ............................. 39  
   2I American Exercise Policies and Valuation ...................... 40  
   2J Is Early Exercise Optimal? ..................................... 44  

3 The Dynamic Programming Approach  
   3A The Bellman Approach .......................................... 61  
   3B First-Order Bellman Conditions ................................ 63  
   3C Markov Uncertainty ............................................ 63  
   3D Markov Asset Pricing ........................................... 64  
   3E Security Pricing by Markov Control ............................ 65  
   3F Markov Arbitrage-Free Valuation ............................... 67  
   3G Early Exercise and Optimal Stopping ........................... 69
# CONTENTS

4 The Infinite-Horizon Setting 79  
4A Markov Dynamic Programming 79  
4B Dynamic Programming and Equilibrium 83  
4C Arbitrage and State Prices 84  
4D Optimality and State Prices 85  
4E Method-of-Moments Estimation 87  

5 The Black-Scholes Model 97  
5A Trading Gains for Brownian Prices 97  
5B Martingale Trading Gains 99  
5C Itô Prices and Gains 100  
5D Itô’s Formula 102  
5E Black-Scholes Option-Pricing Formula 102  
5F Black-Scholes Formula: First Try 104  
5G The PDE for Arbitrage-Free Prices 106  
5H The Feynman-Kac Solution 107  
5I The Multidimensional Case 109  

6 State Prices and Equivalent Martingale Measures 117  
6A Arbitrage 118  
6B Numeraire Invariance 118  
6C State Prices and Doubling Strategies 119  
6D Expected Rates of Return 122  
6E Equivalent Martingale Measures 124  
6F State Prices and Martingale Measures 126  
6G Girsanov and Market Prices of Risk 127  
6H Black-Scholes Again 131  
6I Complete Markets 133  
6J Redundant Security Pricing 136  
6K Martingale Measures From No Arbitrage 137  
6L Arbitrage Pricing with Dividends 140  
6M Lumpy Dividends and Term Structures 142  
6N Martingale Measures, Infinite Horizon 144  

7 Term-Structure Models 151  
7A The Term Structure 152  
7B One-Factor Term-Structure Models 153  
7C The Gaussian Single-Factor Models 156
## CONTENTS

7D The Cox-Ingersoll-Ross Model ........................................... 157
7E The Affine Single-Factor Models .......................................... 158
7F Term-Structure Derivatives ................................................. 160
7G The Fundamental Solution ................................................. 162
7H Multifactor Models .......................................................... 164
7I Affine Term-Structure Models ............................................. 166
7J The HJM Model of Forward Rates ......................................... 168
7K Markovian Yield-Curves and SPDEs ..................................... 171

8 Derivative Pricing ............................................................. 187
8A Martingale Measures in a Black Box .................................... 187
8B Forward Prices ............................................................... 189
8C Futures and Continuous Resettlement .................................. 191
8D Arbitrage-Free Futures Prices ........................................... 192
8E Stochastic Volatility ......................................................... 194
8F Option Valuation by Transform Analysis ................................ 198
8G American Security Valuation ............................................. 202
8H American Exercise Boundaries .......................................... 206
8I Lookback Options ............................................................ 209

9 Portfolio and Consumption Choice ......................................... 223
9A Stochastic Control .......................................................... 223
9B Merton’s Problem ............................................................ 227
9C Solution to Merton’s Problem ............................................. 230
9D The Infinite-Horizon Case .................................................. 233
9E The Martingale Formulation .............................................. 235
9F Martingale Solution .......................................................... 238
9G A Generalization ............................................................. 241
9H The Utility-Gradient Approach .......................................... 242

10 Equilibrium ................................................................. 257
10A The Primitives ............................................................. 257
10B Security-Spot Market Equilibrium ..................................... 258
10C Arrow-Debreu Equilibrium .............................................. 259
10D Implementing Arrow-Debreu Equilibrium ................................ 260
10E Real Security Prices ........................................................ 263
10F Optimality with Additive Utility ....................................... 264
10G Equilibrium with Additive Utility ...................................... 265
CONTENTS

10H The Consumption-Based CAPM ................................. 267
10I The CIR Term Structure ........................................ 269
10J The CCAPM in Incomplete Markets ............................ 272

11 Corporate Securities ............................................. 281
11A The Black-Scholes-Merton Model .............................. 281
11B Endogenous Default Timing .................................... 284
11C Example: Brownian Dividend Growth ......................... 287
11D Taxes and Bankruptcy Costs .................................. 290
11E Endogenous Capital Structure ................................. 292
11F Technology Choice .............................................. 293
11G Other Market Imperfections .................................. 295
11H Intensity-Based Modeling of Default ......................... 297
11I Risk-Neutral Intensity Process ................................. 300
11J Zero-Recovery Bond Pricing .................................... 301
11K Pricing with Recovery at Default ............................. 302
11L Default-Adjusted Short Rate ................................. 304

12 Numerical Methods ............................................... 315
12A Central Limit Theorems ....................................... 315
12B Binomial to Black-Scholes ..................................... 316
12C Binomial Convergence for Unbounded Derivative Payoffs .. 319
12D Discretization of Asset Price Processes ...................... 320
12E Monte Carlo Simulation ....................................... 321
12F Efficient SDE Simulation ..................................... 322
12G Applying Feynman-Kac ....................................... 324
12H Finite-Difference Methods ................................... 325
12I Term-Structure Example ...................................... 329
12J Finite-Difference Algorithms with Early Exercise Options .. 331
12K The Numerical Solution of State Prices ...................... 333
12L Numerical Solution of the Pricing Semi-Group ............... 335
12M Fitting the Initial Term Structure ........................... 337

A Finite-State Probability ........................................ 345

B Separating Hyperplanes and Optimality ....................... 349

C Probability ....................................................... 353
CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>D  Stochastic Integration</td>
<td>359</td>
</tr>
<tr>
<td>E  SDE, PDE, and Feynman-Kac</td>
<td>365</td>
</tr>
<tr>
<td>F  Counting Processes</td>
<td>373</td>
</tr>
<tr>
<td>G  Utility Gradients</td>
<td>381</td>
</tr>
<tr>
<td>H  Itô's Formula with Jumps</td>
<td>387</td>
</tr>
<tr>
<td>I  Ito's Formula for Complex Functions</td>
<td>393</td>
</tr>
<tr>
<td>J  Finite-Difference Code</td>
<td>395</td>
</tr>
</tbody>
</table>
THIS BOOK IS an introduction to the theory of portfolio choice and asset pricing in multiperiod settings under uncertainty. An alternate title might be *Arbitrage, Optimality, and Equilibrium*, because the book is built around the three basic constraints on asset prices: absence of arbitrage, single-agent optimality, and market equilibrium. The most important unifying principle is that any of these three conditions implies that there are “state prices,” meaning positive discount factors, one for each state and date, such that the price of any security is merely the state-price weighted sum of its future payoffs. This idea can be traced to the invention by Arrow (1953) of the general equilibrium model of security markets. Identifying the state prices is the major task at hand. Technicalities are given relatively little emphasis so as to simplify these concepts and to make plain the similarities between discrete- and continuous-time models.

To someone who came out of graduate school in the mid-eighties, the decade spanning roughly 1969–79 seems like a golden age of dynamic asset pricing theory. Robert Merton started continuous-time financial modeling with his explicit dynamic programming solution for optimal portfolio and consumption policies. This set the stage for his 1973 general equilibrium model of security prices, another milestone. His next major contribution was his arbitrage-based proof of the option pricing formula introduced by Fisher Black and Myron Scholes in 1973, and his continual development of that approach to derivative pricing. The Black-Scholes model now seems to be, by far, the most important single breakthrough of this “golden decade,” and ranks alone with the Modigliani and Miller (1958) Theorem and the Capital Asset Pricing Model (CAPM) of Sharpe (1964) and Lintner (1965) in its overall importance for financial theory and practice. A tremendously influential simplification of the Black-Scholes model appeared in the “binomial” option pricing model of Cox, Ross, and Rubinstein (1979), who drew on an
Working with discrete-time models, LeRoy (1973), Rubinstein (1976), and Lucas (1978) developed multiperiod extensions of the CAPM. The “Lucas model” is the “vanilla flavor” of equilibrium asset pricing models. The simplest multiperiod representation of the CAPM finally appeared in Doug Breeden’s continuous-time consumption-based CAPM, published in 1979. Although not published until 1985, the Cox-Ingersoll-Ross model of the term structure of interest rates appeared in the mid-seventies and is still the premier textbook example of a continuous-time general equilibrium asset pricing model with practical applications. It also ranks as one of the key breakthroughs of that decade. Finally, extending the ideas of Cox and Ross (1976) and Ross (1978), Harrison and Kreps (1979) gave an almost definitive conceptual structure to the whole theory of dynamic security prices.

Theoretical developments in the period since 1979, with relatively few exceptions, have been a mopping-up operation. Assumptions have been weakened, there have been noteworthy extensions and illustrative models, and the various problems have become much more unified under the umbrella of the Harrison-Kreps model of equivalent martingale measures. For example, the standard approach to optimal portfolio and consumption choice in continuous-time settings has become the martingale method of Cox and Huang (1989). An essentially final version of the relationship between the absence of arbitrage and the existence of equivalent martingale measures was finally obtained by Delbaen and Schachermayer (1999).

On the applied side, markets have experienced an explosion of new valuation techniques, hedging applications, and security innovation, much of this based on the Black-Scholes and related arbitrage models. No major investment bank, for example, lacks the experts or computer technology required to implement advanced mathematical models of the term structure. Because of the wealth of new applications, there has been a significant development of special models to treat stochastic volatility, jump behavior including default, and the term structure of interest rates, along with many econometric advances designed to take advantage of the resulting improvements in richness and tractability.

Although it is difficult to predict where the theory will go next, in order to promote faster progress by people coming into the field it seems wise to have some of the basics condensed into a textbook. This book is designed to be a streamlined course text, not a research monograph. Much generality is sacrificed for expositional reasons, and there is relatively little emphasis on
mathematical rigor or on the existence of general equilibrium. As its title indicates, I am treating only the theoretical side of the story. Although it might be useful to tie the theory to the empirical side of asset pricing, we have excellent treatments of the econometric modeling of financial data, such as Campbell, Lo, and MacKinlay (1997) and Gourieroux and Jasiak (2000). I also leave out some important aspects of functioning security markets, such as asymmetric information and transactions costs. I have chosen to develop only some of the essential ideas of dynamic asset pricing, and even these are more than enough to put into one book or into a one-semester course.


A reasonable way to teach a shorter course on continuous-time asset pricing out of this book is to begin with Chapter 1 or 2 as an introduction to the basic notion of state prices and then to go directly to Chapters 5 through 11. Chapter 12, on numerical methods, could be skipped at some cost to the student’s ability to implement the results. There is no direct dependence of any results in Chapters 5 through 12 on the first four chapters.

For mathematical preparation, little beyond undergraduate analysis, as in Bartle (1976), and linear algebra is assumed. Some familiarity with Royden (1968) or a similar text on functional analysis and measure theory, would also be useful. Some background in microeconomics would be useful, say Kreps (1990) or Luenberger (1995). Familiarity with probability theory at the level of Jacod and Protter (2000), for example, would also speed things along, although measure theory is not used heavily. In any case, a series of appendices supplies all of the required concepts and definitions from probability theory and stochastic calculus. Additional useful references in this regard are Brémaud (1981), Karatzas and Shreve (1988), Revuz and Yor (1991), and Protter (2004).
Students seem to learn best by doing problem exercises. Each chapter has exercises and notes to the literature. I have tried to be thorough in giving sources for results whenever possible and plead that any cases in which I have mistaken or missed sources be brought to my attention for correction. The notation and terminology throughout is fairly standard. I use $\mathbb{R}$ to denote the real line and $\mathbb{R} = \mathbb{R} \cup \{-\infty, +\infty\}$ for the extended real line. For any set $Z$ and positive integer $n$, I use $Z^n$ for the set of $n$-tuples of the form $(z_1, \ldots, z_n)$ with $z_i$ in $Z$ for all $i$. An example is $\mathbb{R}^n$. The conventions used for inequalities in any context are

- $x \geq 0$ means that $x$ is nonnegative. For $x$ in $\mathbb{R}^n$, this is equivalent to $x \in \mathbb{R}^n_+$;
- $x > 0$ means that $x$ is nonnegative and not zero, but not necessarily strictly positive in all coordinates;
- $x \gg 0$ means $x$ is strictly positive in every possible sense. The phrase “$x$ is strictly positive” means the same thing. For $x$ in $\mathbb{R}^n$, this is equivalent to $x \in \mathbb{R}^n_{++} \equiv \text{int}(\mathbb{R}^n_+)$.  

Although warnings will be given at appropriate times, it should be kept in mind that $X = Y$ will be used to mean equality almost everywhere or almost surely, as the case may be. The same caveat applies to each of the above inequalities. A real-valued function $F$ on an ordered set (such as $\mathbb{R}^n$) is increasing if $F(x) \geq F(y)$ whenever $x \geq y$ and strictly increasing if $F(x) > F(y)$ whenever $x > y$. When the domain and range of a function are implicitly obvious, the notation “$x \mapsto F(x)$” means the function that maps $x$ to $F(x)$; for example, $x \mapsto x^2$ means the function $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = x^2$. Also, while warnings appear at appropriate places, it is worth pointing out again here that for ease of exposition, a continuous-time “process” will be defined throughout as a jointly (product) measurable function on $\Omega \times [0, T]$, where $[0, T]$ is the given time interval and $(\Omega, \mathcal{F}, P)$ is the given underlying probability space.

The first four chapters are in a discrete-time setting with a discrete set of states. This should ease the development of intuition for the models to be found in chapters 5 through 12. The three pillars of the theory, arbitrage, optimality, and equilibrium, are developed repeatedly in different settings. Chapter 1 is the basic single-period model. Chapter 2 extends the results of
Chapter 1 to many periods. Chapter 3 specializes Chapter 2 to a Markov setting and illustrates dynamic programming as an alternate solution technique. The Ho-and-Lee and Black-Derman-Toy term-structure models are included as exercises. Chapter 4 is an infinite-horizon counterpart to Chapter 3 that has become known as the *Lucas model*.

The focus of the theory is the notion of state prices, which specify the price of any security as the state-price weighted sum or expectation of the security’s state-contingent dividends. In a finite-dimensional setting, there exist state prices if and only if there is no arbitrage. The same fact is true in infinite-dimensional settings under technical regularity conditions. Given an agent’s optimal portfolio choice, a state-price vector is given by that agent’s utility gradient. In an equilibrium with Pareto optimality, a state-price vector is likewise given by a representative agent’s utility gradient at the economy’s aggregate consumption process.

Chapters 5 through 11 develop a continuous-time version of the theory. Although the major part of this is based on continuous information, in some of the exercises, and especially in Chapter 11, there is a transition to discontinuous information, that is, settings in which the conditional probability of some events do not adjust continuously with the passage of time. An example is Poisson arrival.

Chapter 5 introduces the continuous-trading model and develops the Black-Scholes partial differential equation (PDE) for arbitrage-free prices of derivative securities. The Harrison-Kreps model of equivalent martingale measures is presented in Chapter 6 in parallel with the theory of state prices in continuous time. Chapter 7 presents models of the term structure of interest rates, including the Black-Derman-Toy, Vasicek, Cox-Ingersoll-Ross, and Heath-Jarrow-Morton models, as well as extensions. Chapter 8 presents specific classes of derivative securities, such as futures, forwards, American options, and lookback options. Chapter 8 also introduces models of option pricing with stochastic volatility. The notion of an “affine” state process is used heavily in Chapters 7 and 8 for its analytical tractability. Chapter 9 is a summary of optimal continuous-time portfolio choice, using both dynamic programming and an approach involving equivalent martingale measures or state prices. Chapter 10 is a summary of security pricing in an equilibrium setting. Included are such well-known models as Breeden’s consumption-based capital asset pricing model and the general equilibrium version of the Cox-Ingersoll-Ross model of the term structure of interest rates. Chapter 11 deals with the valuation of corporate securities, such as debt and equity.
The chapter moves from models based on the capital structure of the corporation, in which default is defined in terms of the sufficiency of assets, to models based on an assumed process for the default arrival intensity. Chapter 12 outlines three numerical methods for calculating derivative security prices in a continuous-time setting: binomial approximation, Monte Carlo simulation of a discrete-time approximation of security prices, and finite-difference solution of the associated PDE for the asset price or the fundamental solution.

In preparing the first edition, I relied on help from many people, in addition to those mentioned above who developed this theory. In 1982, Michael Harrison gave a class at Stanford that had a major effect on my understanding and research goals. Beside me in that class was Chi-fu Huang; we learned much of this material together, becoming close friends and collaborators. I owe him a lot. I am grateful to Niko and Vana Skiadas, who treated me with overwhelming warmth and hospitality at their home on Skiathos, where parts of the first draft were written.

I have benefited from research collaboration over the years with many co-authors, including Antje Berndt, George Constantinides, Qiang Dai, Sanjiv Das, Peter DeMarzo, Amir Dembo, Jean-Dominique Deuschel, Rohan Douglas, Larry Epstein, Mark Ferguson, Damir Filipovic, Nicolae Gârleanu, Mark Garman, John Geanakoplos, Pierre-Yves Geoffard, Peter Glynn, Mike Harrison, Chi-fu Huang, Ming Huang, Matt Jackson, Rui Kan, Nikunj Kapadia, David Lando, Jun Liu, Pierre-Louis Lions, Jin Ma, Andreu Mas-Colell, Andy McLennan, Jun Pan, Laszlo Pedersen, Philip Protter, Rohit Rahi, Tony Richardson, Leando Saita, Walter Schachermayer, David Schranz, Mark Schroder, Wayne Shafer, Ken Singleton, Costis Skiadas, Richard Stanton, Jiongmin Yong Ke Wang, and Bill Zame. I owe a special debt to Costis Skiadas, whose generous supply of good ideas has had a big influence on the result.

I am lucky indeed to have had access to many fruitful research discussions here at Stanford, especially with my colleagues Anat Admati, Steve Boyd, Peter DeMarzo, Amir Dembo, Peter Glynn, Steve Grenadier, Joe Grundfest, Peter Hammond, Mike Harrison, Ayman Hindy, Harrison Hong, Ming Huang, Tsze Lai, Jack McDonald, George Papanicoulao, George Parker, Paul Pfleiderer, Balaji Prabakar, Manju Puri, Tom Sargent, Bill Sharpe, Ken Singleton, Jim Van Horne, Ben VanRoy, and Jeff Zwiebel, and elsewhere with too many others to name.

I am thankful to have had the chance to work on applied financial models with Mike Burger, Bill Colgin, Adam Duff, Stenson Gibner, Elizabeth

I thank Kingston Duffie, Ravi Myneni, Paul Bernstein, and Michael Boulware for coding and running some numerical examples, Linda Bethel for assistance with LaTeX and graphics, and, for production, editing, and typesetting, Laurie Pickert of Archetype and Tina Burke of Technical Typesettings. For assistance with bibliographic research, I am grateful to Yu-Hua Chen, Melissa Gonzalez, David Lee, and Analiza Quiroz. Jack Repcheck and Peter Dougherty have been friendly, helpful, and supportive editors.

Suggestions and comments have been gratefully received from many readers, including Fehmi Ashaboglu, Robert Ashcroft, Irena Asmundsen, Alexandre d’Aspremont, Flavio Auler, Chris Avery, Shahriar Azizpour, Kerry Back, Yaroslav Bazaliy, Ralf Becker, Antje Berndt, Michael Boulware, John Campbell, Andrew Caplin, Karen Chaltikian, Zehao Chen, Victor Chernozhukov, Hung-Ken Chien, Ramakrishnan Chirayathumadom, Seongman Cho, Fai Tong Chung, Chin-Shan Chuan, Howie Corb, Qiang Dai, Eugene Demler, Shijie Deng, Michelle Dick, Phil Dolan, Rod Duncan, Wedad Elmaghraby, Kian Esteghamat, Mark Ferguson, Christian Riis Flor, Prashant Fuloria, John Fuqua, Nicolae Gârleanu, Mark Garmaise, Filippo Ginanni, Michel Grueneberg, Bing Han, Jiarniu Han, Barney Hartmann-Glaeser, Philippe Henrotte, Ayman Hindy, Yael Hochberg, Toshiki Honda, Guillaume Horel, Taiichi Hoshino, Jiangping Hu, Ming Huang, Cristobal Huneeus, Don Iglehart, Michael Intriligator, Farshid Jamshidian, Ping Jiang, Shinsuke Kambe, Rui Kan, Ron Karidi, Don Kim, Felix Kubler, Allan Kulig, Yoichi Kuwana, Piero La Mura, Yingcong Lan, Joe Langsam, Jackie Lee, André Levy, Shujing Li, Xinping Li, Wenzhi Li, Tiong Wee Lim, Jun Liu, Leonid Litvak, Hanno Lustig, Rob McMillan, Rajnish Mehra, Sergei Morozov, Christophe Mueller, Ravi Myneni, Lee Bath Nelson, Yigal Newman, Angela Ng, Kazuhiro Ōhashi, Nelli Oster, Hui Ou-Yang, John Overdeck, Hideo Owen, Caglar Ozden, Mikko Packalen, Jun Pan, Romans Panc, Lasse Pedersen, Albert Perez, Eduardo Perez, Monika Piazzesi, Jorge Picazo, Heracles Polemarchakis, Marius Rabus, Rohit Rahi, Santan Rai, Ramandeep Randhawa, Shikhar Ranjan, Michael Rierson, Amir Sadr, Yuliy Sannikov, Marco Scarsini, Martin Schneider, Christine Shannon, Yong-Seok Shin, Mark Shivers, Björn Sigurdsson, Hersir Sigurgeirsson, Marciano Siniscalchi, Ravi Singh, Ronnie Sircar, Viktor Spivakovskiy, Jun Sun, Lucie Tepla, Sergiy Terentyev, Rajat Tewari, Sverrir Thorvaldsson, Alex Tolstykh, Tunay Tunca, John Uglum, Len Umantsev,
Stijn Van Nieuwerburgh, Laura Veldkamp, Mary Vyas, Jun Yan, Muhamet Yildiz, Nese Yildiz, Ke Wang, Neng Wang, Chao Wei, Wei Wei, Pierre-Olivier Weill, Steven Weinberg, Seth Weingram, Fang Wu, Guojun Wu, Baozhong Yang, Pinghua Young, Assaf Zeevi, and Alexandre Ziegler, with apologies to those whose assistance was forgotten. I am especially grateful to the expert team of Japanese translators of the second edition, Toshiki Honda, Kazuhiko Ohashi, Yoichi Kuwana, and Akira Yamazaki, all of whom are personal friends as well.

For the reader's convenience, the original preface has been revised for this fourth edition. Significant improvements have been made in most chapters. Errors are my own responsibility, and I hope to hear of them and any other comments from readers.
I. Discrete-Time Models

This first part of the book takes place in a discrete-time setting with a discrete set of states. This should ease the development of intuition for the models to be found in Part II. The three pillars of the theory,\textit{ arbitrage}, \textit{optimality}, and \textit{equilibrium}, are developed repeatedly in different settings. Chapter 1 is the basic single-period model, and includes the Capital Asset Pricing Model. Chapter 2 extends the results of Chapter 1 to many periods. Chapter 3 specializes Chapter 2 to a Markov setting and illustrates dynamic programming as an alternate solution technique. The Ho-and-Lee and Black-Derman-Toy term-structure models are included as exercises. Chapter 4 is an infinite-horizon counterpart to Chapter 3 that has become known as the \textit{Lucas model}.

The focus of the theory is the notion of state prices, which specify the price of any security as the state-price weighted sum or expectation of the security’s state-contingent dividends. In a finite-dimensional setting, there exist state prices if and only if there is no arbitrage. The same is true in infinite-dimensional settings under technical regularity conditions. Given an agent’s optimal portfolio choice, a state-price vector is given by that agent’s utility gradient. In an equilibrium with Pareto optimality, a state-price vector is likewise given by a representative agent’s utility gradient at the economy’s aggregate consumption process.
Chapter 1

Introduction to State Pricing

This chapter introduces the basic ideas in a finite-state one-period setting. In many basic senses, each subsequent chapter merely repeats this one from a new perspective. The objective is a characterization of security prices in terms of “state prices,” one for each state of the world. The price of a given security is simply the state-price weighted sum of its payoffs in the different states. One can treat a state price as the “shadow price,” or Lagrange multiplier, for wealth contingent on a given state of the world. We obtain a characterization of state prices, first based on the absence of arbitrage, then based on the first-order conditions for optimal portfolio choice of a given agent, and finally from the first-order conditions for Pareto optimality in an equilibrium with complete markets. State prices are connected with the “beta” model for excess expected returns, a special case of which is the Capital Asset Pricing Model (CAPM). Many readers will find this chapter to be a review of standard results. In most cases, here and throughout, technical conditions are imposed that give up much generality so as to simplify the exposition.

1A Arbitrage and State Prices

Uncertainty is represented here by a finite set \{1, \ldots, S\} of states, one of which will be revealed as true. The \(N\) securities are given by an \(N \times S\) matrix \(D\), with \(D_{ij}\) denoting the number of units of account paid by security \(i\) in state \(j\). The security prices are given by some \(q\) in \(\mathbb{R}^N\). A portfolio \(\theta \in \mathbb{R}^N\) has market value \(q \cdot \theta\) and payoff \(D^\top \theta\) in \(\mathbb{R}^S\). An arbitrage is a
portfolio \( \theta \) in \( \mathbb{R}^N \) with \( q \cdot \theta \leq 0 \) and \( D^\top \theta > 0 \), or \( q \cdot \theta < 0 \) and \( D^\top \theta \geq 0 \). An arbitrage is therefore, in effect, a portfolio offering “something for nothing.” Not surprisingly, it will later be shown that an arbitrage is naturally ruled out, and this gives a characterization of security prices as follows. A state-price vector is a vector \( \psi \) in \( \mathbb{R}^S_+ \) with \( q = D\psi \). We can think of \( \psi_j \) as the marginal cost of obtaining an additional unit of account in state \( j \).

**Theorem.** There is no arbitrage if and only if there is a state-price vector.

**Proof:** The proof is an application of the Separating Hyperplane Theorem. Let \( L = \mathbb{R} \times \mathbb{R}^S \) and \( M = \{(-q \cdot \theta, D^\top \theta) : \theta \in \mathbb{R}^N\} \), a linear subspace of \( L \). Let \( K = \mathbb{R}_+ \times \mathbb{R}^S_+ \), which is a cone (meaning that if \( x \) is in \( K \), then \( \lambda x \) is in \( K \) for each strictly positive scalar \( \lambda \)). Both \( K \) and \( M \) are closed and convex subsets of \( L \). There is no arbitrage if and only if \( K \) and \( M \) intersect precisely at 0, as pictured in Figure 1.1.

Suppose \( K \cap M = \{0\} \). The Separating Hyperplane Theorem (in a version for closed cones that is found in Appendix B) implies the existence of a non-zero linear functional \( F : L \to \mathbb{R} \) such that \( F(z) < F(x) \) for all \( z \) in \( M \) and nonzero \( x \) in \( K \). Since \( M \) is a linear space, this implies that \( F(z) = 0 \) for all \( z \) in \( M \) and that \( F(x) > 0 \) for all nonzero \( x \) in \( K \). The latter fact implies that there is some \( \alpha > 0 \) in \( \mathbb{R} \) and \( \psi \gg 0 \) in \( \mathbb{R}^S \) such that \( F(v,c) = \alpha v + \psi \cdot c \), for any \( (v,c) \in L \). This in turn implies that \(-\alpha q \cdot \theta + \psi \cdot (D^\top \theta) = 0 \) for all \( \theta \) in \( \mathbb{R}^N \). The vector \( \psi/\alpha \) is therefore a state-price vector.

Conversely, if a state-price vector \( \psi \) exists, then for any \( \theta \), we have \( q \cdot \theta = \psi^\top D^\top \theta \). Thus, when \( D^\top \theta \geq 0 \), we have \( q \cdot \theta \geq 0 \), and when \( D^\top \theta > 0 \), we
have \( q \cdot \theta > 0 \), so there is no arbitrage. □

\section*{1B Risk-Neutral Probabilities}

We can view any \( p \) in \( \mathbb{R}^S_+ \) with \( p_1 + \cdots + p_S = 1 \) as a vector of probabilities of the corresponding states. Given a state-price vector \( \psi \) for the dividend-price pair \((D, q)\), let \( \psi_0 = \psi_1 + \cdots + \psi_S \) and, for any state \( j \), let \( \hat{\psi}_j = \psi_j / \psi_0 \). We now have a vector \((\hat{\psi}_1, \ldots, \hat{\psi}_S)\) of probabilities and can write, for an arbitrary security \( i \),

\[
\frac{q_i}{\psi_0} = \hat{E}(D_i) \equiv \sum_{j=1}^{S} \hat{\psi}_j D_{ij},
\]

viewing the normalized price of the security as its expected payoff under specially chosen “risk-neutral” probabilities. If there exists a portfolio \( \theta \) with \( D^\top \theta = (1, 1, \ldots, 1) \), then \( \psi_0 = \theta \cdot q \) is the discount on riskless borrowing and, for any security \( i \), \( q_i = \psi_0 \hat{E}(D_i) \), showing any security’s price to be its discounted expected payoff in this sense of artificially constructed probabilities.

\section*{1C Optimality and Asset Pricing}

Suppose the dividend-price pair \((D, q)\) is given. An \textit{agent} is defined by a strictly increasing \textit{utility function} \( U : \mathbb{R}^S_+ \to \mathbb{R} \) and an \textit{endowment} \( e \) in \( \mathbb{R}^S_+ \). This leaves the \textit{budget-feasible set}

\[
X(q, e) = \{ e + D^\top \theta \in \mathbb{R}^S_+ : \theta \in \mathbb{R}^N, q \cdot \theta \leq 0 \},
\]

and the problem

\[
\sup_{c \in X(q, e)} U(c). \tag{1}
\]

We will suppose for this section that there is some portfolio \( \theta^0 \) with payoff \( D^\top \theta^0 > 0 \). Because \( U \) is strictly increasing, the wealth constraint \( q \cdot \theta \leq 0 \) is then binding at an optimum. That is, if \( c^* = e + D^\top \theta^* \) solves (1), then \( q \cdot \theta^* = 0 \).

**Proposition.** If there is a solution to (1), then there is no arbitrage. If \( U \) is continuous and there is no arbitrage, then there is a solution to (1).

Proof is left as an exercise.
Theorem. Suppose that $c^*$ is a strictly positive solution to (1), that $U$ is continuously differentiable at $c^*$, and that the vector $\partial U(c^*)$ of partial derivatives of $U$ at $c^*$ is strictly positive. Then there is some scalar $\lambda > 0$ such that $\lambda \partial U(c^*)$ is a state-price vector.

Proof: The first-order condition for optimality is that for any $\theta$ with $q \cdot \theta = 0$, the marginal utility for buying the portfolio $\theta$ is zero. This is expressed more precisely in the following way. The strict positivity of $c^*$ implies that for any portfolio $\theta$, there is some scalar $k > 0$ such that $c^* + \alpha D^\top \theta \geq 0$ for all $\alpha$ in $[-k,k]$. Let $g_\theta : [-k,k] \to \mathbb{R}$ be defined by

$$g_\theta(\alpha) = U(c^* + \alpha D^\top \theta).$$

Suppose $q \cdot \theta = 0$. The optimality of $c^*$ implies that $g_\theta$ is maximized at $\alpha = 0$. The first-order condition for this is that $g_\theta'(0) = \partial U(c^*)^\top D^\top \theta = 0$. We can conclude that, for any $\theta$ in $\mathbb{R}^N$, if $q \cdot \theta = 0$, then $\partial U(c^*)^\top D^\top \theta = 0$. From this, there is some scalar $\mu$ such that $\partial U(c^*)^\top D^\top = \mu q$.

By assumption, there is some portfolio $\theta^0$ with $D^\top \theta^0 > 0$. From the existence of a solution to (1), there is no arbitrage, implying that $q \cdot \theta^0 > 0$. We have

$$\mu q \cdot \theta^0 = \partial U(c^*)^\top D^\top \theta^0 > 0.$$

Thus $\mu > 0$. We let $\lambda = 1/\mu$, obtaining

$$q = \lambda D\partial U(c^*),$$

implying that $\lambda \partial U(c^*)$ is a state-price vector.$\quad \blacksquare$

Although we have assumed that $U$ is strictly increasing, this does not necessarily mean that $\partial U(c^*) \gg 0$. If $U$ is concave and strictly increasing, however, it is always true that $\partial U(c^*) \gg 0$.

Corollary. Suppose $U$ is concave and differentiable at some $c^* = e + D^\top \theta^* \gg 0$, with $q \cdot \theta^* = 0$. Then $c^*$ is optimal if and only if $\lambda \partial U(c^*)$ is a state-price vector for some scalar $\lambda > 0$.

This follows from the sufficiency of the first-order optimality conditions for concave objective functions. The idea is illustrated in Figure 1.2. In that figure, there are only two states, and a state-price vector is a suitably normalized nonzero positive vector orthogonal to the set $B = \{D^\top \theta : q \cdot \theta = 0\}$ of budget-neutral consumption adjustments. The first-order condition for optimality of $c^*$ is that movement in any feasible direction away from $c^*$ has
negative or zero marginal utility, which is equivalent to the statement that
the budget-neutral set is tangent at \( c^* \) to the preferred set \( \{ c : U(c) \geq U(c^*) \} \),
as shown in the figure. This is equivalent to the statement that \( \partial U(c^*) \) is
orthogonal to \( B \), consistent with the last corollary. Figure 1.3 illustrates
a strictly suboptimal consumption choice \( c \), at which the derivative vector
\( \partial U(c) \) is not co-linear with the state-price vector \( \psi \).

We consider the special case of an expected utility function \( U \), defined by
a given vector \( p \) of probabilities and by some \( u : \mathbb{R}_+ \to \mathbb{R} \) according to

\[
U(c) = E[u(c)] \equiv \sum_{j=1}^{s} p_j u(c_j).
\]  

(3)

For \( c \gg 0 \), if \( u \) is differentiable, then \( \partial U(c)_j = p_j u'(c_j) \). For this expected
utility function, (2) therefore applies if and only if

\[
q = \lambda E[Du'(c^*)],
\]  

(4)

with the obvious notational convention. As we saw in Section 1B, one can also
write (2) or (4), with the “risk-neutral” probability \( \hat{\psi}_j = u'(c_j^*) p_j / E[u'(c^*)] \),
in the form

\[
\frac{q_i}{\psi_0} = \hat{E}(D_i) \equiv \sum_{j=1}^{s} D_{ij} \hat{\psi}_j, \quad 1 \leq i \leq N.
\]  

(5)
1D Efficiency and Complete Markets

Suppose there are \( m \) agents, defined as in Section 1C by strictly increasing utility functions \( U_1, \ldots, U_m \) and by endowments \( e^1, \ldots, e^m \). An equilibrium for the economy \([ (U_i, e^i), D] \) is a collection \((\theta^1, \ldots, \theta^m, q)\) such that, given the security-price vector \( q \), for each agent \( i \), \( \theta^i \) solves \( \sup_{\theta} U_i(e^i + D^T\theta) \) subject to \( q \cdot \theta \leq 0 \), and such that \( \sum_{i=1}^{m} \theta^i = 0 \). The existence of equilibrium is treated in the exercises and in sources cited in the Notes.

With \( \text{span}(D) \equiv \{ D^T\theta : \theta \in \mathbb{R}^N \} \) denoting the set of possible portfolio payoffs, markets are complete if \( \text{span}(D) = \mathbb{R}^S \), and are otherwise incomplete.

Let \( e = e^1 + \cdots + e^m \) denote the aggregate endowment. A consumption allocation \((c^1, \ldots, c^m)\) in \( (\mathbb{R}_+^S)^m \) is feasible if \( c^1 + \cdots + c^m \leq e \). A feasible allocation \((\hat{c}^1, \ldots, \hat{c}^m)\) is Pareto optimal if there is no feasible allocation \((\hat{c}^1, \ldots, \hat{c}^m)\) with \( U_i(\hat{c}^i) \geq U_i(c^i) \) for all \( i \) and with \( U_i(\hat{c}^i) > U_i(c^i) \) for some \( i \). Complete markets and the Pareto optimality of equilibrium allocations are almost equivalent properties of any economy.

**Proposition.** Suppose markets are complete and \((\theta^1, \ldots, \theta^m, q)\) is an equilibrium. Then the associated equilibrium allocation is Pareto optimal.

This is sometimes known as The First Welfare Theorem. The proof, requiring only the strict monotonicity of utilities, is left as an exercise. We have established the sufficiency of complete markets for Pareto optimality. The necessity of complete markets for the Pareto optimality of equilibrium allocations does not always follow. For example, if the initial endowment...
allocation \((e^1,\ldots,e^m)\) happens by chance to be Pareto optimal, then any equilibrium allocation is also Pareto optimal, regardless of the span of securities. It would be unusual, however, for the initial endowment to be Pareto optimal. Although beyond the scope of this book, it can be shown that with incomplete markets and under natural assumptions on utility, for almost every endowment, the equilibrium allocation is not Pareto optimal.

1E Optimality and Representative Agents

Aside from its allocational implications, Pareto optimality is also a convenient property for the purpose of security pricing. In order to see this, consider, for each vector \(\lambda \in \mathbb{R}^m_+\) of “agent weights,” the utility function \(U_\lambda : \mathbb{R}^S_+ \to \mathbb{R}\) defined by

\[
U_\lambda(x) = \sup_{(c^1,\ldots,c^m)} \sum_{i=1}^m \lambda_i U_i(c^i) \quad \text{subject to } c^1 + \cdots + c^m \leq x. \tag{6}
\]

**Lemma.** Suppose that, for all \(i\), \(U_i\) is concave. An allocation \((c^1,\ldots,c^m)\) that is feasible is Pareto optimal if and only if there is some nonzero \(\lambda \in \mathbb{R}^m_+\) such that \((c^1,\ldots,c^m)\) solves (6) at \(x = e = c^1 + \cdots + c^m\).

**Proof:** Suppose that \((c^1,\ldots,c^m)\) is Pareto optimal. For any allocation \(x\), let \(U(x) = (U_1(x^1),\ldots,U_m(x^m))\). Next, let

\[
U = \{U(x) - U(c) - z : x \in A, z \in \mathbb{R}^m_+ \} \subset \mathbb{R}^m,
\]

where \(A\) is the set of feasible allocations. Let \(J = \{y \in \mathbb{R}^m_+ : y \neq 0\}\). Since \(U\) is convex (by the concavity of utility functions) and \(J \cap U\) is empty (by Pareto optimality), the Separating Hyperplane Theorem (Appendix B) implies that there is a nonzero vector \(\lambda\) in \(\mathbb{R}^m\) such that \(\lambda \cdot y \leq \lambda \cdot z\) for each \(y\) in \(U\) and each \(z\) in \(J\). Since \(0 \in U\), we know that \(\lambda \geq 0\), proving the first part of the result. The second part is easy to show as an exercise. 

**Proposition.** Suppose that markets are complete, that \((\theta^1,\ldots,\theta^m,q)\) is an equilibrium and that, for all \(i\), \(U_i\) is concave. Then there exists some nonzero \(\lambda \in \mathbb{R}^m_+\) such that \((0,q)\) is a (no-trade) equilibrium for the single-agent economy \([\langle U_\lambda,e\rangle,D]\) defined by (6). Moreover, the equilibrium consumption allocation \((c^1,\ldots,c^m)\) solves the allocation problem (6) at the aggregate endowment. That is, \(U_\lambda(e) = \sum_i \lambda_i U_i(c^i)\).
Proof: Since there is an equilibrium, there is no arbitrage, and therefore there is a state-price vector $\psi$. Since markets are complete, this implies that the problem of any agent $i$ can be reduced to

$$ \sup_{c \in \mathbb{R}^S_+} U_i(c) \quad \text{subject to} \quad \psi \cdot c \leq \psi \cdot e^i. $$

We can assume that $e^i$ is not zero, for otherwise $c^i = 0$ and agent $i$ can be eliminated from the problem without loss of generality. By the Saddle Point Theorem of Appendix B, there is a Lagrange multiplier $\alpha_i \geq 0$ such that $c^i$ solves the problem

$$ \sup_{c \in \mathbb{R}^S_+} U_i(c) - \alpha_i \left( \psi \cdot c - \psi \cdot e^i \right). $$

(The Slater condition is satisfied since $e^i$ is not zero and $\psi \gg 0$.) Since $U_i$ is strictly increasing, $\alpha_i > 0$. Let $\lambda_i = 1/\alpha_i$. For any feasible allocation $(x^1, \ldots, x^m)$, we have

$$ \sum_{i=1}^m \lambda_i U_i(c^i) = \sum_{i=1}^m \left[ \lambda_i U_i(c^i) - \lambda_i \alpha_i \left( \psi \cdot c^i - \psi \cdot e^i \right) \right] $$

$$ \geq \sum_{i=1}^m \lambda_i \left[ U_i(x^i) - \alpha_i \left( \psi \cdot x^i - \psi \cdot e^i \right) \right] $$

$$ = \sum_{i=1}^m \lambda_i U_i(x^i) - \psi \cdot \sum_{i=1}^m (x^i - e^i) $$

$$ \geq \sum_{i=1}^m \lambda_i U_i(x^i). $$

This shows that $(c^1, \ldots, c^m)$ solves the allocation problem (6). We must also show that no trade is optimal for the single agent with utility function $U_\lambda$ and endowment $e$. If not, there is some $x$ in $\mathbb{R}^S_+$ such that $U_\lambda(x) > U_\lambda(e)$ and $\psi \cdot x \leq \psi \cdot e$. By the definition of $U_\lambda$, this would imply the existence of an allocation $(x^1, \ldots, x^m)$, not necessarily feasible, such that $\sum_i \lambda_i U_i(x^i) > \sum_i \lambda_i U_i(c^i)$ and

$$ \sum_i \lambda_i \alpha_i \psi \cdot x^i = \psi \cdot x \leq \psi \cdot e = \sum_i \lambda_i \alpha_i \psi \cdot c^i. $$
Putting these two inequalities together, we would have

$$\sum_{i=1}^{m} \lambda_i \left[ U_i(x^i) - \alpha_i \psi \cdot (x^i - e^i) \right] > \sum_{i=1}^{m} \lambda_i \left[ U_i(c^i) - \alpha_i \psi \cdot (c^i - e^i) \right],$$

which contradicts the fact that, for each agent $i$, $(c^i, \alpha_i)$ is a saddle point for that agent’s problem. □

**Corollary 1.** If, moreover, $e \gg 0$ and $U_\lambda$ is continuously differentiable at $e$, then $\lambda$ can be chosen so that $\partial U_\lambda(e)$ is a state-price vector, meaning

$$q = D\partial U_\lambda(e). \quad (7)$$

The differentiability of $U_\lambda$ at $e$ is implied by the differentiability, for some agent $i$, of $U_i$ at $c^i$. (See Exercise 10(C).)

**Corollary 2.** Suppose there is a fixed vector $p$ of state probabilities such that, for all $i$, $U_i(c) = E[u_i(c)] \equiv \sum_{j=1}^{S} p_j u_i(c_j)$, for some $u_i(\cdot)$. Then $U_\lambda(c) = E[u_\lambda(c)]$, where, for each $y$ in $\mathbb{R}^+$,

$$u_\lambda(y) = \max_{x \in \mathbb{R}^m_+} \sum_{i=1}^{m} \lambda_i u_i(x_i) \quad \text{subject to } x_1 + \cdots + x_m \leq y.$$

In this case, (7) is equivalent to $q = E[D u_\lambda'(e)]$.

Extensions of this representative-agent asset pricing formula will crop up frequently in later chapters.

## 1F State-Price Beta Models

We fix a vector $p \gg 0$ in $\mathbb{R}^S$ of probabilities for this section, and for any $x$ in $\mathbb{R}^S$ we write $E(x) = p_1 x_1 + \cdots + p_S x_S$. For any $x$ and $\pi$ in $\mathbb{R}^S$, we take $x\pi$ to be the vector $(x_1\pi_1, \ldots, x_S\pi_S)$.

**Proposition.** A dividend-price pair $(D, q)$ admits no arbitrage if and only if there is some $\pi \gg 0$ in $\mathbb{R}^S$ such that $q = E(D\pi)$.

**Proof:** Given a state-price vector $\psi$, let $\pi_s = \psi_s / p_s$. Conversely, if $\pi$ has the assumed property, then $\psi_s = p_s\pi_s$ defines a state-price vector $\psi$. □
Given \((D, q)\), we refer to any vector \(\pi\) given by this result as a state-price deflator. (The terms state-price density and state-price kernel are often used synonymously with state-price deflator.) For example, the representative-agent pricing model of Corollary 2 of Section 1E shows that we can take \(\pi_s = u'(e_s)\).

For any \(x\) and \(y\) in \(\mathbb{R}^S\), the covariance \(\text{cov}(x, y) \equiv E(xy) - E(x)E(y)\) is a measure of covariation between \(x\) and \(y\) that is useful in asset pricing applications. For any such \(x\) and \(y\) with \(\text{var}(y) \equiv \text{cov}(y, y) \neq 0\), we can always represent \(x\) in the form \(x = \alpha + \beta y + \epsilon\), where \(\beta = \text{cov}(y, x)/\text{var}(y)\), where \(\text{cov}(y, \epsilon) = E(\epsilon) = 0\), and where \(\alpha\) is a scalar. This linear regression of \(x\) on \(y\) is uniquely defined. The coefficient \(\beta\) is called the associated regression coefficient.

Suppose \((D, q)\) admits no arbitrage. For any portfolio \(\theta\) with \(q \cdot \theta \neq 0\), the return on \(\theta\) is the vector \(R^\theta\) in \(\mathbb{R}^S\) defined by \(R^\theta = (D^\top \theta)/q \cdot \theta\). Fixing a state-price deflator \(\pi\), for any such portfolio \(\theta\), we have \(E(\pi R^\theta) = 1\). Suppose there is a riskless portfolio, meaning some portfolio \(\theta\) with constant return \(R^0\). We then call \(R^0\) the riskless return. A bit of algebra shows that for any portfolio \(\theta\) with a return, we have

\[
E(R^\theta) - R^0 = -\frac{\text{cov}(R^\theta, \pi)}{E(\pi)}.
\]

Thus, covariation with \(\pi\) has a negative effect on expected return, as natural from the interpretation of state prices as shadow prices for wealth.

The correlation between any \(x\) and \(y\) in \(\mathbb{R}^S\) is zero if either has zero variance, and is otherwise defined by

\[
\text{corr}(x, y) = \frac{\text{cov}(x, y)}{\sqrt{\text{var}(x) \text{var}(y)}}.
\]

There is always a portfolio \(\theta^*\) solving the problem

\[
\sup_{\theta} \text{corr}(D^\top \theta, \pi). \quad (8)
\]

If there is such a portfolio \(\theta^*\) with a return \(R^*\) having nonzero variance, then it can be shown as an exercise that, for any return \(R^\theta\),

\[
E(R^\theta) - R^0 = \beta_\theta \left[ E(R^*) - R^0 \right], \quad (9)
\]
where
\[ \beta_\theta = \frac{\text{cov}(R^*, R^\theta)}{\text{var}(R^*)}. \]

If markets are complete, then \( R^* \) is of course perfectly correlated with the state-price deflator.

Formula (9) is a state-price beta model, showing excess expected returns on portfolios to be proportional to the excess return on a portfolio having maximal correlation with a state-price deflator, where the constant of proportionality is the associated regression coefficient. The formula can be extended to the case in which there is no riskless return. Another exercise carries this idea, under additional assumptions, to the Capital Asset Pricing Model, or CAPM.

**Exercises**

**Exercise 1.1** The dividend-price pair \((D, q)\) of Section 1A is defined to be weakly arbitrage-free if \( q \cdot \theta \geq 0 \) whenever \( D^\top \theta \geq 0 \). Show that \((D, q)\) is weakly arbitrage-free if and only if there exists (“weak” state prices) \( \psi \in \mathbb{R}_+^S \) such that \( q = D\psi \). This fact is known as Farkas’s Lemma.

**Exercise 1.2** Prove the assertion in Section 1A that \((D, q)\) is arbitrage-free if and only if there exists some \( \psi \in \mathbb{R}_+^S \) such that \( q = D\psi \). Instead of following the proof given in Section 1A, use the following result, sometimes known as the Theorem of the Alternative: Stiemke’s Lemma. Suppose \( A \) is an \( m \times n \) matrix. Then one and only one of the following is true:

(a) There exists \( x \) in \( \mathbb{R}_+^n \) with \( Ax = 0 \).
(b) There exists \( y \) in \( \mathbb{R}_+^m \) with \( y^\top A > 0 \).

**Exercise 1.3** Show for \( U(c) \equiv E[u(c)] \) as defined by (3) that (2) is equivalent to (4).

**Exercise 1.4** Prove the existence of an equilibrium as defined in Section 1D under these assumptions: There exists some portfolio \( \theta \) with payoff \( D^\top \theta > 0 \) and, for all \( i \), \( e^i \gg 0 \) and \( U_i \) is continuous, strictly concave, and strictly increasing. This is a demanding exercise, and calls for the following general result.

Kakutani’s Fixed Point Theorem. Suppose \( Z \) is a nonempty convex compact subset of \( \mathbb{R}^n \), and for each \( x \) in \( Z \), \( \varphi(x) \) is a nonempty convex compact subset of \( Z \). Suppose also that \( \{(x, y) \in Z \times Z : x \in \varphi(y)\} \) is closed. Then there exists \( x^* \) in \( Z \) such that \( x^* \in \varphi(x^*) \).

**Exercise 1.5** Prove Proposition 1D. Hint: The maintained assumption of strict monotonicity of \( U_i(\cdot) \) should be used.
Exercise 1.6  Suppose that the endowment allocation \((e^1, \ldots, e^m)\) is Pareto optimal. (A) Show, as claimed in Section 1D, that any equilibrium allocation is Pareto optimal. (B) Suppose that there is some portfolio \(\theta\) with \(D\top\theta > 0\) and, for all \(i\), that \(U_i\) is concave and \(e^i \gg 0\). Show that \((e^1, \ldots, e^m)\) is itself an equilibrium allocation.

Exercise 1.7  Prove Proposition 1C. Hint: A continuous real-valued function on a compact set has a maximum.

Exercise 1.8  Prove Corollary 1 of Proposition 1E.

Exercise 1.9  Prove Corollary 2 of Proposition 1E.

Exercise 1.10  Suppose, in addition to the assumptions of Proposition 1E, that (a) \(e = e^1 + \cdots + e^m\) is in \(\mathbb{R}^S_+\); (b) for all \(i\), \(U_i\) is concave and twice continuously differentiable in \(\mathbb{R}^S_{++}\); (c) for all \(i\), \(c^i\) is in \(\mathbb{R}^S_{++}\) and the Hessian matrix \(\partial^2 U(c^i)\), which is negative semi-definite by concavity, is in fact negative definite. Property (c) can be replaced with the assumption of \textit{regular preferences}, as defined in a source cited in the Notes. (A) Show that the assumption that \(U_\lambda\) is continuously differentiable at \(e\) is justified and, moreover, that for each \(i\) there is a scalar \(\gamma_i > 0\) such that \(\partial U_\lambda(e) = \gamma_i \partial U_i(c^i)\). (This co-linearity is known as “equal marginal rates of substitution,” a property of any Pareto optimal allocation.) Hint: Use the following: Implicit Function Theorem. Suppose for given \(m\) and \(n\) that \(f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n\) is \(C^k\) (\(k\) times continuously differentiable) for some \(k \geq 1\). Suppose also that the \(n \times n\) matrix \(\partial_y f(\bar{x}, \bar{y})\) of partial derivatives of \(f\) with respect to its second argument is nonsingular at some \((\bar{x}, \bar{y})\). If \(f(\bar{x}, \bar{y}) = 0\), then there exist scalars \(\epsilon > 0\) and \(\delta > 0\) and a \(C^k\) function \(Z : \mathbb{R}^m \to \mathbb{R}^n\) such that if \(\|x - a\| < \epsilon\), then \(f(x, Z(x)) = 0\) and \(\|Z(x) - b\| < \delta\).

(B) Show that the negative-definite part of condition (c) is satisfied if \(e \gg 0\) and, for all \(i\), \(U_i\) is an expected utility function of the form \(U_i(c) = E[u_i(c)]\), where \(u_i\) is strictly concave with an unbounded derivative on \((0, \infty)\).

(C) Obtain the result of part (A) without assuming the existence of second derivatives of the utilities. (You would therefore not exploit the Hessian matrix or Implicit Function Theorem.) As the first (and main) step, show the following. Given a concave function \(f : \mathbb{R}^S_+ \to \mathbb{R}\), the superdifferential of \(f\) at some \(x \in \mathbb{R}^S_+\) is
\[
\partial f(x) = \{z \in \mathbb{R}^S : f(y) \leq f(x) + z \cdot (y - x), \quad y \in \mathbb{R}^S_+\}.
\]
For any feasible allocation \((c^1, \ldots, c^m)\) and \(\lambda \in \mathbb{R}^m_+\) satisfying \(U_\lambda(e) = \sum_i \lambda_i U_i(c^i)\),
\[
\partial U_\lambda(e) = \sum_{i=1}^m \lambda_i \partial U_i(c^i)\]
Exercise 1.11 (Binomial Option Pricing) As an application of the results in Section 1A, consider the following two-state \((S = 2)\) option-pricing problem. There are \(N = 3\) securities:

(a) a stock, with initial price \(q_1 > 0\) and dividend \(D_{11} = Gq_1\) in state 1 and dividend \(D_{12} = Bq_1\) in state 2, where \(G > B > 0\) are the “good” and “bad” gross returns, respectively;

(b) a riskless bond, with initial price \(q_2 > 0\) and dividend \(D_{21} = D_{22} = Rq_2\) in both states (that is, \(R\) is the riskless return and \(R^{-1}\) is the discount);

(c) a call option on the stock, with initial price \(q_3 = C\) and dividend \(D_{3j} = (D_{1j} - K)^+ \equiv \max(D_{1j} - K, 0)\) for both states \(j = 1\) and \(j = 2\), where \(K \geq 0\) is the exercise price of the option. (The call option gives its holder the right, but not the obligation, to pay \(K\) for the stock, with dividend, after the state is revealed.)

(A) Show necessary and sufficient conditions on \(G, B,\) and \(R\) for the absence of arbitrage involving only the stock and bond.

(B) Assuming no arbitrage for the three securities, calculate the call-option price \(C\) explicitly in terms of \(q_1, G, R, B,\) and \(K\). Find the state-price probabilities \(\hat{\psi}_1\) and \(\hat{\psi}_2\) referred to in Section 1B in terms of \(G, B,\) and \(R\), and show that \(C = R^{-1} \hat{E}(D_3)\), where \(\hat{E}\) denotes expectation with respect to \((\hat{\psi}_1, \hat{\psi}_2)\).

Exercise 1.12 (CAPM) In the setting of Section 1D, suppose \((c^1, \ldots, c^m)\) is a strictly positive equilibrium consumption allocation. For any agent \(i\), suppose utility is of the expected-utility form \(U_i(c) = E[u_i(c)]\). For any agent \(i\), suppose there are fixed positive constants \(c_i\) and \(b_i\) such that, for any state \(j\), we have \(c^j < c\) and \(u_i(x) = x - b_i x^2\) for all \(x \leq c\).

(A) In the context of Corollary 2 of Section 1E, show that \(u'_\lambda(e) = k - Ke\) for some positive constants \(k\) and \(K\). From this, derive the CAPM

\[ q = AE(D) - Bcov(D,e), \]  

(10)
for positive constants \(A\) and \(B\), where \(cov(D,e) \in \mathbb{R}^N\) is the vector of covariances between the security dividends and the aggregate endowment.

Suppose for a given portfolio \(\theta\) that each of the following is well defined:

- the return \(R^\theta \equiv D^T \theta/q \cdot \theta\);
- the return \(R^M\) on a portfolio \(M\) with payoff \(D^T M = e\);
- the return \(R^0\) on a portfolio \(\theta^0\) with \(cov(D^T \theta^0, e) = 0\);
- \(\beta_\theta = cov(R^\theta, R^M)/\text{var}(R^M)\).

The return \(R^M\) is sometimes called the market return. The return \(R^0\) is called the zero-beta return and is the return on a riskless bond if one exists. Prove the “beta” form of the CAPM

\[ E(R^\theta - R^0) = \beta_\theta E(R^M - R^0). \]  

(11)
Chapter 1. State Pricing

(B) Part (A) relies on the completeness of markets. Without assuming complete markets, but supposing that the equilibrium allocation \((c^1, \ldots, c^m)\) is strictly positive, show that the same beta form (11) applies, provided we extend the definition of the market return \(R^M\) to be the return on any portfolio solving

\[
\sup_{\theta \in \mathbb{R}^N} \text{corr}(R^\theta, e). \tag{12}
\]

For complete markets, \(\text{corr}(R^M, e) = 1\), so the result of part (A) is a special case.

(C) The CAPM applies essentially as stated without the quadratic expected-utility assumption provided that each agent \(i\) is strictly variance-averse, in that \(U_i(x) > U_i(y)\) whenever \(E(x) = E(y)\) and \(\text{var}(x) < \text{var}(y)\). Formalize this statement by providing a reasonable set of supporting technical conditions.

We remark that a common alternative formulation of the CAPM allows security portfolios in initial endowments \(\hat{\theta}^1, \ldots, \hat{\theta}^m\) with \(\sum_{i=1}^m \hat{\theta}^i_j = 1\) for all \(j\). In this case, with the total endowment \(e\) redefined by \(e = \sum_{i=1}^m (e^i + D^i \hat{\theta}^i)\), the same CAPM (11) applies. If \(e^i = 0\) for all \(i\), then even in incomplete markets, \(\text{corr}(R^M, e) = 1\), since (12) is solved by \(\theta = (1, 1, \ldots, 1)\). The Notes provide references.

Exercise 1.13 An Arrow-Debreu equilibrium for \([U_i, e^i, D]\) is a nonzero vector \(\psi\) in \(\mathbb{R}^S_+\) and a feasible consumption allocation \((c^1, \ldots, c^m)\) such that for each \(i\), \(c^i\) solves \(\sup_c U_i(c)\) subject to \(\psi \cdot c^i \leq \psi \cdot e^i\). Suppose that markets are complete, in that \(\text{span}(D) = \mathbb{R}^S\). Show that \((c^1, \ldots, c^m)\) is an Arrow-Debreu consumption allocation if and only if it is an equilibrium consumption allocation in the sense of Section 1D.

Exercise 1.14 Suppose \((D, q)\) admits no arbitrage. Show that there is a unique state-price vector if and only if markets are complete.

Exercise 1.15 (Aggregation) For the “representative-agent” problem (6), suppose for all \(i\) that \(U_i(c) = E[u(c)]\), where \(u(c) = c^\gamma / \gamma\) for some nonzero scalar \(\gamma < 1\).

(A) Show, for any nonzero agent weight vector \(\lambda \in \mathbb{R}^m_+\), that \(U_\lambda(c) = E[kc^\gamma / \gamma]\) for some scalar \(k > 0\) and that (6) is solved by \(c^i = k_i c^\gamma\) for some scalar \(k_i \geq 0\) that is nonzero if and only if \(\lambda_i\) is nonzero.

(B) With this special utility assumption, show that there exists an equilibrium with a Pareto efficient allocation, without the assumption that markets are complete, but provided that, for all \(i\), \(e^i\) is in \(\text{span}(D)\). Calculate the associated equilibrium allocation.

Exercise 1.16 (State-Price Beta Model) This exercise is to prove and extend the state-price beta model (9) of Section 1F.

(A) Show problem (8) is solved by any portfolio \(\theta\) such that \(\pi = D^\top \theta + \epsilon\), where \(\text{cov}(\epsilon, D^j) = 0\) for any security \(j\), where \(D^j \in \mathbb{R}^S\) is the payoff of security \(j\).

(B) Given a solution \(\theta\) to (8) such that \(R^\theta\) is well defined with nonzero variance, prove (9).
(C) Reformulate (9) for the case in which there is no riskless return by redefining $R^0$ to be the expected return on any portfolio $\theta$ such that $R^\theta$ is well defined and $\text{cov}(R^\theta, \pi) = 0$, assuming such a portfolio exists.

Exercise 1.17  Prove the following.

Riesz Representation Theorem. Suppose $F : \mathbb{R}^S \to \mathbb{R}$ is linear. Then there is a unique $\pi$ in $\mathbb{R}^S$ such that, for all $x$ in $\mathbb{R}^S$, we have $F(x) = E(\pi x)$.

The following hint is unnecessary in this simple setting but allows the result to be extended to a broad variety of spaces called Hilbert spaces. Given a vector space $L$, a function $(\cdot | \cdot) : L \times L \to \mathbb{R}$ is called an inner product for $L$ if, for any $x$, $y$, and $z$ in $L$ and any scalar $\alpha$, we have the five properties:

(a) $(x | y) = (y | x)$
(b) $(x + y | z) = (x | z) + (y | z)$
(c) $(\alpha x | y) = \alpha(x | y)$
(d) $(x | x) \geq 0$
(e) $(x | x) = 0$ if and only if $x = 0$.

Suppose a finite-dimensional vector space $L$ has an inner product $(\cdot | \cdot)$. (This defines a special case of a Hilbert space.) Two vectors $x$ and $y$ are defined to be orthogonal if $(x | y) = 0$. For any linear subspace $H$ of $L$ and any $x$ in $L$, it can be shown that there is a unique $y$ in $H$ such that $(x - y | z) = 0$ for all $z$ in $H$. This vector $y$ is the orthogonal projection in $L$ of $x$ onto $H$, and solves the problem $\text{min}_{b \in H} \|x - b\|$. Let $L = \mathbb{R}^S$. For any $x$ and $y$ in $L$, let $(x | y) = E(xy)$. We must show that given a linear functional $F$, there is a unique $\pi$ with $F(x) = (\pi | x)$ for all $x$. Let $J = \{x : F(x) = 0\}$. If $J = L$, then $F$ is the zero functional, and the unique representation is $\pi = 0$. If not, there is some $z$ such that $F(z) = 1$ and $(z | x) = 0$ for all $x$ in $J$. Show this using the idea of orthogonal projection. Then show that $\pi = z/(z | z)$ represents $F$, using the fact that for any $x$, we have $x - F(x)z \in J$.

Exercise 1.18  Suppose there are $m = 2$ consumers, $A$ and $B$, with identical utilities for consumption $c_1$ and $c_2$ in states 1 and 2 given by $U(c_1, c_2) = 0.2\sqrt{c_1} + 0.5 \log c_2$. There is a total endowment of $e_1 = 25$ units of consumption in state 1.

(A)  Suppose that markets are complete and that, in a given equilibrium, consumer $A$'s consumption is 9 units in state 1 and 10 units in state 2. What is the total endowment $e_2$ in state 2?

(B)  Continuing under the assumptions of part (A), suppose there are two securities. The first is a riskless bond paying 10 units of consumption in each state. The second is a risky asset paying 5 units of consumption in state 1 and 10 units in state 2. In equilibrium, what is the ratio of the price of the bond to that of the risky asset?
Exercise 1.19  There are two states of the world, labeled 1 and 2, two agents, and two securities, both paying units of the consumption numeraire good. The risky security pays a total of 1 unit in state 1 and pays 3 units in state 2. The riskless security pays 1 unit in each state. Each agent is initially endowed with half of the total supply of the risky security. There are no other endowments. (The riskless security is in zero net supply.) The two agents assign equal probabilities to the two states. One of the agents is risk-neutral, with utility function \( E(c) \) for state-contingent consumption \( c \), and can consume negatively or positively in both states. The other, risk-averse, agent has utility \( E(\sqrt{c}) \) for nonnegative state-contingent consumption. Solve for the equilibrium allocation of the two securities in a competitive equilibrium.

Exercise 1.20  Consider a setting with two assets \( A \) and \( B \), only, both paying off the same random variable \( X \), whose value is non-negative in every state and non-zero with strictly positive probability. Asset \( A \) has price \( p \), while asset \( B \) has price \( q \). An arbitrage is then a portfolio \( (\alpha, \beta) \in \mathbb{R}^2 \) of the two assets whose total payoff \( \alpha X + \beta X \) is non-negative and whose initial price \( \alpha p + \beta q \) is strictly negative, or whose total payoff is non-zero with strictly positive probability and always non-negative, and whose initial price is negative or zero.

(A) Assuming no restrictions on portfolios, and no transactions costs or frictions, state the set of arbitrage-free prices \( (p, q) \). (State precisely the appropriate subset of \( \mathbb{R}^2 \).)

(B) Assuming no short sales \( (\alpha \geq 0 \text{ and } \beta \geq 0) \), state the set of arbitrage-free prices \( (p, q) \).

(C) Now suppose that \( A \) and \( B \) can be short sold, but that asset \( A \) can be short sold only by paying an extra fee of \( \phi > 0 \) per unit sold short. There are no other fees of any kind. Provide the obvious new definition of “no arbitrage” in precise mathematical terms, and state the set of arbitrage-free prices.

Notes
The basic approach of this chapter follows Arrow (1953), taking a general-equilibrium perspective originating with Walras (1877). Black (1995) offers a perspective on the general equilibrium approach and a critique of other approaches. Textbook treatments of financial economics covering the themes of Chapter 1 include those of Cvitanić and Zapatero (2004), Eichberger and Harper (1997), Huang and Litzenberger (1988), LeRoy and Werner (2001), and Lengwiler (2004).

1A: The state-pricing implications of no arbitrage found in Section 1A originate with Ross (1978).

1B: The idea of “risk-neutral probabilities” apparently originates with Arrow (1970), a revision of Arrow (1953), and appears as well in Drèze (1971).

1C: This material is standard.

1D: Proposition 1D is the First Welfare Theorem of Arrow (1951) and Debreu (1954). The generic inoptimality of incomplete-markets equilibrium allocations can be gleaned

1E: The “representative-agent” approach goes back, at least, to Negishi (1960). The existence of a representative agent is no more than an illustrative simplification in this setting, and should not be confused with the more demanding notion of aggregation of Gorman (1953) found in Exercise 15. In Chapter 10, the existence of a representative agent with smooth utility, based on Exercise 11, is important for technical reasons.


Additional Topics: Ross (1976) introduced the arbitrage pricing theory, a multifactor model of asset returns that, in terms of expected returns, can be thought of as an extension of the CAPM. In this regard, see also Bray (1994a), Bray (1994b) and Gilles and LeRoy (1991). Balasko and Cass (1986) and Balasko, Cass, and Siconolfi (1990) treat equilibrium with constrained participation in security trading. See, also, Hara (1994).

Debreu (1972) provides a notion of regular preferences that substitutes for the existence of a negative-definite Hessian matrix of each agent’s utility function at the equilibrium allocation. For more on regular preferences and the differential approach to general equilibrium, see Mas-Colell (1985) and Balasko (1989). Kreps (1988) reviews the theory of choice and utility representations of preferences. For Farkas’s and Stiemke’s Lemmas, and other forms of the Theorem of the Alternative, see Gale (1960).

Arrow and Debreu (1954) and, in a slightly different model, McKenize (1954) are responsible for a proof of the existence of complete-markets equilibria. Debreu (1982) surveys the existence problem. Standard introductory treatments of general equilibrium theory are given by Debreu (1959) and Hildenbrand and Kirman (1989). In this setting, with incomplete markets, Polemarchakis and Siconolfi (1993) address the failure of existence unless one has a portfolio \( \theta \) with payoff \( D^\top \theta > 0 \). Geanakoplos (1990) surveys other literature on the existence of equilibria in incomplete markets, some of which takes the alternative of defining security payoffs in nominal units of account, while allowing consumption of multiple commodities. Most of the literature allows for an initial period of consumption before the realization of the uncertain state. For a survey, see Magill and Shafer (1991). Additional results on incomplete markets equilibrium include those of Araujo and Monteiro (1989), Berk (1997), Boyle and Wang (1999), and Weil (1992).

For related results in multiperiod settings, references are cited in the Notes of Chapter 2. The superdifferendiability result of Exercise 10(C) is due to Skiadas (1995).
Hellwig (1996), Mas-Colell and Monteiro (1996), and Monteiro (1996) have shown existence of equilibrium with a continuum of states. Geanakoplos and Polemarchakis (1986) and Chae (1988) show existence in a model closely related to that studied in this chapter. Grodal and Vind (1988) and Yamazaki (1991) show existence with alternative formulations. With multiple commodities or multiple periods, existence is not guaranteed under any natural conditions, as shown by Hart (1975), who gives a counterexample. For these more delicate cases, the literature on generic existence is cited in the Notes of Chapter 2.

The binomial option-pricing formula of Exercise 11 is from an early edition of Sharpe (1985), and is extended in Chapter 2 to a multiperiod setting. The hint given for the demonstration of the Riesz representation exercise is condensed from the proof given by Luenberger (1969) of the Riesz-Fréchet Theorem: For any Hilbert space $H$ with inner product $(\cdot | \cdot)$, any continuous linear functional $F : H \to \mathbb{R}$ has a unique $\pi$ in $H$ such that $F(x) = (\pi | x), x \in H$. The Fixed Point Theorem of Exercise 4 is from Kakutani (1941).