Rational Equilibrium Asset-Pricing Bubbles in Continuous Trading Models

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We study rational equilibrium asset-pricing bubbles in an economic environment in which agents are allowed to trade continuously, including as special cases some models from financial economics. For positive net supply assets, we present new necessary and sufficient conditions for the absence of bubbles in complete and incomplete markets equilibria with several types of borrowing constraints. For zero net supply assets, including financial derivatives with finite maturities, we show that bubbles can generally exist and have properties different from their discrete-time, infinite-horizon counterparts. We introduce a probabilistic approach to studying bubbles, generalizing analogs of existing results in the discrete-time bubbles literature. Journal of Economic Literature Classification Numbers: D50, G12, G13. © 2000 Academic Press

1. INTRODUCTION

It is often argued that the absence of arbitrage guarantees that the equilibrium market price of any asset will equal its fundamental value, defined as the appropriately discounted present value of the asset’s future dividends. For a finite-dimensional economy, this is implied by the

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Fundamental Theorem of Asset Pricing, which asserts the equivalence between the absence of arbitrage, the existence of a probability measure under which every asset's discounted price is a martingale, and the existence of an optimum for a hypothetical agent who prefers more to less. For certain infinite-dimensional economies, this version of the Fundamental Theorem of Asset Pricing does not hold, suggesting that equilibrium prices in these economies may deviate from their fundamental values; i.e., equilibrium prices may have bubbles. For discrete-time, infinite-horizon economies, the possibility that bubbles can exist has been extensively studied; see, for example, [2, 9, 13, 17, 23, 32, 36].

We study rational asset-pricing bubbles in a finite-horizon continuous-trading economy in which agents face wealth constraints. Wealth constraints will be important for ruling out arbitrage opportunities in the form of doubling strategies, which are essentially sequences of bets that win for sure in finite time by doubling up after a loss. A model of continuous trade in which agents prefer more to less requires some assumption to make doubling strategies infeasible, for otherwise there can be no equilibrium. Wealth constraints are effective in ruling out doubling strategies [11, 15], but they still allow suicide strategies, which throw away wealth for sure by essentially running a doubling strategy in reverse. Suicide strategies are sometimes regarded as economically irrelevant because an agent who prefers more to less would never choose a suicide strategy as part of an optimal plan. What we show in this paper is that asset prices themselves may contain suicide strategies in the form of bubbles—asset prices can be decomposed into a fundamental value and a suicide strategy—and they will be economically relevant because agents cannot exploit the apparent arbitrage given a wealth constraint.

The continuous-trade economy is a rich new environment for studying bubbles and yields several new economic insights. The fact that asset-pricing bubbles can exist in continuous-trade economies has not been previously studied partly because of a misconception that some notion of the absence of arbitrage should imply the existence of an equivalent change of probability that makes every discounted asset price a martingale, a result which would rule out all asset-pricing bubbles. However, in a continuous-trade economy with wealth constraints, a price system can represent a potential equilibrium—and hence will be arbitrage-free—even if no such change of measure exists. Moreover, wealth constraints can drive a wedge
between the market price of an asset and the amount of initial wealth needed to replicate the payouts of an asset. In particular, our results demonstrate that some standard asset-pricing techniques, such as computing replicating costs or using risk-neutral measures, can provide misleading conclusions for economies in which wealth constraints are important.

Our results typically apply to one of two types of assets, either those in positive net supply or those in zero net supply. For assets in positive net supply, we present several conditions (some of which are new) that rule out bubbles on equilibrium prices of these assets under varying assumptions about market completeness and wealth constraints. As Santos and Woodford [32] do for discrete-time economies, we conclude that absence of bubbles for assets in positive net supply is fairly robust to various economic assumptions. However, we will point out several fundamental economic differences between our conditions and those currently in the literature.

For assets in zero net supply, there are no theoretical conditions that rule out asset-pricing bubbles. These bubbles share some characteristics of their discrete-time counterparts, but they also have some unique properties. For example, continuous-time bubbles can burst with probability one by any deterministic time prior to the end of trade, and they may be uniformly bounded across states of nature. These properties are important because they permit assets with bounded prices and finite lifespans, like bonds and put option contracts, to have bubbles. This suggests that asset-pricing bubbles on zero net supply assets may be more ubiquitous and more difficult to detect in empirical data than has been previously thought, and that obtaining clear econometric restrictions from this class of models may be difficult given the resultant wide-ranging price indeterminacy. Our results suggest that some of the current intuition about bubbles follows more from the common mathematical framework of existing models than from pure economics.

In our study of asset-pricing bubbles, we use probabilistic techniques from martingale theory. Using these techniques helps us to simplify some of our analysis and proofs; indeed, we use them to present a new necessary and sufficient condition for the absence of bubbles on positive net supply assets for general wealth constraints and possibly incomplete markets. We conjecture that these techniques could be used to develop analogs of some

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4 One intuition is that the former property follows from a remapping of a discrete-time economy into one in which trading opportunities are more and more frequent near a finite horizon. This intuition is incomplete, however, because it would not explain how a bubble can exist on an asset that matures before the limit point (e.g., at the midpoint of the trading interval). A better intuition will follow from the fact that in discrete time the standard transversality condition on wealth—although enforced only at the horizon—prevents finite maturity bubbles, but there is no analog of a transversality condition in continuous-trade models (i.e., wealth constraints must be enforced continuously to be effective).
of our results for the discrete-time case and to make the proofs of some existing results more transparent.

We state our main assumptions in Section 2 and our notion of equilibrium in Section 3. Section 4 presents our main results for complete markets and a particular choice of wealth constraint, and Section 5 presents our results for markets that are possibly incomplete and for general wealth constraints. In Section 6, we present examples that highlight the importance of our assumptions, and the Appendix contains our proofs.

2. THE MODEL

2.1. Primitives

We consider an economy in which all economic activity takes place on a finite time interval \([0, T]\) or \([0, T]\). (We do not formally treat the case where \(T\) is infinite, but with minor modifications our analysis would include this case.) Economic uncertainty is described by a probability space \((\Omega, \mathcal{F}, P)\), on which is defined a standard \(d\)-dimensional Brownian motion \(Z\). Information arrival is represented by a filtration \(\{\mathcal{F}_t\}_{t\in[0,T]}\), which we take to be the filtration generated by the Brownian motion \(Z\) and augmented by all \(P\)-nullsets. This model is standard for many applications in financial economics.

The economy contains \(I\) individuals, each of whom is completely described by preferences over intermediate and terminal consumption, private endowments, and initial endowments of securities. For a given agent \(i\), an intermediate consumption plan \(c^i = \{c^i(t) : t \in [0, T]\}\) is admissible if it is nonnegative, progressively measurable, and finite almost surely, meaning that \(c^i(t, \omega) < \infty\) almost surely on \([0, T] \times \Omega\). Cumulative consumption at time \(t\) is given by \(\int_0^t c^i(s) \, ds\). A terminal consumption plan \(C^i\) is admissible if it is \(\mathcal{F}_T\)-measurable and if it is \(P\)-almost surely non-negative and finite.

Preferences for intermediate and terminal consumption for agent \(i\) are represented by a real-valued utility function \(U^i(c, C)\). Although we always assume that agents prefer more consumption to less, we sometimes interpret this in different ways. Each of our results will apply one of three cases: that in which (i) preferences are strictly increasing only in intermediate consumption (i.e., \(U^i\) is monotonic only in its first argument), (ii) preferences are strictly increasing only in terminal consumption (i.e., \(U^i\) is monotonic only in its second argument), or (iii) preferences are strictly increasing in both intermediate and terminal consumption (i.e., \(U^i\) is monotonic in both arguments). We make the standard assumption that agents are indifferent to consumption plans that are almost everywhere identical.
In our general analysis, we permit but do not require $U'$ to have additional properties such as convexity, concavity, continuity, and the like. We sometimes assume that preferences have the von Neumann–Morgenstern representation

$$U'(c, C) = E\left[ \int_0^T u'(c^i(t), t) \, dt + V'(C^i) \right],$$  \hspace{1cm} (1)$$

where the functions $u': \mathbb{R}_+ \times [0, T) \to \mathbb{R}$ and $V': \mathbb{R}_+ \to \mathbb{R}$ are almost surely continuous. Depending on whether $u'$, $V'$, or both are nontrivial, preferences will be strictly monotonic only over intermediate consumption, strictly monotonic only over terminal consumption, or strictly monotonic over both types of consumption.

Each agent $i$ receives a finite nonnegative rate of private endowment $e^i(t)$ over the time interval $[0, T)$ and a finite nonnegative private terminal endowment $e^i$ at time $T$. The corresponding cumulative endowment process is

$$\int_0^T e^i(s) \, ds + e^i \mathbf{1}_{\{t = T\}}(t),$$

where $\mathbf{1}_{\{t = T\}}(t)$ denotes the indicator function which equals 1 when $t = T$ and equals zero otherwise. The process $e^i(t)$ is assumed to be progressively measurable, and $e^i$ is assumed to be $\mathcal{F}_T$-measurable. To avoid unnecessary complications, we assume that when preferences are monotonic only over intermediate consumption, then $e^i$ equals zero, and when preferences are monotonic only over terminal consumption, then $e^i$ equals zero. Agents also receive initial security endowments, which are described in the next section.

Remark 2.1. We sometimes ignore the superscript $i$ when we refer to a particular agent if there is no ambiguity.

2.2. The Financial Market

The asset market consists of $K + 1$ securities. The first security represents locally riskless borrowing and lending. Its price $B$ represents the value of a unit investment in the asset from time 0 to $t$. We assume that $B$ is a continuous predictable finite variation process with $B(0) = 1$. Each agent is endowed with $x_i$ units of the riskless asset, and we assume that

$$\sum_{i=0}^{I} x_i = 0,$$

meaning that the riskless asset is in zero net supply.
The remaining $k = 1, ..., K$ assets are risky and represent claims to risky dividend flows. The cumulative dividend process $D_k$ for a given risky asset $k$ is described by

$$D_k(t) = \int_0^t \delta_k(s) \, ds,$$

where $\delta_k$ is a finite nonnegative adapted process. The corresponding gains process $G_k$ is given by the continuous semimartingale

$$G_k(t) = S_k(t) + D_k(t),$$

where $S_k(t)$ denotes the nonnegative and adapted price of the $i$th risky security at time $t$. We interpret $S_k(T)$ as the liquidating dividend, and we assume that it is finite and nonnegative. We sometimes use the discounted gains process $\tilde{G}_k$, which is defined by

$$\tilde{G}_k(t) = S_k(t) - B(t) + \int_0^t \delta_k(s) \, ds.$$  

(2)

To simplify notation, we use the vector notation $G(t) = [G_1(t), ..., G_K(t)]^T$, $S(t) = [S_1(t), ..., S_K(t)]^T$, and $D(t) = [D_1(t), ..., D_K(t)]^T$ to denote the gains processes, the risky asset prices, and the cumulative dividends, respectively. (Superscript $T$ denotes transpose.)

Each agent $i$ is initially endowed with $\pi^i_k$ units of risky asset $k$. The net supply $\pi_k$ of asset $k$ is described by

$$\pi_k = \sum_{i=0}^f \pi^i_k,$$

and we assume that each $\pi_k$ is nonnegative. Using vector notation, we can represent each agent’s asset endowment by $\pi^i = [\pi^i_1, ..., \pi^i_K]$ and the aggregate supply of the assets by $\pi = [\pi_1, ..., \pi_K]$.

Agents manage their asset portfolio continuously through time to finance consumption. Agent $i$'s wealth at time $t$ is given by

$$W_i(t) = \pi^i(t) B(t) + \pi^i(t) S(t),$$  

(3)

where $\pi^i(t) = [\pi^i_1(t), ..., \pi^i_K(t)]$ and $\pi^i$ are adapted $\mathbb{R}^K$-valued and $\mathbb{R}$-valued processes, respectively. Portfolio strategies are assumed to be self-financing, meaning that

$$W_i'(t) = W_i(0) + \int_0^t (\pi^i(s) - \pi^i(s)) \, ds + \int_0^t \alpha^i(s) \, dB(s) + \int_0^t \pi^i(s) \, dG(s).$$  

(4)
and $C_i' \leq W^i(T) + \varepsilon'$, where $W^i(0) = \pi' + \pi'S(0)$ is the market value of individual $i$'s initial securities endowments.

To ensure that trading gains are well defined, we require admissible trading strategies to be progressively measurable and to satisfy conditions that make gains from trade well defined as stochastic integrals. These conditions are standard integrability restrictions assumed in models of continuous trade. In the notation of [28, Chap. IV], we require each agent to choose strategies from the set $L(B, G)$.

3. VIABILITY AND EQUILIBRIUM

We focus only on prices that represent potential models of competitive equilibrium, so we now develop the structure necessary to identify such prices. We first describe the consumption and investment choice problem for a given agent. In general, we assume each agent $i$ must honor a wealth constraint described by

$$\left(\forall t \in [0, T]\right) W^i(t) \geq -L^i(t), \quad P\text{-almost surely,} \quad (5)$$

where $L^i$ is a finite and progressively measurable process. We interpret (5) as a constraint that a monitor, possibly a regulatory agency or other trading partners, could enforce given rational expectations about prices and endowments. In particular, $L^i$ may depend on equilibrium prices and future endowments, but it may not depend on consumption plans or trading strategies. Later we sometimes make special assumptions about $L^i$.

Here is the choice problem for agent $i$.

**Problem 3.1.** Choose admissible consumption plans $(c^i, C^i)$ and an admissible trading strategy $\pi^i$ to maximize utility $U^i(c^i, C^i)$ subject to $W^i(T) + \varepsilon \geq C^i$, the dynamic budget constraint (4), and the wealth constraint (5).

A price system for which there exists a solution to Problem 1 for some hypothetical agent who takes prices as given is viable (in the sense of [14]) as a model of economic equilibrium. Properties of viable price systems are important because they tell us what our economic models predict for asset pricing and consumption-investment behavior.

Under additional assumptions we will make, the wealth constraint (5) has the dual role of enforcing market clearing and making doubling strategies infeasible. While there are other constraints that sometimes make doubling strategies infeasible (e.g., short-sales constraints, a restriction to
simple strategies, $L^p$-integrability restrictions), we study a class of constraints that includes (but is not limited to) those which (i) are implementable in the sense described above, (ii) do not introduce market imperfections beyond that of making doubling strategies infeasible, and (iii) are well defined for all the potential models of equilibrium we consider.

To determine whether a price system is viable, we need to define an appropriate notion of arbitrage given our assumptions. Here is the definition we will use. We denote by $\ell$ the Lebesgue measure on $[0, T]$.

**Definition 3.1.** An arbitrage opportunity is an admissible consumption plan $(c, C)$ financed by an investment strategy $\pi$ with the property that the corresponding wealth process $W$

1. requires no endowments ($W(0) = 0$, $c' \equiv 0$, and $c' \equiv 0$) and permits nonnegative consumption ($\ell \times P(c \geq 0) = 1$ and $P(C \geq 0) = 1$),

2. generates positive consumption with positive probability ($P(\int_0^T c(t) \, dt > 0) > 0$ if agents’ preferences are increasing in intermediate consumption or $P(C > 0) > 0$ if agents’ preferences are increasing in terminal consumption), and

3. is pathwise nonnegative ($\forall t \in [0, T], W(t) \geq 0$, almost surely).

The economic spirit of arbitrage is that it is inconsistent with viability because adding it as a net trade to any feasible consumption plan improves utility and continues to be feasible. As stressed in [26], remaining within this spirit requires an arbitrage strategy to maintain nonnegative wealth when agents face wealth constraints like (5), for otherwise adding an arbitrage as a net trade to some consumption plans will be infeasible. Our notion of arbitrage is the appropriate one given our assumptions about choice because it is equivalent to the existence of an optimum for a hypothetical agent who prefers more to less (see [26]).

We always work with prices that are arbitrage-free; however, this by itself is a very weak restriction for continuous-trade models. For example, Loewenstein and Willard [26] show that a price system may be arbitrage-free even if there is no reasonable sense in which state prices are positive. In this paper, we focus only on equilibria with positive state prices, so we make the following additional assumption.

**Assumption 3.3.1.** Assume there exists a local martingale $M$ with $M(0) = 1$ such that, for each $k = 1, \ldots, K$, the deflated price process defined by

$$M(t) \frac{S_k(t)}{B(t)} + \int_0^t M(s) \frac{\delta_k(s)}{B(s)} \, ds$$

(6)
is a local martingale. Moreover, assume that the corresponding process $\rho$ defined by

$$\rho(t) = \frac{M(t)}{B(t)}$$

is positive (i) on $[0, T)$ if agents’ preferences are monotonic only over intermediate consumption or (ii) on $[0, T]$ if agents’ preferences are monotonic over terminal consumption.

Conditions on preferences that would justify Assumption 3.3.1 as a consequence of optimality are given in [26]. We call $\rho$ a state-price representation if it has these properties. Unlike the familiar state-price density in financial economics, a state-price representation is not generally a density because it may not correspond to any equivalent change of probability. A state-price representation will be a state-price density if and only if $M$ is a positive martingale on $[0, T]$, in which case $E[M(T)] = 1$. However, since $M$ is generally a nonnegative local martingale, it is a supermartingale and $E[M(t)] \leq 1$ for all $t$. When $M$ fails to be a martingale, it must be that $E[\sup_{t \in [0, T]} M(t)] = \infty$, so in some sense state prices are quite large on a small probability set. One of our purposes in this paper is to study the economic significance of this fact (see in particular Section 4.6).

We now need a definition of equilibrium. For this, we denote the aggregate intermediate and terminal endowments by

$$e(t) = \sum_{i=1}^{I} e^i(t) + \bar{\pi}(t) \quad \text{and} \quad e = \sum_{i=1}^{I} e^i + \bar{\pi}S(T),$$

respectively. Note that these include net claims to dividends.

**Definition 3.2.** An equilibrium consists of

1. a vector of risky asset prices $S$ and a continuous finite-variation process $B$ for which there exists a corresponding state-price representation $\rho$ which is positive on $[0, T]$ (resp., $[0, T]$) if agents have preferences monotonic over terminal consumption (resp., if agents have preferences monotonic only over intermediate consumption), and

2. a set of admissible consumption-investment plans $\{(c^i, C^i, \pi^i, \bar{\pi}^i); i = 1, \ldots, I\}$ such that (a) for each agent $i$, the consumption-investment

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5 A process $M$ is a local martingale if there is a sequence of stopping times $\tau_n \uparrow T$, $P$-a.s., such that $M(\tau_n \wedge t)$ is a martingale. A martingale is always a local martingale, but the reverse is not true. A nonnegative local martingale is a supermartingale and a nonnegative local martingale with $E[\sup_{t \in [0, T]} M(t)] < \infty$ is a martingale. See [21] for additional discussion.
strategy \((c', C', \pi', x')\) solves Problem 1 for a given \(L'\) in the wealth constraint (5) and (b) consumption and asset markets clear, meaning that
\[
\sum_{i=1}^{I} c'(t) = e(t), \quad \sum_{i=1}^{I} C' = \epsilon.
\]
and
\[
\sum_{i=1}^{I} \pi'(t) = 0, \quad \text{and} \quad \sum_{i=1}^{I} \pi'(t) = \bar{\pi}.
\]
These conditions are understood to hold almost everywhere in the appropriate sense.

4. BUBBLES IN COMPLETE MARKETS EQUILIBRIA

We begin by taking as given a complete markets equilibrium and assuming a particular form of the wealth constraint (5). While not all of our results in this section depend on these assumptions, making them here helps to clarify the economics of bubbles in continuous-trade economies while avoiding the extra notation and structure needed for studying more general economies. We study topics specific to incomplete markets and general wealth constraints in Section 5.

4.1. Complete Markets Equilibria

We first define what we mean by a complete markets equilibrium. Let \((S, B)\) be a viable price system, and let \(M\) be a local martingale positive on \([0, T]\) that makes deflated prices (6) a local martingale. By Ito’s lemma, the corresponding discounted gains process \(G\) in (2) has the representation
\[
dG_k(t) = \mu_k(t) dt + \sigma_k(t) dZ(t)
\]
for some real-valued process \(\mu_k\) and some \(\mathbb{R}^d\)-valued process \(\sigma_k\).

We now make a complete-markets assumption that says that the volatility matrix \(\sigma(t) = [\sigma_1(t), \ldots, \sigma_d(t)]^T\) spans \(\mathbb{R}^d\) at almost every time \(t\). This condition is often used to guarantee that a given admissible consumption plan is financed by an admissible trading strategy, a result which we formalize later. Recall that \(\ell\) denotes the Lebesgue measure on \([0, T]\).

**Assumption 4.4.1.** For all \(x \in \mathbb{R}^d\), there exists \(\pi \in \mathbb{R}^k\) such that \(\pi \sigma(t, \omega) = x\), \(\ell \otimes P\)-a.s.
Next, we make an assumption about the form of the wealth constraint (5). In this section, we always assume that \( L_i \) is given by

\[
L_i(t) = E\left[\rho(T) \varepsilon + \int_t^T \rho(s) \varepsilon'(s) \, ds \mid \mathcal{F}_t\right] + X'B(t). \tag{8}
\]

The first component of \( L_i \) reflects the present value of future endowments, possibly permitting the agent to borrow against future anticipated income. The second component permits additional borrowing if \( X_i > 0 \) or restricts borrowing against anticipated future income if \( X_i < 0 \). This constraint can be enforced by a monitor who has rational expectations about future endowments and equilibrium prices.

To make this constraint effective, we assume that the value of agents’ endowments are finite in the following sense.

**Assumption 4.4.2.** The value of the aggregate endowment, defined as

\[
E\left[\rho(T) \varepsilon(T) + \int_0^T \rho(s) \varepsilon(s) \, ds\right],
\]

is finite.

Our first lemma demonstrates that a state-price representation is useful for defining a static form of an agent’s budget constraint in that a state-price representation is a stochastic discount factor (or pricing kernel) for future consumption.

**Lemma 4.1.** Let \( \rho \) be a given state-price representation for an economy in which Assumption 4.4.2 holds. Suppose that each agent \( i \) must honor the wealth constraint (5) with \( L_i \) given by (8). Then any admissible consumption and investment plan \((c_i, C_i, \pi_i, x')\) satisfying the dynamic budget constraint (4) also satisfies the static budget constraint

\[
E\left[\int_0^T \rho(s)(c'(s) - e'(s)) \, ds + \rho(T)(C' - c')\right] \leq W(0) + X'(1 - E[M(T)]). \tag{9}
\]

The key feature of this static budget constraint is that it is nonlinear when \( X_i \) is nonzero and \( E[M(T)] < 1 \). As discussed in [26] and in Proposition 4.3, when \( X_i \) is positive, the constant part provides a measure of the value of a limited arbitrage opportunity taken to its maximum scale given the wealth constraint. An agent with access to credit will always use a limited arbitrage opportunity as part of an optimal strategy, and the constant \( X'(1 - E[M(T)]) \) reflects the resulting higher effective wealth.
available to the agent. However, these limited arbitrages require negative wealth, and “arbitraging” them away will not be possible given a wealth constraint. When $X'$ is negative, the constant provides a measure of the reverse of this limited arbitrage, or a suicide strategy, which the agent would be forced to hold in the presence of a bubble on the riskless asset (see Example 6.3).

Remark 4.1. While in this section we are assuming at markets are complete—and hence that the state-price representation is unique. Lemma 4.1 also applies to incomplete markets with multiple state-price representations. In the latter case, inequality (9) would hold for every state-price representation, a fact we use in our later analysis of general markets.

Our next result, which follows directly from Lemma 4.1, shows that there are no arbitrage opportunities or doubling strategies.

Corollary 4.1. Under the assumptions of Lemma 4.1, there are no arbitrage opportunities (even if $P(\rho(T) = 0) > 0$). In particular there are no feasible doubling strategies.

Our next result provides a converse to Lemma 4.1: in a complete market, any consumption plan satisfying the static-budget equation can be financed by a feasible trading strategy.

Lemma 4.2. Let $\rho$ be a given state-price representation for an economy in which Assumptions 4.4.1 and 4.4.2 hold. Then (i) if $\rho$ is strictly positive on $[0, T]$, then there exists a feasible trading strategy that finances any given admissible consumption plan $(c', C')$ that satisfies (9) with equality, and (ii) if $\rho$ is strictly positive on $[0, T)$, then there exists a feasible trading strategy that finances any given admissible intermediate consumption plan $c'$ that satisfies (9) with equality.

We now show that any state-price representation can be used to construct an equilibrium for a single agent economy with carefully chosen endowments.

Lemma 4.3. Let $(S, B)$ be a price system for which there exists a state-price representation $\rho$. Assume that $\rho$ is strictly positive on $[0, T)$ and that Assumptions 3.3.1, 4.4.1, and 4.4.2 hold. Then we can choose a hypothetical agent who prefers more to less and appropriate endowments so that $(S, B)$ represent equilibrium prices in an economy populated only by this agent.

This result is important because it says that, given our assumptions about wealth constraints, assuming existence of a state-price representation implies the existence of some economy which supports this choice as a
competitive equilibrium. For this reason, we simply work with a given state-price representation for the remainder of this section.

4.2. Complete-Markets Asset-Pricing Bubbles

We now present a precise definition of an asset-pricing bubble in our setting. Our definition is consistent with the traditional definition of an asset-pricing bubble used, for example, in [9, 23, 32, 36].

Recall that the deflated risky asset prices in (6) are nonnegative local martingales, and because of this they are also supermartingales [21, Problem 1.5.19]. This implies that

\[
\rho(t) S_k(t) + \int_0^t \rho(s) \delta_k(s) \, ds \geq E \left[ \rho(T) S_k(T) + \int_0^T \rho(s) \delta_k(s) \, ds \mid \mathcal{F}_t \right]
\]

for each asset \( k \). On \([0, T)\), we can equivalently write

\[
S_k(t) \geq E \left[ \rho(T) S_k(T) + \int_t^T \rho(s) \delta_k(s) \, ds \mid \mathcal{F}_t \right] / \rho(t), \tag{10}
\]

which says that the current price of the risky asset \( k \) must be at least as large as its expected present value of its future dividends.

What is important about inequality (10) is that it may be strict. Thus we can think of the asset price as being composed of two parts: on \([0, T)\), there is a fundamental value given by

\[
f_k(t) = E \left[ \rho(T) S_k(T) + \int_t^T \rho(s) \delta_k(s) \, ds \mid \mathcal{F}_t \right] / \rho(t),
\]

and a bubble component given by

\[
b_k(t) = S_k(t) - f_k(t).
\]

If inequality (10) is strict at some time \( t < T \), then we say that asset \( k \)'s price has a bubble. A bubble therefore represents the amount by which the equilibrium price of an asset exceeds the present value of its payouts.

\[\text{A different definition based on charges in the dual pricing operator is used in [13]. The difference between that definition and the one we use is subtle. We do not address this, but see [17].}\]

\[\text{The definition of the bubble's value at time } T \text{ can be subtle because } \rho(T) \text{ may equal zero with positive probability if preferences are monotonic only over intermediate consumption. A natural choice would be to use the appropriate limit on the set } \rho(T) > 0 \text{ and } S_k(T^-) \text{ on the set } \rho(T) = 0. \text{ Notice that } S_k(T) = S_k(T^-) \text{ from our continuity assumptions.}\]
We can also define a bubble on the price of the riskless asset in a similar fashion. This bubble, which we denote by \( b_0 \), is given on \([0, T)\) by
\[
b_0(t) = B(t) - \frac{\mathbb{E}[\rho(T) B(T) | \mathcal{F}_t]}{\rho(t)} = \frac{M(t) - \mathbb{E}[M(T) | \mathcal{F}_t]}{\rho(t)}
\]
and \( b_0(T) = 0 \). A bubble exists on the riskless asset if and only if \( M \) is a local martingale that is not a martingale. Note also that having a bubble on the riskless asset is a necessary condition for the constant part of the pricing rule defined in (9) to be nonzero.

We now present some examples of continuous-time bubbles.

4.3. Examples of Asset-Pricing Bubbles

This section contains several examples of asset-pricing bubbles which are consistent with the existence of equilibrium when agents must honor wealth constraints. We begin by presenting an example of a local martingale that is not a martingale and some proofs.

Example 4.1. Let \( s \) be some deterministic time in the interval \((0, T]\), and let \( Z \) be a one-dimensional Brownian motion. Let \( \mu, r, \) and \( \sigma \) be positive constants with \( \mu > r \), and define the process \( \eta_s \) by
\[
\eta_s(t) = 1_{[0, s)}(t) \exp \left( - \int_0^t \frac{\mu - r}{\sigma(s - u)} \, dZ(u) - \frac{1}{2} \int_0^t \frac{(\mu - r)^2}{\sigma^2(s - u)} \, du \right).
\]
Then \( \eta_s \) is a continuous nonnegative local martingale with \( \eta_s(t) = 0 \) almost surely for all \( t \in [s, T] \). In particular, \( \eta_s \) is not a martingale. That \( \eta_s \) is continuous follows from [29, Exercise IV.3.25]. Because we have \( \mathbb{E}[\eta_s(t)] = 0 \) on \([s, T]\) and \( \eta_s(0) = 1 \), the local martingale \( \eta_s \) cannot be a martingale.

Here is our first example of an equilibrium asset-pricing bubble. In this example, we use the local martingale \( \eta_s \) to put a bubble on the price of a redundant security.

Example 4.2. Consider an economy that consists of a riskless asset \( B \) and two risky assets with prices denoted by \( S_1 \) and \( S_2 \). The risky assets pay no intermediate dividends, and the market is dynamically complete when trading is restricted to the assets \( B \) and \( S_1 \). Let \( \rho \) be a state-price representation that makes \( \rho S_1 \) a martingale (which implies that \( b_1(t) \equiv 0 \)). Suppose that \( S_2 \) equals
\[
S_2(t) = S_1(t) + \frac{\eta_s(t)}{\rho(t)},
\]
where $\eta_s$ is the local martingale defined in (11) with $0 < s \leq T$. Then $S_1$ and $S_2$ pay the same dividends, and $\rho S_2$ is a local martingale. Thus $\rho$ is also a state-price representation for the price system $(B, S_1, S_2)$, and there is a bubble on the price of the second risky security since $b_2(t) = \eta_s(t)/\rho(t)$.

In Example 4.2, $S_2$ represents a tracking portfolio which mimics the payouts of $S_1$. Because $\rho$ is a state-price representation for the price system $(B, S_1, S_2)$, we see that having a bubble on $S_2$ is consistent with the existence of equilibrium. In such an equilibrium, an agent would like to sell short $S_2$ to fund purchases of $S_1$, but cannot do so at an arbitrary scale without violating a finite wealth constraint.

We now provide an example of a bubble on the riskless asset.

**Example 4.3.** Suppose that $B(t) = \exp(rt)$ for some nonnegative constant $r$ and that there is also a single risky asset paying no intermediate dividends. Suppose that the risky asset’s price has the representation $dS/S = \mu dt + \sigma(T-t)^{1/2}dZ(t)$, where $\mu$ and $\sigma$ are constants. The local price of risk, $\theta$, is given by

$$
\theta(t) = \frac{\mu - r}{\sigma(T-t)^{1/2}},
$$

and the local martingale $M$, defined by $M(t) = \eta_p(t)$, is positive on $[0, T)$ and makes the corresponding discounted gains process a local martingale. Hence the price system is arbitrage-free. In this example, $M(T)$ equals zero almost surely, so the entire price of the riskless asset consists of a bubble; i.e., $b_0(t) = B(t)$.

In Example 4.3, $\rho$ is a state-price representation for $(B, S)$ even though $\rho(T) = 0$. In [26], we show that a price system with this property is viable for a class of preferences which includes those monotonic only over intermediate consumption.

### 4.4. General Properties of Finitely Lived Asset-Pricing Bubbles

Here are some general properties of continuous-time asset-pricing bubbles.

**Proposition 4.1.** Let $(B, S)$ be a viable price system, and let $\rho$ be a state-price representation. Assume $\rho(T)$ is positive almost surely. Then the following are properties of any equilibrium asset-pricing bubble (including those on the riskless asset):

1. A bubble is almost everywhere nonnegative, and it is almost everywhere equal to zero after the time at which it first hits zero. Moreover,
it is possible for a bubble to burst with probability one by any deterministic
time in (0, T).

2. A bubble \( b_k \) is identically equal to zero if \( \rho b_k \) is dominated by a
Class D stochastic process.\(^8\) In particular, this implies that the condition

\[
E\left[ \sup_{u \in [t, T]} \rho(u) b_k(u) \right] = \infty
\]

holds over any interval \([t, T]\) on which \( b_k \) is nonzero with positive prob-
ability.

3. The following inequality holds for any bubble \( b_k \):

\[
P\left[ \sup_{t \in [0, T]} \rho(t) b_k(t) \geq \lambda \right] \leq \frac{1}{\lambda} b_k(0). \tag{12}
\]

4. It is possible for a bubble to be uniformly bounded across paths, even
if it is a bubble on a risky asset.

Note we do not assume that markets are complete in Proposition 4.1, so
its conclusions also hold for bubbles in incomplete markets. (However, we
will need to slightly modify our definition of a bubble for incomplete
markets in Section 5.) The first statement implies that a bubble cannot start
after the initial trading date and that a bubble cannot restart after it has
burst (i.e., after it has hit zero). These are well-known properties of asset-
pricing bubbles in the discrete-time infinite-horizon literature (see, e.g.,
[9]).

The second part of the statement is important because it tells us that
even assets with finite maturities, such as many derivative securities, can
have bubbles. Example 4.2 provides an example in which the bubble can
burst at any time \( s \) prior to the end of the trading horizon. This is our first
indication that continuous-time bubbles are different from discrete-time
ones since a standard backwards induction argument shows that discrete-
time bubbles cannot burst by a deterministic date prior to the end of the
trading horizon.

At first glance, this result may not seem surprising, for one can simply
remap an infinite horizon economy into a finite horizon economy (e.g.,
count each period \( n \) as \( 2^{-n} \) decades) to obtain an economy in which a bub-
ble bursts for sure in finite time. However, our result is different because
the trading horizon is fixed: given a fixed trading horizon of length \( T \), a

\(^8\) A stochastic process \( Y \) is a Class D process if the family of random variables \( \{ Y(\tau) \} \) is
uniformly integrable, where \( \tau \) ranges over all stopping times with values in \([0, T]\). A local
martingale is a martingale whenever it is Class D \([29, \text{Proposition IV.1.7}] \).
bubble can burst at *any* deterministic time \( s \in (0, T) \).\(^9\) The technical difference is really that continuous-time permits us to define bubbles using true \( L^1 \)-local martingales. This is not possible for discrete-time bubbles because every discrete-time \( L^1 \)-local martingale is a martingale [30, Theorem II.54.1]. Said differently, continuous-time bubbles only need to satisfy a local martingale property, in contrast to the usual martingale property for discrete-time bubbles (see [32, Eq. 2.4]).

The second statement of Proposition 4.1 tells us something about the explosiveness of an asset-pricing bubble. For example, one implication is that we cannot uniformly bound \( \rho b_k \) by a constant over its remaining life unless \( b_k = 0 \) over that interval. What is important is that the value of the bubble is large relative to the state-price representation, which can be thought of as representing agents’ intertemporal marginal rates of substitution. Loosely speaking, a bubble must either be very large when state prices are very low, or it must be sufficiently large when state prices are very high. However, inequality (12) tells us that, while \( \rho b_k \) must be large, the probability of observing a path along which it becomes large is small. This is an analog of the statement in [9] that a bubble is “empirically plausible only if... the probability is small that a bubble would become arbitrarily large.”

The final statement is also about the difference between bubbles in discrete-time and continuous-trade economies. For example, it is often argued that assets with known price limits (e.g., bonds or put options) cannot have bubbles, because otherwise the prices would violate these bounds with positive probability. This is not true when continuous trade is permitted. Later we present an example of a bounded bubble and identify a sufficient condition for all bubbles to be unbounded.

4.5. Bubbles on the Prices of Risky Assets

We want to determine whether there exist assumptions about the primitives of the economy that rule out equilibrium asset-pricing bubbles. Our first result along this line is a positive one, saying that, given complete markets and the form of the wealth constraints we are assuming, there cannot exist bubbles on assets in positive net supply if agents are allowed to borrow at least up to the present value of their endowments.

\(^9\)The difference is highlighted by the following example. Let \( \{s_n\} \) be any (possibly infinite) increasing sequence of times in \((0, T)\), and let \( \{b^n\} \) be a sequence of bubbles defined as in Example 4.2 such that \( b^n_s \) bursts deterministically at time \( s_n \) and \( \sum b^n_s(0) < \infty \). Then, for a given redundant asset \( k \) in zero net supply with fundamental value \( f_k \), the market price \( f_k + \sum b^n_k \) will also be an equilibrium price in many cases (as we show later in this section), but there is no simple mapping from a discrete-time economy that would produce such a bubble.
Theorem 4.1. Assume that \( L^i(t) \) is given by (8), and assume that each \( X^i \) is nonnegative. In any equilibrium in which Assumptions 3.3.1, 4.4.1, and 4.4.2 hold, there cannot be any bubbles on any risky asset in positive net supply.

Theorem 4.1 follows from the intuition that bubbles cannot have aggregate wealth effects if agents’ wealth constraints are sufficiently loose. Note that Lemmas 4.1 and 4.2 imply that in a complete market each optimizing agent chooses consumption plans so that (9) holds with equality; i.e.,

\[
E \left[ \rho(T) C^i + \int_0^T \rho(s) c^i(s) \, ds \right] = \pi^i S(0) + \bar{\pi} + E \left[ \rho(T) a^i + \int_0^T \rho(s) e^i(s) \, ds \right] + X^i \left( 1 - E[M(T)] \right). \tag{13}
\]

Summing over all agents and using the market clearing conditions (7a), we get

\[
E \left[ \rho(T) \pi S(T) + \int_0^T \rho(s) \pi \delta(s) \, ds \right] - \pi S(0) = (1 - E[M(T)]) \sum_{i=1}^T X^i. \tag{14}
\]

Because \( \pi \geq 0 \), inequality (10) implies that the left-hand side of (14) is non-positive, but the right-hand side must also be nonnegative because each \( X^i \) is nonnegative and \( E[M(T)] \leq 1 \). Thus having an equilibrium implies that both sides must equal zero; in particular, whenever \( \pi^i_k > 0 \), we have

\[
E \left[ \rho(T) S_k(T) + \int_0^T \rho(s) \delta_k(s) \, ds \right] = S_k(0). \tag{15}
\]

Because each deflated price process (6) is a nonnegative local martingale, equality (15) implies that each deflated price must be a martingale for an asset \( k \) in positive net supply [21, Exercise 1.3.25]; hence, \( b_k(t) \equiv 0 \) for all such assets.

For assets in zero net supply, we may be unable to rule out rational asset-pricing bubbles. Our next result shows that sometimes bubbles can be added or deleted from the prices of these assets without causing wealth effects. Thus there is a sense of price indeterminacy for assets in zero net supply.

Proposition 4.2. Consider an equilibrium in which agents are initially endowed with nonnegative amounts of each asset. Then for any zero net
supply asset except the risk free asset, we can add or delete bubbles without affecting the equilibrium allocation if investment opportunities do not change after doing so.

From (13), we see that adding bubbles in the manner of Example 4.2 or deleting bubbles by substituting fundamental values in place of an original asset does not have wealth effects on any asset \( k \) for which \( \pi'_k = 0 \) for each agent \( i \). If investment opportunities do not change, then agents choose the same consumption allocations regardless of whether an asset in zero net supply has a bubble. In Section 6, we present an example in which deleting a bubble changes the span of the market and, consequently, adversely affects the Pareto optimality of the equilibrium.

Derivatives are important examples of assets held in zero net supply. The traditional approach of derivative pricing is to set it equal to what its no-arbitrage price would be if it were introduced as a redundant asset in zero net supply. Our next result shows that a no-arbitrage price may include a bubble, and it may be nonunique.

**Corollary 4.2.** Given a complete-markets equilibrium, we can add redundant (derivative) securities in zero net supply at prices that either do or do not include bubbles. Moreover, for some economies, trade in these securities can be nontrivial even if there is a bubble.

A common argument against bubbles on redundant assets is that agents would not trade these assets because there would exist a feasible trading strategy that seems superior. However, this is not the case here because the wealth constraint makes superior strategies infeasible. Another argument against bubbles on redundant assets is that agents would be able to arbitrage these bubbles away. Corollary 4.2 indicates that both of these arguments are flawed. Trade in these securities can be nontrivial because they are redundant: adding redundant securities with or without bubbles may introduce an indeterminacy in equilibrium trading strategies but not in equilibrium consumption allocations. Intuitively, an agent can generate a fixed payout using many different feasible strategies given a redundant security, and market clearing will be preserved since counterparties can offset this agent’s positions using appropriate feasible replicating strategies.

For economies like ours, Corollary 4.2 is also a negative result for the traditional technique of pricing derivative securities using a risk-neutral measure or a stochastic discount factor. When agents must honor wealth constraints, prices of derivative securities are essentially indeterminate in this class of continuous-time economies, and naïve applications of these techniques to actual markets where wealth constraints are important might generate inaccurate conclusions. Justification for using these pricing techniques must come from outside this standard competitive continuous-time model.
4.6. Bubbles on the Price of the Riskless Asset

We now state some special properties of bubbles on the riskless asset.

**Proposition 4.3.** Let \((B, S)\) be a viable price system, and let \(p = M/B\) be the corresponding state-price representation. The following are properties of bubbles on the riskless asset:

1. A bubble on the price of the riskless asset exists if and only if 
\[
E[M(T)] < 1.
\]
There cannot be a bubble on the riskless asset if \(p\) is a state-price density.

2. Any bubble on the riskless asset is bounded by the price of the riskless asset. In particular, if \(B\) is bounded, then any bubble on its price is bounded.

3. There is a bubble on the riskless asset if and only if there exists an admissible trading strategy that (i) requires no endowment \((W'(0) = 0, e' \equiv 0, \epsilon' \equiv 0)\), (ii) provides positive consumption \(((c', C') \neq (0, 0))\), and (iii) requires a finite amount of negative discounted wealth with positive probability \(P((\forall t \in [0, T]) W(t) \geq 0 < 1\) and \(P((\forall t \in [0, T]) W(t)/B(t) \geq -\gamma) = 1\) for some constant \(\gamma > 0\)).

The first statement follows from the definition of \(b_0\) and from the fact that a supermartingale \(M\) with \(E[M(T)] = 1\) is a martingale [21, Problem 1.3.25]. The existence of a state-price density would imply the existence of an equivalent change of probability measure with density \(M(T) dP\). (However, it is important to note that it would not guarantee an equivalent change of probability under which every discounted asset price is a martingale (i.e., there might not exist an equivalent martingale measure) because other assets in zero net supply might have bubbles under the change of measure; see Proposition 4.2 and Corollary 4.2.) Note that the absence of bubbles on the riskless asset is not a sufficient condition for \(p\) to be a state-price density. The second statement follows from the definition of \(b_0\).

The admissible trading strategy described in the third statement is called a free snack in [26], and its existence follows from Lemma 4.2. Although a free snack costs nothing and provides profit with no chance of loss, it is not an arbitrage because it requires negative wealth. A wealth constraint like (5) prevents an agent from holding an arbitrarily large position in such strategies; hence, their existence can be compatible with equilibrium. Our next result describes to what extent free snacks may be included in equilibrium trading strategies.
Corollary 4.3. Assume that $X^i \geq 0$ for all $i = 1, \ldots, I$. If there is a bubble on the risk-free asset, then the existence of equilibrium requires $X^i = 0$ for all $i$. Moreover if $\sum X^i > 0$, then the existence of equilibrium requires $E[M(T)] = 1$, and there would be no bubble on the risk-free asset.

Corollary 4.3 captures the intuition that an agent $i$ can take a long position in a free snack only if there is a second agent $j$ who is forced to take the offsetting position. From the second agent’s perspective, this is a suicide strategy and would be inconsistent with optimality if $X^j \leq 0$. Thus if $X^i$ is nonnegative for each agent $i$, then agents cannot undertake free snacks in equilibrium. More precisely the result follows from the fact that both sides of inequality (14) must be zero in equilibrium and that a bubble on the riskless asset exists only if $E[M(T)] < 1$.

Proposition 4.3 says that a bubble on the riskless asset is bounded if the bond price is bounded. We now show that a bubble on the riskless asset can induce bounded bubbles on risky assets.

Example 4.4. Let $(B, S_1)$ be a viable price system, and suppose that the market is complete. Let $\rho = \frac{M}{B}$ be the corresponding state-price representation, and suppose $M$ is strictly positive on $[0, T]$ and $E[M(T)] < 1$. Suppose that $B$ is bounded above, $S_1$ pays no intermediate dividends, and $b_1(t) = 0$. Introduce a redundant risky asset paying no intermediate dividends at the price

$$S_2(t) = \pi S_1(t) + \alpha B(t),$$

where $\pi$ and $\alpha$ are positive constants. There is a bubble on $S_2$, and its value is

$$b_2(t) = \alpha B(t) \left(1 - \frac{E[M(T) \vert \mathcal{F}_t]}{M(t)}\right).$$

Because $B$ is bounded, $b_2$ is bounded.

The intuition of Example 4.4 can be used to construct more elaborate examples; for instance, Loewenstein and Willard [26] use put-call parity to construct a bounded bubble on a put option. However, as our next result shows, bounded bubbles are primarily related to bubbles on the riskless asset.

Proposition 4.4. Let $\rho$ be a state-price representation positive on $[0, T]$, and assume that there are no bubbles on the riskless asset. Then any discounted equilibrium asset-pricing bubble $b_k/B$ cannot be uniformly bounded from above. In particular if $B$ is bounded below away from zero, then $b_k$ itself cannot be uniformly bounded from above.
Proposition 4.4 states an analog of the familiar result that an asset-pricing bubble must have a growth rate higher than that of the locally riskless asset [37]; however, Example 4.4 illustrates that our assumption that there are no bubbles on the riskless asset is important for this result.

Here is the intuition of Proposition 4.4. First, the absence of bubbles on the riskless asset and a strictly positive state-price representation imply that there exists an equivalent change of measure $Q$ with Radon–Nikodym derivative $dQ/dP = M(T)$. Second, we have $E[dQ/dP | \mathcal{F}_t] = M(t)$, and because $pb_k^q$ is a local martingale under the probability measure $P$, the process $b_k/B$ is a local martingale under $Q$ [28]. Thus if $b_k/B$ is dominated by a process that is Class D under $Q$ (e.g., a uniformly bounded process), then $b_k/B$ must be a martingale with constant expectation [29, Proposition IV.1.7]. Since $b_k/B$ is nonnegative and $b_k(T)/B(T) = 0$ almost surely, this is possible only if $b_k \equiv 0$. Hence $b_k/B$ cannot be uniformly bounded under these assumptions, and $b_k$ itself cannot be uniformly bounded if $1/B$ is bounded above.

The following proposition gives a sufficient condition for ruling out the possibility of a bubble on the price of the riskless asset.

**PROPOSITION 4.5.** Suppose that there exists some agent $i$ with von Neumann–Morgenstern preferences (1) with $V^i \equiv 0$. Suppose that $u^i(c, t)$ is continuously differentiable on $(0, \infty)$ in its first argument, is concave, and satisfies Inada conditions (where all these conditions hold almost everywhere). Moreover, assume that this agent has a solution, $\bar{c}$, to Problem 1, and assume there exists a positive constant $a$ such that the marginal utility $u^i_\bar{c}$ satisfies $u^i_\bar{c}(\bar{c}(t), t) \leq a$, almost surely. If the price of the riskless asset $B$ is a Class D process, then there is no bubble on the riskless asset in equilibrium.

In Propositions 4.4 and 4.5, we make assumptions about $B$ that should be determined in equilibrium. We now identify a class of economies for which $B$ will be bounded above and below away from zero in equilibrium.

**COROLLARY 4.4.** Assume that the aggregate endowment $e$ satisfies

\begin{equation}
\frac{de(t)}{dt} = \mu_e(t) dt + \sigma_e(t) dZ(t)
\end{equation}

for some bounded predictable processes $\mu_e$ and $\sigma_e$, and assume that $e$ satisfies $0 < \delta < e(t) \leq \Lambda$ for some positive constants $\delta$ and $\Lambda$. Suppose also that all agents’ preferences are state independent and satisfy the conditions of Proposition 4.5. Then there is no bubble on the riskless asset, and any bubble on a zero net supply asset cannot be uniformly bounded.
These assumptions about preferences hold, for example, if the preferences are of the HARA class. The assumption that $\mu_e$ and $\sigma_e$ are bounded processes is used to put a uniform bound on the equilibrium interest rate, and it can be relaxed somewhat.

5. BUBBLES IN EQUILIBRIUM MODELS

The preceding results apply to complete markets equilibria in which each wealth constraint (5) has the special form (8). In general, however, whether or not an asset market is complete is determined in equilibrium (not by assumption), and we may be interested in other types of borrowing constraints. In this section, we study equilibrium asset-pricing bubbles when markets are possibly incomplete or agents face general borrowing constraints.

5.1. Incomplete Markets Equilibria

We first address a complication for defining a notion of asset-pricing bubbles in incomplete markets. Our definition of equilibrium, presented in Definition 3.2, remains appropriate for the incomplete markets case, as does the choice problem given in Problem 1. However, the defining feature of an incomplete market is that there exist infinitely many state-price representations in an arbitrage-free market.

Because of this, the appropriately discounted present value of future dividends is ambiguous, since it depends on the state-price representation used in calculating it. Moreover, the absence of bubbles might also be ambiguous, because prices might have bubbles relative to one state-price representation and no bubbles relative to another. It is generally not possible to rule out the existence of such bubbles with respect to an arbitrary state-price representation; however, in some cases, we can prove that there exists some state-price representation relative to which there are no bubbles on assets in positive net supply. We present results along this line after we develop some more notation.

Let $\mathcal{K}$ be the set of state-price representations implied by a given price system. For a given state-price representation $\rho \in \mathcal{K}$ and a given asset $k$, we associate a fundamental value $f^\rho_k$ relative to $\rho$ by

$$f^\rho_k(t) = \frac{\mathbb{E}\left[\rho(T) S_k(T) + \int_t^T \rho(s) \delta_k(s) ds \mid \mathcal{F}_t\right]}{\rho(t)}.$$  

Because there should be no confusion, we suppress $\mathcal{K}$'s dependence on the given price system. Karatzas et al. [20] characterize $\mathcal{K}$, but we do not need it for our analysis.
and a bubble component relative to $\rho$, which is given by

$$b_\xi(t) = S_\xi(t) - f_\xi(t).$$

The dependence of $f_\xi$ and $b_\xi$ on $\rho$ indicates their dependence on the given state-price representation. If markets were known to be complete, then $\mathscr{X}$ would be a singleton and the fundamental and bubble values would be unambiguous and would equal the values defined in our complete markets analysis.

5.2. Nonexistence of Bubbles on Positive Net Supply Assets

Recall that the general form of agents’ wealth constraints are given by

$$((\forall t \in [0, T]) W^i(t) \geq -L^i(t)), \quad P\text{-almost surely.}$$

(We have repeated (5) here for convenience.) In this section, we leave $L^i$ unspecified to learn more about the interaction between wealth constraints and the existence of bubbles. However, we need enough structure to ensure that the wealth constraints at least make doubling strategies infeasible. (Recall that there is no hope of equilibrium when agents prefer more to less if there is a feasible doubling strategy.) We also assume a finite value of the aggregate endowment to ensure that $\int_0^T \rho(s) (e'(s) - e'(s)) \, ds$ exists. Here is the assumption we use.

**Assumption 5.5.1.** For a given state-price representation $\rho \in \mathscr{X}$, the collateral value $L^i$ ensures that the process

$$\rho(t) W^i(t) + \int_0^t \rho(s) (e'(s) - e'(s)) \, ds$$

is uniformly integrable from below and for all admissible trading and consumption strategies. Moreover, the value of the aggregate endowment is finite relative to $\rho$:

$$E \left[ \int_0^T \rho(s) e(s) \, ds + p(T) e \right] < \infty.$$

In this section, we will always work with Assumption 5.5.1. It is important to note that wealth constraints are still important for our results in this section through our assumption, even though they do not appear directly.

Our next result says that, under Assumption 5.5.1, each feasible discounted wealth process is a supermartingale and that, by implication, doubling strategies are infeasible.
Lemma 5.1. Let $\rho \in \mathcal{X}$ be a given state-price representation for which Assumption 5.5.1 holds. For any admissible consumption plan $(c^i, C^i)$ financed by a trading strategy satisfying the dynamic budget equation (4), the discounted wealth process in (17) is a supermartingale, and the following inequality holds:

$$E\left[\rho(T)(C^i - \epsilon^i) + \int_0^T \rho(t)(c^i(t) - e^i(t)) \, dt\right] \leq W^i(0).$$

(18)

In particular, no arbitrage opportunities exist (even if $P(\rho(T) = 0) > 0$), and there are no feasible doubling strategies.

Lemma 5.1 is an analog of Lemma 4.1 for possibly incomplete markets. Note that wealth may not satisfy Assumption 5.5.1 if $X^i > 0$ when $L_i$ has the form given in (8).

Here is our main result for this section.

Theorem 5.1. In a given equilibrium, let $\{W^i : i = 1, \ldots, I\}$ denote agents' optimal wealth processes, and let $\rho \in \mathcal{X}$ be a given state-price representation such that Assumption 5.5.1 holds. Then there are no bubbles on assets in positive net supply relative to $\rho$ if and only if each agent's discounted optimal wealth process (17) is a Class D martingale.

Theorem 5.1 says that whether agents' discounted wealth processes are additionally martingales at an optimum is the key for determining whether we can rule out bubbles relative to a given system of state prices. It also provides a necessary and sufficient condition for ruling out bubbles that applies to either complete- or incomplete-markets equilibria, and it does not depend on the exact form of the wealth constraints (provided that Assumption 5.5.1 holds). In Section 6.3, we present an example that helps to illustrate the intuition of Theorem 5.1.

Our next result is useful for showing a relation between absence of bubbles and a statement about the equilibrium allocations.

Corollary 5.1. In a given equilibrium, suppose that for each agent $i$ there exists a state-price representation $\rho^i \in \mathcal{X}$ such that

$$\rho^i(t) W^i(t) + \int_0^t \rho^i(s)(c^i(s) - e^i(s)) \, ds$$

is a Class D martingale. Additionally assume that there exists a positive constant $\gamma$ and some $j \in \{1, \ldots, I\}$ such that

$$(\forall i = 1, \ldots, K) \rho^i(t) \geq \gamma p^j(t).$$

(19)
Then there are no bubbles on assets in positive net supply relative to the state-price representation \( \rho^i \).

Condition (19) places a lower bound on the ratio \( \rho'/\rho^i \). If, for instance, each agent \( i \)'s preferences have a von Neumann–Morgenstern representation (1) in which \( V' \equiv 0 \) and \( u_i'(c'(t, \omega), t) = \bar{\lambda}' \rho'(t, \omega) \), then (19) indicates that there are no bubbles on assets in positive net supply relative to \( \rho^i \). This equilibrium allocation is not wildy different from a Pareto-optimal allocation in the sense that one agent cannot optimally consume a large amount while another agent optimally consumes very little. The assumption of Corollary 5.1 often holds whenever wealth constraints are specified so that they do not bind in equilibrium, but this is not necessary. See [17] for a related result for a discrete-time economy with no uncertainty.

One may suspect a finite supremum value of aggregate endowment over all state-price representations implies the absence of bubbles on positive net supply assets for some state-price representation. This is true, for example, in discrete-time economies [32, Theorem 3.1], but our next result suggests that it may not hold for continuous-trade economies.

**Corollary 5.2.** Suppose that there exists some admissible trading strategy that finances an admissible consumption plan \((c, C)\) with the following properties: \((c, C)\) dominates some positive fraction of the net dividends; i.e., there exists some \( \gamma > 0 \) such that

\[
c(t) \geq \gamma \bar{\pi} \delta(t) \quad \text{and} \quad C \geq \gamma \bar{\pi} S(T)
\]

hold almost surely, and the supremum value of the \((c, C)\) is attained by some state-price representation \( \rho^* \); i.e.,

\[
\sup_{\rho \in \mathcal{X}} E \left[ \rho(T) C + \int_0^T \rho(t) c(t) dt \right] = E \left[ \rho^*(T) C + \int_0^T \rho^*(t) c(t) dt \right] < \infty.
\]

Then there are no bubbles on assets in positive net supply relative to \( \rho^* \).

The primary difference between the discrete-time result of [32] and our Corollary 5.2 is that we additionally need to assume that the supremum value of some dominating consumption plan is attained by some state-price representation.

\[\text{[11]}\] In this case, \( \rho' \) would be related to the minimax equivalent martingale measure of [15] and [7] (except that there is not necessarily an equivalent martingale measure here) and the results of [20].
6. ADDITIONAL EXAMPLES

This section contains examples highlighting the role of some of our assumptions.

6.1. Wealth Effects Caused by Introducing Bubbles

An assumption of Proposition 4.2 is that each agent's initial endowment of the assets is nonnegative. As this example illustrates, this assumption is important because otherwise introducing bubbles can have wealth effects. There are two important conclusions from the example: (i) in some circumstances, there can be indeterminacy not only in asset prices but also in equilibrium (Pareto optimal) allocations, and (ii) there can be a transfer of wealth from an agent initially endowed with a short position in an asset with a bubble (e.g., the issuer with asset) to those with positive endowments of the asset, because the former agent may sometimes need to trade out of the short position to maintain a wealth constraint.

Suppose that there are two agents in the economy, each with von Neumann–Morgenstern preferences (1) described by $u_i(c) = \log(c)$ and $V_i(C) \equiv 0$. The aggregate endowment is described by

$$e(t) = \exp((\mu - \frac{1}{2}\sigma^2) t + \sigma Z(t)),$$

where $\mu$ and $\sigma$ are positive constants. As specified below, each agent will receive a specified constant fraction of this intermediate endowment and no terminal endowment. Each agent $i$ must maintain a wealth constraint for which $L_i$ has the form (8) with $X_i = 0$; i.e., each agent is allowed to borrow only up to the present value of his share of the aggregate endowment. The asset market consists of a risky asset and a riskless bond, each of which is in zero net supply. The risky asset pays intermediate dividends $\delta(t) = e(t)$ and no terminal dividend. The price of the riskless bond is described by

$$B(t) = 1 + \int_{t_0}^t r(s) B(s) \, ds.$$  

Each agent’s initial endowment of the bond equals zero.

Case 6.1. Suppose that each agent has endowments $e_i(t) = \frac{1}{2}e(t)$. Then the state-price representation is $p(t) = 1/\rho(t)$, and each agent consumes one half of the aggregate endowment. Equilibrium asset prices are

$$S(t) = \frac{1}{\rho(t)} E \left[ \int_t^T \rho(s) e(s) \, ds \mid \mathcal{F}_t \right] = \frac{1}{\rho(t)} (T - t) e(t),$$

where $\rho(t)$ is the state-price density. (21)
and
\[ B(t) = \exp((\mu - \sigma^2)t). \] (22)

In Case 6.1, the equilibrium price of the risky asset does not have a bubble. In our next case, we show that the equilibrium is unchanged if we modify the initial endowments so that they include some shares of the risky asset.

Case 6.2. Suppose that the first agent has a private endowment of \( e^1(t) = \frac{1}{4}e(t) \) and is endowed with \( \frac{1}{4} \) shares of the risky asset. Suppose that the second agent has a private endowment of \( e^2(t) = \frac{3}{4}e(t) \) and is endowed with \( -\frac{1}{4} \) shares of the risky asset. Then the state-price representation and asset prices defined in (21) and (22) are also equilibrium prices here, and each agent again consumes one half of the aggregate endowment.

In Case 6.2, agents’ consumption allocations are unchanged because reallocating the endowments has no wealth effects. We can see this through the agents’ static budget equations, which are given by

\[
E\left[ \int_0^T \rho(t) c^1(t) \, dt \right] \leq \frac{1}{4} E\left[ \int_0^T \rho(t) e(t) \, dt \right] + \frac{1}{4} S(0)
\]

and

\[
E\left[ \int_0^T \rho(t) c^2(t) \, dt \right] \leq \frac{1}{4} E\left[ \int_0^T \rho(t) e(t) \, dt \right] - \frac{1}{4} S(0)
\]

Because these are the same as their counterparts in Case 6.1, the agents choose the same consumption plans as before, and equilibrium asset prices do not change.

In the next case, we show that agents’ consumption allocations may not be independent of their endowments when the equilibrium price of the risky asset includes a bubble. For this, we will need to define new equilibrium asset prices that include a bubble. We do this in the same manner described in the proof of Proposition 4.2: we define the new risky asset prices \( \hat{S} \) by

\[
\hat{S}(t) = S(t) + \frac{\eta x(T)}{\rho(t)},
\]
where $\eta_T$ is defined in Example 4.1, $\gamma \in [0, 2T]$ is a constant, and $S$ is given by (21). We take the bond price to be that in (22).

We now illustrate that adding the bubble causes wealth effects.

Case 6.3. Assume the same initial endowments as in Case 6.2. The price system defined using $(S, B)$ is an equilibrium price system, and the agents' static budget equations are given by

$$E\left[ \int_0^T \rho(t) \, c^1(t) \, dt \right] \leq \frac{1}{4} E\left[ \int_0^T \rho(t) \, e(t) \, dt \right] + \frac{1}{4} S(0)$$

and

$$E\left[ \int_0^T \rho(t) \, c^2(t) \, dt \right] \leq \frac{1}{4} E\left[ \int_0^T \rho(t) \, e(t) \, dt \right] - \frac{1}{4} S(0).$$

If there is no bubble (i.e., if $\gamma = 0$), then the agents' optimal consumption is the same as in Cases 6.1 and 6.2. If $\gamma \in (0, 2T)$, then Agent 1 acquires more wealth at the expense of Agent 2. If $\gamma = 2T$, then Agent 1 possesses all of the wealth in the economy. Equilibrium consumption is described by $c^1(t) = (1 + \gamma/2T) \, e(t)/2$ and $c^2(t) = e(t) - c^1(t)$.

Note that in Case 3 the consumption allocations are Pareto optimal, even though there is an asset-pricing bubble.

6.2. Bubbles and Asset Span

We now present an example to show (i) that adding or deleting an asset-pricing bubble can affect the span of a asset market, (ii) that trade in a asset can be nontrivial even if it has a bubble, and (iii) that deleting a bubble on an asset (even if it is zero net supply) can make equilibrium allocations Pareto inefficient. The example complements Proposition 4.2, which says that adding or deleting bubbles has no consequence on equilibrium allocations if the market remains complete after doing so and if changing asset prices has no wealth effects.

In this example, a two-dimensional standard Brownian motion $Z = (Z_1, Z_2)$ represents uncertainty. Suppose that there are two agents ($i = 1, 2$) with von Neumann–Morgenstern preferences (1) described by $u(c, t) = \log(c)$ and $V'(C) = 0$. Each agent’s endowment is given by $e^i(t) = 1/(2M_0(t) \, M_i(t))$, where

$$M_0(t) = \exp \left( -\frac{\mu}{\sigma} Z_1(t) - \frac{1}{2} \frac{\mu^2}{\sigma^2} t \right)$$
and
\[ M_i(t) = \exp \left( \int_0^t \gamma_i(s) \, dZ_2(s) - \frac{1}{2} \int_0^t (\gamma_i(s))^2 \, ds \right). \]

We assume that the processes \( \gamma_1 \) and \( \gamma_2 \) are progressively measurable, differ on a set of \( \ell \times P \)-positive measure, and make \( M_1 \) and \( M_2 \) bounded above and bounded away from zero almost surely. We additionally assume that each \( \gamma_i \) is measurable with respect to the augmented filtration generated by \( Z_2 \). Each agent \( i \) must maintain a wealth constraint in which \( L^i \) is given by (8) with \( X^i = 0 \).

**Example 6.1.** Let \( \rho \) be defined by
\[ \rho(t) = \frac{1}{c_1(t) + c_2(t)} = \frac{2M_0(t) M_1(t) M_2(t)}{M_1(t) + M_2(t)}. \]

Consider the following three assets in zero net supply. The first asset is locally riskless with price
\[ B(t) = \exp \left( \int_0^t r(s) \, ds \right), \]
where \( r \) is given by \( dr = -\rho \, dt + \sigma \rho \, dZ \) (explicit representations for \( r \) and \( \sigma \) can be calculated using Ito’s lemma). The second asset is a risky asset paying only a terminal dividend with price
\[ S_1(t) = \exp \left( \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma Z_1(t) \right). \]

The final asset is also risky, pays no dividends, and has price
\[ S_2(t) = \frac{\eta_t(t)}{\rho(t)}, \]
where \( \eta_t \) is defined as in Example 4.1 (where we substitute \( Z_2 \) for \( Z \)). Then these are equilibrium prices that make markets complete over intermediate consumption, and corresponding equilibrium consumption is given by \( c_1(t) = e(t)/\lambda \) and \( c_2(t) = (\lambda - 1) e(t)/\lambda \), where
\[ \frac{1}{\lambda} = E \left[ \int_0^T \frac{M_2(s)}{M_1(s) + M_2(s)} \, ds \right]. \]

In particular, the equilibrium allocation is Pareto optimal.
In this example, the price of the second asset is a pure bubble; i.e., \( b_2(t) \equiv S_2(t) \). The fact that the second asset has a bubble is what completes the market. We remark that the \( \rho \) defined above is the state-price representation for this market. If we were to delete the bubble from this asset, its price would identically equal zero because it pays no dividends. Our next example shows that deleting the bubble affects equilibrium allocations by changing the span of the asset market.

**Example 6.2.** Consider the same dividend structure assumed in the previous example, but now take the corresponding prices to be

\[
B(t) \equiv 1, \quad S_1(t) = \exp \left( \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma Z_1(t) \right),
\]

and \( S_2(t) \equiv 0 \). These are equilibrium prices for an economy in which markets are incomplete over intermediate consumption and in which \( c_1(t) = e_1(t) \) and \( c_2(t) = e_2(t) \). In particular, no trade takes place, there are no asset-pricing bubbles, and the equilibrium allocation is not Pareto optimal.

This example suggests others in which bubbles can help to complete the market. For example, in an incomplete market, one might be able to introduce derivative securities on existing assets at prices that include bubbles that depend on uncertainty not spanned by the primitive assets. In this case, bubbles would help to improve the Pareto efficiency of equilibrium allocations and would improve the hedging opportunities of market participants.

### 6.3. Bubble on a Positive Net Supply Asset

Theorem 5.1 suggests circumstances under which a bubble might exist on an asset in positive net supply in a complete-markets economy. Our next example identifies such a circumstance.

**Example 6.3.** Assume that there are two agents for whom \( u' \equiv 0 \) and \( V'(C) = \log(C) \). Assume that \( B \equiv 1 \) and that there is a single risky asset with price

\[
S(t) = \frac{1 - X(\rho(t) - E[\rho(T) | \mathcal{F}_t])}{\rho(t)}
\]

where \( X \) is a strictly negative real number and \( \rho \) is a strictly positive local martingale with \( \rho(0) = 1 \) and \( \rho(T) = S^{-1}(T) \). Note that \( E[\rho(T)] < 1 \) by assumption, and the market will be complete under reasonable assumptions about \( S(T) \).
Each agent is endowed with one share of the risky asset and receives no private endowments. Also suppose that each agent must observe the wealth constraint

\[ \rho(t) W_i(t) \geq -X(\rho(t) - E[\rho(t) | \mathcal{F}_t]). \]

(This says that each agent must maintain a lower bound on wealth that is strictly positive (because \( X \) is negative); i.e., \( L' < 0 \).) Neither agent trades in equilibrium, and each consumes the terminal payout \( S(T) \). The value of the aggregate endowment is finite, markets clear, and the allocation is Pareto optimal. However, the risky asset's price has a bubble, even though it is in positive net supply.

To see that holding the stock is optimal for the agents in Example 6.3, observe that

\[ E[\rho(T) W_i(T)] \leq S(0) - X(1 - E[\rho(T)]) \]

holds with equality when agent \( i \)'s wealth process satisfies \( W_i(T) = S(T) \) (the inequality comes from Lemma 4.1). The existence of the bubble is consistent with Theorem 5.1 because deflated wealth is a true supermartingale and not a martingale. (The conclusion of Theorem 4.1 does not hold here because in the example we assume that each \( X_i \) is non-negative.) In essence, the wealth constraint in the example forces each agent to hold the bubble as a suicide strategy in equilibrium.

7. CONCLUSION

We have studied conditions under which rational asset-pricing bubbles may or may not exist for a popular class of continuous-time economies. One implication of our results is that a complete theory of asset pricing for continuous-trade economies with wealth constraints requires a complete understanding of how securities originate and how agents coordinate on prices, especially given the degree of price indeterminacy for assets in zero net supply that our results suggest. These issues are not addressed by standard competitive equilibrium models, and conclusions drawn from them about actual economies in which wealth constraints are important may be misleading or false. Including more complex features, such as differential information or limited rationality, only worsens the problem (see, e.g., [35]), and will not help us to understand the fundamental features of these continuous-trade economies. Although we have studied a particular version of the continuous-trade model, it should be clear that the techniques apply to more general economies, including, e.g., those with different...
models of uncertainty, discontinuous information arrival, and overlapping
generations.

APPENDIX: PROOFS

Proof of Lemma 4.1. Recall that $p = M/B$ for some local martingale $M$.
We first claim that the process $N$, defined by

$$N(t) = p(t) W(t) + \int_0^t p(s) (c(s) - e'(s)) \, ds,$$

is a local martingale for each admissible consumption-investment strategy.
(Recall that we sometimes drop the superscript $i$ when there is no
ambiguity.) We shall prove this in a moment, but for now we accept this
and continue with our proof.

By assumption, $N$ and $M$ are both local martingales, so there exists an
increasing sequence of stopping times $\{\tau_n\}$ almost surely such that

$$E[N(\tau_n) + XM(\tau_n)] = W(0) + X$$  \hspace{1cm} (A.2)

for each $n$. By assumption, $W$ satisfies (5) with $L'$ given by (8), so

$$\rho(t) W(t) + XM(t) \geq -E \left[ \rho(T) e' + \int_t^T \rho(s) e'(s) \, ds \big| \mathcal{F}_t \right]$$

$$\geq -E \left[ \rho(T) e' + \int_0^T \rho(s) e'(s) \, ds \big| \mathcal{F}_t \right]$$

for all $t \in [0, T]$. Under Assumption 4.4.2, the random variables $\rho(\tau_n)$
$W(\tau_n) + XM(\tau_n)$ are uniformly integrable from below ([3, Corollary 4.2.3],
applied to the sequence $\max\{ -\rho(\tau_n) W(\tau_n) - XM(\tau_n), 0 \}$). Fatou's lemma
[3, Theorem 4.2.2] and the fact that $W(T) + e' \geq C$ imply that

$$E[\rho(\tau_n) W(\tau_n) + XM(\tau_n)] \geq E[\rho(T) W(T)] + XE[M(T)]$$

$$\geq E[\rho(T)(C - e')] + XE[M(T)].$$

Moreover, the monotone convergence theorem [3, Corollary 4.2.2] yields

$$\lim_{n \to \infty} E \left[ \int_0^{\tau_n} \rho(s) c(s) \, ds \right] = E \left[ \int_0^T \rho(s) c(s) \, ds \right]$$
and
\[ \lim_{n \to \infty} E \left[ \int_0^\tau \rho(s) \, e'(s) \, ds \right] = E \left[ \int_0^\tau \rho(s) \, e'(s) \, ds \right]. \]

Then inequality (9) follows directly from (A.2), since
\[ W(0) + X = \lim_{n \to \infty} E[N(\tau_n) + XM(\tau_n)] \geq E[N(T) + XM(T)] \]
\[ \geq E \left[ \rho(T)(C - e') + \int_0^T (c(t) - e'(t)) \, dt \right] + XM(T). \]

Our proof will be finished after we verify our claim that \( N \) is a local martingale. Consider any admissible consumption-investment plan \((c, C)\). Because \( \rho \) is a state-price representation, we know that \( Y \), defined by
\[ Y(t) = \rho(t) \, S(t) + \int_0^t \rho(s) \, \delta(s) \, ds, \]
is a local martingale. Integrating by parts yields
\[ Y(t) = \rho(0) \, S(t) + \int_0^t \rho(s) \, dG(s) + \int_0^t S(s) \, dp(s) + \langle G, \rho \rangle(t). \]
(Here \( \langle \cdot, \cdot \rangle \) denotes quadratic covariation.) Again integrating by parts and using the self-financing condition (4), we also have
\[ N(t) = W(0) + \int_0^t \rho(s) \, x(s) \, dB(s) + \int_0^t \rho(s) \, \pi(s) \, dG(s) \]
\[ + \int_0^t W(s) \, dp(s) + \langle W, \rho \rangle(t), \]
where
\[ \langle W, \rho \rangle(t) = \int_0^t \pi(s) \, d\langle G, \rho \rangle(s). \]

Because
\[ M(t) = \rho(t) \, B(t) = 1 + \int_0^t \rho(s) \, dB(s) + \int_0^t B(s) \, dp(s), \]
we may write
\[ \int_0^t \pi(s) \, dM(s) = \int_0^t \pi(s) \, dB(s) + \int_0^t \pi(s) \, d\rho(s). \]

Putting all this together, we find that
\[ N(t) = W(0) + \int_0^t \pi(s) \, dM(s) + \int_0^t \pi(s) \, dY(s). \tag{A.3} \]

Because \( N \) can be represented as a stochastic integral with respect to a local martingale, it is itself a local martingale, as claimed.

**Proof of Corollary 4.** The proof follows directly from Lemma 4.1.

**Proof of Lemma 4.2.** The same proof works for both cases (i) and (ii) except in case (ii) we set \( t' = 0 \). (In case (ii), recall that \( t' = 0 \).) Recall that \( \rho = M/B \) for some local martingale \( M \). Let \((c, C)\) be an admissible consumption plan satisfying the static budget equation (4.1). Define the \( \mathcal{F}_\tau \)-measurable random variable \( \zeta \) by
\[ \zeta = \rho(T)(C - e') + \rho(T) \frac{X}{B(T)} + \int_0^T \rho(s)(c(s) - e'(s)) \, ds. \]

By assumption, \( E[\zeta] = W(0) + X \). By the martingale representation theorem \([21, \text{Problem 3.4.17}]\), there exists a progressively measurable process \( \psi \) such that \( \int_0^T ||\psi(t)||^2 \, dt < \infty \), \( P \)-almost surely, and
\[ E[\zeta | \mathcal{F}_t] = W(0) + X + \int_0^t \psi(s) \, dZ(s). \]

Define the process \( \tilde{W}(t) \) by
\[ \tilde{W}(t) = \frac{E[\zeta | \mathcal{F}_t] - \int_0^t \rho(s)(c(s) - e'(s)) \, ds}{\rho(t)}. \]

We set the actual wealth process \( W \) equal to
\[ W(t) = \tilde{W}(t) - XB(t). \tag{A.4} \]

We now show that this choice of \( W \) satisfies the wealth constraint (5). Note that
\[
\rho(t) \tilde{W}(t) = E[\rho(T)(C - e') + XM(T) | \mathcal{F}_t] \\
+ E \left[ \int_t^T \rho(s)(c(s) - e'(s)) \, ds | \mathcal{F}_t \right],
\]
which yields
\[
\rho(t) W(t) = E \left[ \rho(T)(C - \varepsilon') + \int_0^T \rho(s)(c(s) - e'(s)) \, ds \mid \mathcal{F}_t \right] \\
+ \mathbb{X} E[ M(T) \mid \mathcal{F}_t] - M(t).
\]

That \( W \) satisfies (5) now follows from the nonnegativity of \( M, C, \) and \( c \).

Finally, we must show that there is an admissible portfolio trading strategy that finances \( W \). For this, we make the claim that \( N \), which is defined in (A.1) for an arbitrary consumption plan \((c, C)\), satisfies
\[
N(t) = W(0) + \int_0^T M(s) \left( \pi(s) \sigma(s) - \frac{W(s)}{B(s)} \theta(s) \right) \, dZ(s), \quad (A.5)
\]
where \( \theta \) is a progressively measurable process satisfying
\[
M(t) = 1 - \int_0^T M(s) \theta(s) \, dZ(s), \quad (A.6)
\]
and \( \int_0^T \| \theta(t) \|^2 \, dt < \infty, P\)-almost surely (such a \( \theta \) exists by the martingale representation theorem).

For now, we take (A.5) as given and proceed with the proof. From (A.5), we note that an arbitrary wealth process \( W \) must satisfy
\[
d\rho(t) W(t) = M(t) \left( \pi(t) \sigma(t) - \frac{W(t)}{B(t)} \theta(t) \right) \, dZ(t) - \rho(t)(c(t) - e'(t)) \, dt,
\]
and we know that our choice of \( W \) in (A.4) satisfies
\[
d\rho(t) W(t) = (\psi(t) + XM(t) \theta(t)) \, dZ(t) - \rho(t)(c(t) - e'(t)) \, dt.
\]
Given Assumption 4.4.1 we can choose \( \pi \) to solve
\[
M(t) \left( \pi(t) \sigma(t) - \frac{W(t)}{B(t)} \theta(t) \right) = \psi(t) + XM(t) \theta(t),
\]
and we set
\[
\pi(t) = \frac{W(t) - \pi(t) S(t)}{B(t)}, \quad (A.7)
\]
This is the required trading strategy.
We now prove our claim that $N$ satisfies (A.5). First, note that in Lemma 4.1, we showed that $N$ satisfies (A.3), and we work with that representation. Second, define the discounted dividend process $\hat{D}$ by

$$
\hat{D}_k(t) = \int_0^t \frac{\delta_k(s)}{B(s)} ds.
$$

Suppressing the time dependence and using the usual vector notation and Ito’s lemma, we have

$$
dY = d(\mathcal{MG}) + \hat{D}dM = -\hat{G}M\theta dZ + M(\mu dt + \sigma dZ) - \hat{D}M\theta dZ
$$

$$
= - (\hat{G} + \hat{D}) M\theta dZ + M\sigma dZ = M \left( \sigma - \frac{S}{B} \theta \right) dZ. \tag{A.8}
$$

In (A.8), we have used the fact $\mathcal{MG}$ is a local martingale, and this requires $\theta = \mu \sigma$ (a local martingale cannot have nonzero drift). Because $\pi$ must satisfy (A.7) and $M$ satisfies (A.6), $N$ satisfies (A.5) and our proof is complete.

**Proof of Lemma 4.3.** We show that there is an equilibrium for an economy populated by an agent with von Neumann–Morgenstern preferences (1) with an intermediate utility function $u$ that is state independent, is bounded, satisfies Inada conditions, and is continuously differentiable. We assume that $V' \equiv 0$ denote the inverse of $u_i(x, t)$ by $I_1$, and choose endowments

$$
e^i(t) = I_1(t, \rho(t)).
$$

Setting $\pi = 0$ and $X = 0$ in (8), we have an equilibrium.

**Proof of Proposition 4.1.** (Statement 1) The first part follows from the supermartingale property of the process $\rho_k$ [21, Problem 1.3.29]. The second part is demonstrated by Example 4.2.

(Statement 2) If $\rho_k$ is dominated by a Class D process, then $\rho_k$ is a Class D local martingale. But a Class D local martingale is a martingale [29, Proposition IV.1.7], so $\rho_k$ must have constant expectation. Because $\rho(T) > 0$ almost surely, we know that $\rho(T) b_k(T) = 0$, $P$-almost surely, so we must have $E[\rho(t) b_k(t)] = 0$ for all $t$. This implies that $\rho(t) b_k(t) = 0$, so we must have $b_k(t) = 0$ almost surely for all $t$. In addition, if $E[\sup_{u \in [t, T]} \rho(u) b_k(u)] < \infty$ for all $s \in [t, T]$, then $\rho_k$ is again a martingale [28, Theorem I.6.47], and we can apply the argument in the preceding paragraph to show that $b_k \equiv 0$.

(Statement 3) Inequality (12) follows from the maximal inequality for nonnegative supermartingales [21, Theorem 1.3.8].
(Statement 4) An example of a bounded bubble is given in Example 4.4.

**Proof of Theorem 4.1.** See the discussion following the statement.

**Proof of Proposition 4.2.** See the discussion following the statement.

**Proof of Corollary 4.2.** The result is an immediate consequence of Proposition 4.2.

**Proof of Proposition 4.3.** (Statement 1) See the discussion following the statement.

(Statement 2) Using the definition of \( b_0 \), we can write

\[
b_0(t) = B(t) \left(1 - \frac{E[M(T) | \mathcal{F}_t]}{M(t)}\right).
\]

Clearly this is bounded by the bond price \( B \).

(Statement 3) The proof is a direct application of Lemmas 4.1 and 4.2.

**Proof of Corollary 4.3.** See the discussion following the statement.

**Proof of Proposition 4.4.** See the discussion following the statement.

**Proof of Proposition 4.5.** Given an optimum for this agent, we have

\[
\rho(t) b_0(t) = \lambda^* B(t).
\]

for some constant \( \lambda \), so \( \rho \) is bounded above by some constant \( \lambda^* \). By definition,

\[
\rho(t) b_0(t) \leq \lambda^* B(t).
\]

By Proposition 4.1, \( b_0 \) must be zero if \( B \) is a Class D process.

**Proof of Corollary 4.4.** We claim that \( B \) is bounded above and away from zero under these assumptions. (Hence \( 1/B \) will be trivially Class D.) Given this, then our proof follows immediately from proofs of Propositions 4.1, 4.4, and 4.5.

We now prove our claim. First, we note that because markets are complete, the first welfare theorem holds, so we can find weights \( \gamma \) such that the optimal consumption plans \( c^* \) solve the representative agent problem

\[
U(c, t) = \max_{\sum_i \tilde{c}_i(t) \leq c(t)} \sum_i \gamma_i u_i(\tilde{c}_i(t), t).
\] (A.9)
(See, e.g., [10] or [18]). Now note that \( \rho(t) = \gamma U_e(\epsilon, t) \) for a constant \( \gamma \) and that the price of the riskless asset is given by

\[
B(t) = \exp \left( \int_0^t r(s) \, ds \right),
\]

where \( r \) satisfies

\[
r(t) = - \left[ \frac{U_{\epsilon t}(\epsilon(t), t)}{U_{\epsilon}(\epsilon(t), t)} + \mu_{\epsilon}(t) \frac{U_{\epsilon t}(\epsilon(t), t)}{U_{\epsilon}(\epsilon(t), t)} + \frac{1}{2} \| \sigma_{\epsilon}(t) \|^2 \frac{U_{\epsilon \epsilon}(\epsilon(t), t)}{U_{\epsilon}(\epsilon(t), t)} \right].
\]

As functions of their first arguments, \( U_{\epsilon t}, U_{\epsilon}, U_{\epsilon \epsilon}, U_{\epsilon} \) are bounded on \([d, \infty)\) because they are continuously differentiable on \((0, \infty)\). Because \( \mu_{\epsilon} \) and \( \sigma_{\epsilon} \) are bounded, \( r \) is bounded; consequently, so is \( B \). Every bounded process is Class D, so \( \rho b_0 \) is a Class D process and, hence, identically equal to zero.

**Proof of Lemma 5.1.** Apply the proof of Lemma 4.1 with minor notation changes.

**Proof of Theorem 5.1** We first prove sufficiency. Suppose that there exists a state-price representation \( \rho \) such that the process in (17) is a Class D martingale for each agent \( i \). Summing over all agents and using the market-clearing conditions (7a) and (7b), we have

\[
\rho(t) \pi S(t) + \int_0^t \rho(s) \pi \delta(s) \, ds
\]

is a Class D martingale. Thus

\[
\pi S(0) = E \left[ \rho(T) \pi S(T) + \int_0^T \rho(t) \pi \delta(t) \, dt \right],
\]

and there is no bubble on any asset \( k \) for which \( \pi_k > 0 \) (see Theorem 4.1).

We now prove necessity. Suppose that there exists a state-price representation \( \rho \) relative to which there exist no equilibrium pricing bubbles on any asset \( k \) for which \( \pi_k > 0 \). From the supermartingale property of \( \rho(t) W(t) + \int_0^t \rho(s) \epsilon'(s) - c'(s) \, ds \), inequality (18) must hold at each agent’s optimum. Summing over all agents and using market clearing conditions (7a) and (7b), we find that

\[
\pi S(0) \geq E \left[ \rho(T) \pi S(T) + \int_0^T \rho(t) \pi \delta(t) \, dt \right]. \tag{A.10}
\]
By hypothesis, inequality (A.10) holds with equality, and this can be true only if

\[ E\left[ \rho(T)(C' - c') + \int_0^T \rho(s)(c'(s) - e'(s)) \, ds \right] = W'(0) \]

for each \( i \). Thus the expression in (17) must be a martingale for each \( i \).

**Proof of Corollary 5.1.** Our assumptions imply that

\[ \rho'(t) W'(t) + \int_t^0 \rho'(s)(c'(s) - e'(s)) \, ds \]

\[ \geq \gamma \rho'(t) W'(t) + \int_t^0 \rho'(s)(c'(s) - e'(s)) \, ds \]

is true for each agent \( j \). Thus

\[ \rho'(t) W'(t) + \int_t^0 \rho'(s)(c'(s) - e'(s)) \, ds \]

is a martingale because it is a local martingale dominated by a Class D martingale [29, Proposition IV.1.7]. Taking \( \rho = \rho' \) in Theorem 5.1 finishes the proof.

**Proof of Corollary 5.2.** We prove Corollary 5.2 by showing that if its assumptions hold, then \( \rho^*(t) \pi S(t) + \int_0^t \rho^*(s) \pi \delta(s) \, ds \) will be dominated by a Class D martingale. The conclusion then follows directly from Proposition 4.1.

For \( s > t \), define \( \rho_s(s) = \rho(s)/\rho^*(t) \). Note that the process

\[ \rho^*(t) \sup_{\rho_s \in \mathcal{K}_t} E\left[ \rho_s(T) C + \int_t^T \rho_s(s) c(s) \, ds \bigg| \mathcal{F}_t \right] + \int_t^0 \rho^*(s) c(s) \, ds \quad (A.11) \]

is a supermartingale, where \( \mathcal{K}_t \) denotes the collection of state-price representations from \( \mathcal{K} \) which agree with \( \rho^* \) on \([0, t]\) and \( \sup \) denotes essential supremum.\(^{12}\) (See, e.g., [22, Chapter 5.6]). Our claim will be proven once we show that this process is really a uniformly integrable

\(^{12}\)The essential supremum of a family of measurable functions \( \{g_\alpha, \alpha \in A\} \) is denoted by \( g = \sup_{\alpha \in A} g_\alpha \) and is defined by (i) \( g \) is measurable, (ii) \( g \geq g_\alpha \), and (iii) for any \( h \) satisfying (i) and (ii), \( h \geq g \) ([3]).
martingale. To see this, note that the supermartingale property and our assumption that the supremum is attained by $\rho^*$ imply

$$E\left[ \rho^*(t) \sup_{r \in K} E \left[ T_r C + \int_r^T \rho_i(s) c(s) \, ds \mid \mathcal{F}_r \right] + \int_r^T \rho^*(s) c(s) \, ds \right]$$

$$\leq E\left[ \rho^*(T) C + \int_0^T \rho^*(s) c(s) \, ds \right].$$

On the other hand, by the definition of essential supremum, we know that this inequality holds with equality. Therefore, the process (A.11) is a supermartingale with constant expectation, and it must therefore be a uniformly integrable martingale [21, Exercise 1.3.25; 29, Proposition IV.1.7].

REFERENCES