Injecting rational bubbles

Narayana Kocherlakota*

Department of Economics, University of Minnesota, Minneapolis, MN 55455, USA, Federal Reserve Bank of Minneapolis, NBER

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Abstract

This paper proves two theorems about economies with a finite number of infinitely lived agents who trade a complete set of one-period Arrow securities and several infinitely lived securities at each date, subject to short-sales constraints. The first theorem in the paper considers an equilibrium to an economy of this kind. It proves that there exists another economy with perturbed short-sales constraints in which there is an allocation-equivalent equilibrium in which asset prices have a bubble. The second theorem extends to the result to the case in short-sales constraints are endogenously determined in the sense of Alvarez and Jermann [Efficiency, equilibrium, and asset pricing with risk of default, Econometrica 68 (2000) 775–797].

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1. Introduction

The term bubble refers to a situation in which the market value of a financial asset exceeds the present value of the dividend stream associated with that asset. Many observers believe that bubbles are important elements of real-world asset markets (for example, in Japan in the 1980s and in the United States in the 1990s). However, despite the widespread belief in the existence of bubbles in the real world, it is difficult to construct model economies in which bubbles exist in equilibrium.

The problem with generating bubbles is an intuitive one. In equilibrium, it is individually optimal for agents to exhaust their wealth. As long as the present value of consumption in the
economy is finite, this implies that the present value of consumption equals the total market value of all assets in the economy (including both financial and human assets). But if an asset has a bubble, its market value exceeds the present value of its underlying cash flows. It follows that the present value of consumption exceeds the present value of all income in the economy, which is a contradiction of equilibrium.

Santos and Woodford [5] exploit this basic intuition to show that bubbles can only exist in complete market economies with a finite number of rational infinitely lived agents if the present value of aggregate consumption is infinite. (Tirole [6] has a similar result for overlapping generations economies.) This means in turn that bubbles cannot exist in any economy in which there is some infinitely lived asset with a dividend stream that grows at the same long-run rate as aggregate consumption: such an asset would have an infinite price.

In this paper, I prove two theorems about bubbles. The first theorem considers an economy in which a finite number of infinitely lived agents trade some infinitely lived assets and a complete set of one-period Arrow securities at each date. At each date and state, agents face (possibly personalized) solvency constraints that require their financial wealth to be greater than some exogenous bound. Consider an equilibrium in this economy; the equilibrium has a unique asset-pricing kernel because markets are complete. (I impose no requirement that the solvency constraints bind in this equilibrium.) Pick any infinitely lived asset, and let $\varepsilon_t$ be a bubble—that is, a non-negative stochastic process that satisfies:

$$\varepsilon_t = E_t \{m_{t+1} \varepsilon_{t+1}\}.$$

I prove the following theorem. Suppose that for each agent $j$, we perturb agent $j$’s solvency constraints upwards by $a^j_t \varepsilon_t$ at each date and state, where $a^j_t$ is agent $j$’s initial holdings of the infinitely lived asset. Then there is an equilibrium in this perturbed economy such that the asset’s price contains the bubble. The new equilibrium has the same consumption, the same asset pricing kernel, and all solvency constraints bind in exactly the same dates and states. I term this result the bubble equivalence theorem.

Note that the result applies even if the present value of aggregate consumption is finite. Hence, the bubble equivalence theorem is in apparent conflict with Santos and Woodford’s [5] analysis. The difference lies in the nature of the solvency constraints in the perturbed economy. Santos and Woodford [5] assume that solvency constraints impose a non-positive floor on agents’ financial wealth. Because they move with the bubble term, the perturbed solvency constraints used in the bubble equivalence theorem necessarily impose a positive floor on some agents’ financial wealth in some dates and states. It is only in this way that the new solvency constraints prevent agents from doing the infinitely lived arbitrage that would eliminate equilibrium bubbles. Nonetheless, the old and new solvency constraints are binding in exactly the same dates and states. In this sense, despite their unusual nature, the new solvency constraints differ only in empirically irrelevant ways from the solvency constraints in the original economy.

In the bubble equivalence theorem, solvency constraints are treated as exogenous. In my second result, I extend the theorem to allow for endogenous solvency constraints of the form described by Alvarez and Jermann [1]. As in Kehoe and Levine [3], Alvarez and Jermann assume that agents are free to leave the economy at any date and state and receive some exogenously specified (but possibly date and state contingent) level of utility. They use this exogenous outside option to endogenize the agents’ solvency constraints. In particular, at each date, the floor $\omega^j_t$ on agent $j$’s financial wealth is set in equilibrium so that agent $j$ is just indifferent between having financial wealth $\omega^j_t$ and leaving the economy.
I prove the following theorem. Consider any Alvarez–Jermann equilibrium. Pick any infinitely lived asset and any bubble ε. As before, adjust all agents’ solvency constraints upward by \( a_j^0 \varepsilon \) at each date and state, where \( a_j^0 \) is the initial holdings of the asset by agent \( j \). The bubble equivalence theorem tells us that, with these solvency constraints, there is a bubble in the equilibrium price of the infinitely lived asset. The new result is that at any date \( t \), an agent in this economy is indifferent between financial wealth \( \omega_j^t + a_j^t \varepsilon_t \) and leaving the economy. Thus, the perturbed solvency constraints are actually part of an Alvarez–Jermann equilibrium.

Why is an agent indifferent between having \( \omega_j^t + a_j^t \varepsilon_t \) and leaving the economy at date \( t \)? The reason is that his future solvency constraints are also adjusted upwards by \( \varepsilon \). Because \( \varepsilon \) is a risk-adjusted martingale, an agent with \( \omega_j^t (z^t) + \varepsilon_t (z^t) \) in history \( z^t \) can only satisfy his future solvency constraints by investing all of \( \varepsilon_t (z^t) \) and consuming none of it. Hence, he is indifferent between having \( \omega_j^t (z^t) + \varepsilon_t (z^t) \) in the new equilibrium and having \( \omega_j^t (z^t) \) in the old equilibrium.

2. The economy: exogenous solvency constraints

Consider an infinite horizon world with \( J \) agents; time is indexed by the natural numbers. In each period \( t \), all agents learn the realization of a random variable \( z_t \) with finite support \( Z \). Let \( z^t \) denote the history \((z_1, \ldots, z_t)\), and let \( \Pr(z^t) \) denote the unconditional probability of that history. I assume that \( \Pr(z^t_1 = z^*_1) = 1 \) and that \( \Pr(z^*_2, z^*_3, \ldots, z^*_t) > 0 \) for all \((z^*_2, z^*_3, \ldots, z^*_t)\) in \( Z^{t-1} \) (so that agents know that \( z^*_1 = z^*_2 \) in period 1). I use the notation \( z^{t+s} \geq z^t \) to refer to histories \( z^{t+s} \) such that \( \Pr(z^{t+s} | z^t) > 0 \).

There is a single consumption good. Let

\[
C = \{(c_t)_{t=1}^{\infty} | c_t : Z^t \to R_+\}
\]

be the commodity space, where \( R_+ \) is the non-negative real numbers. Agent \( j \) has preferences representable by the utility function \( U_j \) over \( C \). I assume that preferences remain stable over time. Agent \( j \)'s utility function is strictly increasing in its various components.

Agent \( j \) has an endowment process \( y^j \) in \( C \). There are also \( N \) tradeable long-lived assets. These assets are in unit total supply. Asset \( n \) has payoff process \( d_n \) in \( C \). Agent \( j \) is initially endowed with \( a_{n1}^j \) units of asset \( n \), where \( a_{n1}^j \geq 0 \) for all \( j, n \), and \( \sum_{j=1}^{J} a_{n1}^j = 1 \) for all \( n \).

Trading works as follows. In each period \( t \), agents can trade current consumption, the \( N \) long-lived assets, and a complete set of state-contingent claims to consumption next period. However, each agent faces a solvency constraint. In particular, for each agent \( j \), there exists a stochastic process \( \omega_j^t = (\omega_j^t)_{t=2}^{\infty} \) where \( \omega_j^t : Z^t \to [\infty, 0] \). The total value of agent \( j \)'s holdings of the \( N \) long-lived assets and the various state-contingent claims, assessed immediately after \( z_t \) is realized, must exceed \( \omega_j^t (z^t) \). Note that I allow \( \omega_j^t (z^t) \) to equal \(-\infty\), so that there may be no limit on borrowing in a particular date and state.

An equilibrium in this economy has five components \((c, a, b, p, q)\), where

\[
c \in C^J,
\]
\[
a = ((a_j^t)_{t=1}^{\infty})_{j=2}^{\infty}, a_j^t : Z^{t-1} \to R,
\]
\[
b = ((b_j^t)_{t=1}^{\infty})_{j=2}^{\infty}, b_j^t : Z^t \to R,
\]
\[
p = (p_n)_{n=1}^{N}, p_n \in C,
\]
\[
q = (q_t)_{t=2}^{\infty}, q_t : Z^t \to R_+.
\]
The first component is the allocation of consumption across agents. The second component describes the allocation of long-lived assets across agents. The third component is the allocation of state-contingent claims to consumption. The fourth component is the price of the long-lived assets, and the final component is the price of the one-period ahead state-contingent claims to consumption.

A quintuplet \((c^*, a^*, b^*, p^*, q^*)\) is an equilibrium, given \(\omega\), if it satisfies the following conditions. The first is individual optimality for each \(j\), so that \((c^*_j, a^*_j, b^*_j)\) is a solution to the maximization problem

\[
\max_{(c^j, a^j, b^j)} U^j(c^j)
\]

s.t.

\[
c^*_j(z^t) + \sum_{n=1}^{N} a^*_n(z^t) p^*_n(z^t) + \sum_{z' \in Z} b^*_t(z^t, z') q^*_t(z^t, z') \\
\leq y^*_j(z^t) + \sum_{n=1}^{N} a^*_n(z^t-1) (p^*_n(z^t) + d^*_n(z^t)) + b^*_t(z^t) \quad \text{for all } t, z^t,
\]

\[
\sum_{n=1}^{N} a^*_n(z^t-1) (p^*_n(z^t) + d^*_n(z^t)) + b^*_t(z^t) \geq \omega^*_t(z^t) \quad \text{for all } z^t, t > 1,
\]

\[
c^*_j(z^t) \geq 0, a^*_n(z_0) \equiv a^*_1, b^*_1(z^*_1) = 0.
\]

The second condition is market-clearing for consumption, long-lived assets, and state-contingent claims

\[
\sum_{j=1}^{J} c^*_j(z^t) = \sum_{j=1}^{J} y^*_j(z^t) + \sum_{n=1}^{N} d^*_n(z^t) \quad \text{for all } t, z^t,
\]

\[
\sum_{j=1}^{J} a^*_n(z^t-1) = 1 \quad \text{for all } n, t, z^t-1,
\]

\[
\sum_{j=1}^{J} b^*_j(z^t) = 0 \quad \text{for all } t, z^t.
\]

This paper is about bubbles. I use a standard definition of the asset price’s fundamental and its bubble.

**Definition 1.** Let \((c^*, a^*, b^*, p^*, q^*)\) be an equilibrium given \(\omega\). The fundamental \(p^*_n(z^t)\) of asset \(n\) is defined to be

\[
p^*_n(z^t) = \lim_{T \to \infty} \sum_{s=1}^{T} \sum_{z^t+s \geq z^t} \left\{ \prod_{r=1}^{s} q^*_t(z^t+r) \right\} d_{n,t+s}(z^t+s)
\]

and its bubble \(\varepsilon_n\) is defined to be

\[
\varepsilon_n = p^*_n - p^{f*}_n.
\]

In this definition, I use the state-prices \(q\) to discount the future dividends.
3. A representative agent example

In this section, I show how it is possible to always inject bubbles into a representative agent economy. Suppose all agents have preferences over $C$ representable by the utility function

$$U(c) = \sum_{t=1}^{\infty} \sum_{z^t \in Z^t} \Pr(z^t) \beta^{t-1} u(c_t(z^t)).$$

Each agent is endowed with a single unit of a long-lived asset with constant dividend $d$, and each agent has a constant endowment $y$. We begin with an economy in which the agents’ solvency constraint $\omega = -y \lambda/(1 - \beta), \lambda \geq 0$.

In this economy, the agents’ budget set is given by

$$c_t(z^t) + a_{t+1}(z^t) p_t(z^t) + \sum_{z'} q_{t+1}(z^t, z') b_{t+1}(z^t, z')$$

$$\leq y + a_t(z^{t-1}) (p_t(z^t) + d) + b_t(z^t),$$

$$a_t(z^{t-1}) (p_t(z^t) + d) + b_t(z^t) + \lambda y/(1 - \beta) \geq 0,$$

$$b_1 = 0,$$

$$a_1(z_0) = 1.$$

In any equilibrium, $c_t(z^t) = y + d, b_t(z^t) = 0, a_{t+1}(z^t) = 1$ for all $t, z^t$. The solvency constraints do not bind in any equilibrium. Hence, $q_t(z^t) = \beta \Pr(z^t|z^{t-1})$ for all $t$. There is an equilibrium in which

$$p_t(z^t) = \beta d/(1 - \beta).$$

Is there any other equilibrium? If there is, it must take the form

$$p'_t(z^t) = \beta d/(1 - \beta) + \epsilon_t(z^t),$$

where $\epsilon_t$ is a non-negative process that satisfies

$$\epsilon_t(z^t) = \beta \sum_{z'} \epsilon_{t+1}(z^t, z') \Pr(z|z^t).$$

But note that there is a necessary transversality condition to the agents’ problem

$$\lim_{t \to \infty} \sum_{z^t \in Z^t} \Pr(z^t) \beta^t (p_t(z^t) + d + \lambda y(1 - \beta)^{-1}) = 0$$

and this necessary condition can only be satisfied by setting $\epsilon_t(z^t) = 0$ for all $t, z^t$. Intuitively, if $\epsilon_t(z^t)$ is positive, all agents will want to sell off the asset and buy consumption instead. With this specification of $\omega$, the bubble equilibria cannot survive.

However, if we perturb the solvency constraints in the right way, it is possible to inject bubbles. Pick an arbitrary bubble process $\epsilon$, and define

$$\hat{p}_t(z^t) = \beta d/(1 - \beta) + \epsilon_t(z^t),$$

$$\hat{\omega}_t(z^t) = -\lambda y/(1 - \beta) + \epsilon_t(z^t).$$
The agents’ solvency constraint still never binds. However, the agents’ transversality condition now becomes

$$\lim_{t \to \infty} \sum_{z^t \in Z^t} \beta^t (\hat{p}_t(z^t) + d - \varepsilon_t(z^t) + \lambda y (1 - \beta)^{-1}) = 0$$

and this is satisfied.

What is going on in this equilibrium? There is a bubble in the asset price. The agents would like to exploit this bubble by permanently lowering their holdings of the asset by \(\delta\). But this is not possible with a solvency constraint of the form \(\hat{\omega}\). There is a positive probability that for some \(t\), \(\varepsilon_t\) grows sufficiently large that

$$d(1 - \delta)/(1 - \beta) + \varepsilon(1 - \delta) + \lambda y/(1 - \beta) - \varepsilon$$

falls below zero. Hence, permanently lowering asset holdings by \(\delta\) would lead to the agent’s solvency constraint being violated. In other words, the agent’s solvency constraint never binds—but it is impossible for the agent to permanently lower his holdings of the asset. This makes a bubble possible.

The solvency constraint is designed so that, in some states of the world, agents are required to have a positive amount of financial wealth. The realism of this requirement may be doubted by some readers. However, note that for a given bubble \(\varepsilon\), the probability of these states can be made arbitrarily small by setting \(\lambda\) sufficiently large.

4. Bubble equivalence theorem

In this section, I return to the general class of economies described in Section 1. I first prove that any bubble must be a non-negative risk-adjusted martingale. I then prove the bubble equivalence theorem. It considers an arbitrary equilibrium and an arbitrary non-negative risk-adjusted martingale \(\varepsilon\). Given these objects, is possible to construct solvency constraints such that there is a new equilibrium in which an infinitely lived asset has a bubble \(\varepsilon\) in its price. The new equilibrium and the old equilibrium are equivalent in the sense that the consumption allocations are the same, and the solvency constraints bind in exactly the same dates and states.

Let \((c, a, b, p, q)\) be an equilibrium given \(\omega\). As before, define

\(\varepsilon_{nt}(z^t) = p_{nt}(z^t) - p_{nt}^f(z^t)\)

to be the bubble on asset \(n\), where its fundamental is

\[p_{nt}^f(z^t) = \lim_{T \to \infty} \sum_{s=1}^{T} \sum_{z^{t+s} \geq z^t} \left( \prod_{r=1}^{s} q_{t+r}(z^{t+r}) \right) d_{n,t+s}(z^{t+s}).\]

We can show that the bubble \(\varepsilon\) must satisfy two key properties. The first property is that, as originally established by Diba and Grossman [2], the bubble \(\varepsilon\) must be non-negative.

**Proposition 2.** The bubble \(\varepsilon_{nt}(z^t) \geq 0\) for all \(t, z^t\).

**Proof.** Suppose \((c, a, b, p, q)\) is an equilibrium given \(\omega\) and in history \(z^t\):

\[p_{nt}(z^t) > \sum_{z'} q_{t+1}(z', z') \{ p_{n,t+1}(z', z') + d_{n,t+1}(z', z') \}.\]
Then, any agent $j$ can improve his welfare by choosing $(c', a', b')$, where in history $z'$
\[
c_j^{'}(z') = c_j(z') + \eta_{t+1}^j(z') p_{nt}(z') - \eta_{t+1}^j(z') \\
\times \sum_{z'} q_{t+1}(z', z')(p_{n,t+1}(z', z') + d_{n,t+1}(z', z'))
\]
\[
b_{t+1}^j(z', z') = b_{t+1}^j(z', z') + [p_{n,t+1}(z', z') + d_{n,t+1}(z', z')]\eta_{t+1}^j(z')
\]
\[
a_{t+1}^j(z') = a_{t+1}^j(z') - \eta_{t+1}^j(z')
\]
for some positive $\eta_{t+1}^j(z')$ and $(c', a', b')$ equals $(c, a, b)$ for all other histories. The same arbitrage argument can be applied if the inequality is reversed. Hence
\[
p_{nt}(z') = \sum_{z'} q_{t+1}(z', z')\{p_{n,t+1}(z', z') + d_{n,t+1}(z', z')\}.
\]

We can iterate this difference equation forwards to get that for all $T$
\[
p_{nt}(z') = \sum_{s=1}^{T} \sum_{z^{t+s+1} \geq z'} \left\{ \prod_{r=1}^{s} q_{t+r}(z^{t+r}) \right\} d_{n,t+s}(z^{t+s}) \\
+ \sum_{z^{t+T} \geq z'} \left\{ \prod_{r=1}^{T} q_{t+r}(z^{t+r}) \right\} p_{n,t+T}(z^{t+T}) \\
\geq \sum_{s=1}^{T} \sum_{z^{t+s+1} \geq z'} \left\{ \prod_{r=1}^{s} q_{t+r}(z^{t+r}) \right\} d_{n,t+s}(z^{t+s})
\]
which implies in turn that
\[
p_{nt}(z') \geq p_{nt}^f(z').
\]
This proves the proposition. \(\square\)

The second restriction is that the bubble must be a martingale under the equivalent martingale measure defined by $q$.

**Proposition 3.** \(\sum_{z'} q_{t+1}(z', z')e_{n,t+1}(z', z') = e_{nt}(z').\)

**Proof.** The same arbitrage argument as in the proof of Proposition 1 implies that
\[
\sum_{z'} q_{t+1}(z', z')(p_{n,t+1}(z', z') + d_{n,t+1}(z', z')) = p_{nt}(z').
\]

Also, we can show that
\[
\sum_{z^{t+1} \geq z'} q_{t+1}(z^{t+1})p_{n,t+1}^f(z^{t+1})
\]
\[
= \sum_{z^{t+1} \geq z'} q_{t+1}(z^{t+1}) \lim_{T \to \infty} \sum_{s=1}^{T} \sum_{z^{t+s+1} \geq z^{t+1}} \left( \prod_{r=1}^{s} q_{t+r}(z^{t+r+1}) \right) d_{n,t+s+1}(z^{t+s+1})
\]
\[
\lim_{T \to \infty} \sum_{s=1}^{T} \sum_{z' \geq z_t} \left( \prod_{r=1}^{s+1} q_{t+r}(z_{t+r}) \right) d_{n,t+1+s}(z_{t+1+s}) = p_{nt}(z_t) - \sum_{z' \geq z_t} d_{n,t+1}(z_{t+1})q_{t+1}(z_{t+1})
\]

which completes the proof of the proposition. □

The bubble equivalence theorem is a partial converse to Propositions 1 and 2. Suppose we start with an economy with no bubbles and state-prices \(q\). Given any non-negative stochastic process \(\varepsilon\) that is a “risk-adjusted” martingale under \(q\), it is possible to perturb the solvency constraints \(\omega\) so that the new economy has a consumption-equivalent equilibrium in which asset \(n\) has a bubble equal to \(\varepsilon\).

More precisely, let \((c^*, a^*, b^*, p^*, q^*)\) be an equilibrium given \(\omega\). The solvency constraint \(\omega\) may equal \(-\infty\) in some dates and states. The value of agent \(j\)’s wealth, \(\sum_{n=1}^{N} a_{nt}^j(z_{t-1})(p_{nt}^*(z_t) + d_{nt}(z_t)) + b_{nt}^j(z_t)\), is finite for all \(t, z_t\); otherwise, agent \(j\)’s allocation \(c_{t}^j\) is not individually optimal for him. Define

\[
\omega_t^j(z_t') = \max\left( \sum_{n=1}^{N} a_{nt}^j(z_t-1)\{p_{nt}^*(z_t) + d_{nt}(z_t')\} + b_{nt}^j(z_t'), \omega_t^j(z_t) \right).
\]

Then, \((c^*, a^*, b^*, p^*, q^*)\) is an equilibrium given \(\omega'\) and \(\omega_t^j(z_t')\) is finite for all \(t, z_t\). Hence, there is no loss in generality in restricting attention to cases in which the solvency constraints \(\omega\) is finite in all dates and states for all \(j\).

Then, we can prove the bubble equivalence theorem.

**Theorem 4.** Let \((c^*, a^*, b^*, p^*, q^*)\) be an equilibrium given \(\omega\), where \(\omega_t^j(z_t') > -\infty\) for all \(j, t, z_t\). Let \(\varepsilon = ((\varepsilon_{nt})_{t=1}^{\infty})_{n=1}^{N}\) be any element of \(C^N\) such that

\[
\varepsilon_{nt}(z_t') = \sum_{z' \geq z_t} q_{t+1}^*(z_t', z_t') \varepsilon_{nt+1}(z_t', z_t')
\]

for all \(n, t, z_t\) and \(\varepsilon_{nt}(z_t')\). Then, \((c^*, a^*, \hat{b}, \hat{p}, q^*)\) is an equilibrium given \(\hat{\omega}\), where:

\[
\hat{p}_{nt}(z_t') = p_{nt}^*(z_t') + \varepsilon_{nt}(z_t'),
\]

\[
\hat{b}_{nt}^j(z_t') = b_{nt}^j(z_t') + \sum_{n=1}^{N} \varepsilon_{nt}(z_t')(a_{nt}^j - a_{nt}^j(z_t'-1)),
\]

\[
\hat{\omega}_t^j(z_t') = \omega_t^j(z_t') + \sum_{n=1}^{N} \varepsilon_{nt}(z_t')a_{nt}^j.
\]

**Proof.** To prove the theorem, I first need to show that the triple \((c^*, a^*, \hat{b})\) satisfies market-clearing. But this is trivial because

\[
\sum_{j=1}^{J} \hat{b}_{nt}^j(z_t') = 0 - \sum_{n=1}^{N} \varepsilon_{nt}(z_t') + \sum_{n=1}^{N} \varepsilon_{nt}(z_t') = 0.
\]
Next, I need to show that \((c^{j*}, a^{j*}, \hat{b}^j)\) is individually optimal, given prices \((\hat{p}, q^*)\) and solvency constraints \(\hat{\varnothing}^j\). I do this in two steps. In step 1, I demonstrate that \((c^{j*}, a^{j*}, \hat{b}^j)\) is budget-feasible, given prices \((\hat{p}, q^*)\) and solvency constraints \(\hat{\varnothing}^j\). In terms of the flow constraint

\[
c_i^j(z^t) + \sum_{n=1}^N a_{i,t+1}^{j*}(z') \hat{p}_{nt}(z') + \sum_{z' \in Z} \hat{b}_{i+1}^j(z', z') q_{t+1}^{j*}(z', z') - y_i^j(z^t) - \sum_{n=1}^N a_{nt}^{j*}(z') (\hat{p}_{nt}(z') + d_{nt}(z^t)) - \hat{b}_i^j(z') = 0
\]

where the last equality follows from the restriction that the current value of \(e_n\) equals the present value of next period’s \(e_n\). Hence, \((c^{j*}, a^{j*}, \hat{b}^j)\) satisfies the flow constraint.

In terms of the solvency constraint

\[
\sum_{n=1}^N a_{nt}^{j*}(z') (\hat{p}_{nt}(z') + d_{nt}(z^t)) + \hat{b}_i^j(z') - \hat{\varnothing}_i^j(z') = 0
\]

\[
= \sum_{n=1}^N a_{nt}^{j*}(z') (p_{nt}^*(z') + d_{nt}(z^t)) + b_i^{j*}(z') - \omega_i^j(z')
\]

\[
+ \sum_{n=1}^N a_{nt}^{j*}(z') e_{nt}(z^t) + \sum_{n=1}^N (a_{nt}^j - a_{nt}^{j*}(z') e_{nt}(z^t) - \sum_{n=1}^N a_{nt}^j e_{nt}(z^t) \geq 0.
\]

Hence, we have completed step 1; \((c^{j*}, a^{j*}, \hat{b}^j)\) is budget-feasible given \((\hat{p}, q^*)\) and \(\hat{\varnothing}^j\).

In step 2, I need to show that the set of budget-feasible consumption allocations defined by \((\hat{p}, q^*)\) and \(\hat{\varnothing}^j\) is no larger than the set of budget-feasible allocations defined by \((p^*, q^*)\) and \(\omega^j\). Let \((c^{j'}, a^{j'}, b^{j'})\) be an element of the budget set defined by \((\hat{p}, q^*)\) and \(\hat{\varnothing}^j\). Then, I claim that \((c^{j'}, a^{j'}, b^{j''})\) is an element of the budget set defined by \((p^*, q^*)\) and \(\omega^j\), where

\[
b_i^{j''}(z') = b_i^{j'}(z') - \sum_{n=1}^N (a_{n1}^j - a_{nt}^{j'}(z') c_{nt}(z^t)).
\]

First, we can check the flow constraint

\[
c_i^{j'}(z') + \sum_{n=1}^N a_{i,t+1}^{j'}(z') p_{nt}^*(z') + \sum_{z' \in Z} b_{i+1}^{j''}(z', z') q_{t+1}^{j*}(z', z')
\]
\[-y_t(z_t) - \sum_{n=1}^{N} a_{nt}^{j'}(z_{t-1})\{p_{nt}^*(z_t') + d_{nt}(z_t')\} - b_t^{j''}(z_t') \leq \ - \sum_{n=1}^{N} a_{nt}^{j'}(z_t')e_{nt}(z_t') - \sum_{n=1}^{N} \sum_{z_t'=1}^{N} (\bar{a}_{n1}^{j} - a_{nt}^{j'}(z_t'))e_{n,t+1}(z_t', z_t') \\
+ \sum_{n=1}^{N} a_{nt}^{j'}(z_{t-1})e_{nt}(z_t') + \sum_{n=1}^{N} (\bar{a}_{n1}^{j} - a_{nt}^{j'}(z_{t-1}))e_{nt}(z_t') = 0.\]

And, then we can check that the solvency constraint is satisfied
\[
\sum_{n=1}^{N} a_{nt}^{j'}(z_t')e_{nt}(z_t') + b_t^{j''}(z_t') - \omega_t^{j}(z_t') \\
\geq \ - \sum_{n=1}^{N} a_{nt}^{j'}(z_{t-1})e_{nt}(z_t') - \sum_{n=1}^{N} (\bar{a}_{n1}^{j} - a_{nt}^{j'}(z_{t-1}))e_{nt}(z_t') \\
+ \sum_{n=1}^{N} a_{nt}^{j}e_{nt}(z_t') = 0.
\]

Hence, we have verified that the budget set in the “hatted” economy is no larger than the budget set in the original economy. It follows that \((c_t^*, a_t^*, \hat{b})\) is individually optimal, given prices \((\hat{p}, q^*)\) and solvency constraints \(\hat{\omega}^j\). The theorem is proved. \(\Box\)

The reason for the name of the theorem is that as long as \(\epsilon_{nt}(z_t')\) is non-zero for some \((t, z_t')\), then asset \(n\) is not bubble-free in the second, hatted, economy. To see this, first note that we can roll the equation
\[
\epsilon_{nt}(z_t') = \sum_{z_t'} q_{t+1}(z_t', z_t')e_{n,t+1}(z_t', z_t')
\]
forwards in time to find that
\[
\epsilon_{nt}(z_t') = \sum_{z_t'+T \geq z_t'} \left\{ \prod_{r=1}^{T} q_{t+r}(z_t'+r) \right\} e_{n,t+T}(z_t^{T+T}).
\]

This implies that
\[
\lim_{T \to \infty} \sum_{z_t'+T \geq z_t'} \left\{ \prod_{r=1}^{T} q_{t+r}(z_t'+r) \right\} \hat{p}_{n,t+T}(z_t^{T+T}) = \lim_{T \to \infty} \sum_{z_t'+T \geq z_t'} \left\{ \prod_{r=1}^{T} q_{t+r}(z_t'+r) \right\} \left[ p_{n,t+T}(z_t^{T+T}) + \epsilon_{n,t+T}(z_t^{T+T}) \right]
\]
\[
\geq \lim_{T \to \infty} \sum_{z_t'+T \geq z_t'} \left\{ \prod_{r=1}^{T} q_{t+r}(z_t'+r) \right\} \epsilon_{n,t+T}(z_t^{T+T}) = \epsilon_{nt}(z_t').
\]

and so asset \(n\) has a bubble.
Obviously, the hatted economy and the original economy differ in the specification of their solvency constraints. In the hatted economy, agents are more tightly restricted in their ability to short-sell. However, it is important to emphasize that these “tighter” borrowing constraints do not bind the agents any more than the solvency constraints in the original economy. In particular

\[
\sum_{n=1}^{N} a_{nt}^j (z^{l-1}) \{p_{nt}(z^l) + d_{nt}(z^l)\} + \hat{b}_t^j (z^l) - \hat{\alpha}_t^j (z^l) \\
= \sum_{m=1}^{M} a_{nt}^j (z^{l-1}) \{p_{nt}^*(z^l) + \varepsilon_{nt}(z^l) + d_{nt}(z^l)\} \\
+ b_t^j (z^l) + \sum_{n=1}^{N} (\bar{a}_{n1}^j - a_{nt}^j (z^{l-1})) \varepsilon_{nt}(z^l) \\
- \omega_t^j (z^l) - \sum_{n=1}^{N} \bar{a}_{n1}^j \varepsilon_{nt}(z^l) \\
= \sum_{n=1}^{N} a_{nt}^j (z^{l-1}) \{p_{nt}^*(z^l) + d_{nt}(z^l)\} + b_t^j (z^l) - \omega_t^j (z^l) \quad \text{for all } t, z^l.
\]

The solvency constraints hold with equality in the hatted economy if and only if they hold with equality in the starred economy.

Thus, the bubble equivalence theorem says that it is always possible to perturb the solvency constraints in a given economy so as to create a bubble in the economy. Importantly, though, we would never actually see the impact of the perturbation in equilibrium, because the new solvency constraints bind exactly when the old ones do.

5. Bubble equivalence theorem: endogenous solvency constraints

I now show that the bubble equivalence theorem is valid, even if solvency constraints are determined endogenously as in Alvarez and Jermann [1]. I make the following changes in the specification of the economy and equilibrium described in Section 2. I specialize preferences to be additively separable over dates and states, so that agent \(j\) has preferences representable by the utility function

\[
\sum_{t=1}^{\infty} \beta_t^j u^j (c_{jtt}(z^l)) \Pr(z^l)
\]

over \(C\). (This is not essential, but it does make the definition of equilibrium somewhat more transparent.) More importantly, I now assume that in any date and state, any agent can choose to leave the economy permanently. If agent \(j\) leaves the economy in history \(z^l\), he receives continuation utility \(U_j(z^l)\).

This outside option endogenizes solvency constraints. I model this formally as Alvarez–Jermann [1] do. In particular, an A–J equilibrium in this economy has six components \((c, a, b, p, q, \omega)\), where

\[
c \in C^J, \\
a = (\{a_t^j\}_{j=1}^{\infty})_{t=2}, a_t^j : Z^{t-1} \rightarrow R,
\]

over \(Z^{t-1}\).
\[
\begin{align*}
\mathbf{b} &= ( ( b_j^l )_{j=1}^J )_{l=2}^\infty, \quad b_j^l : Z^l \to R, \\
\mathbf{p} &= ( p_n )_{n=1}^N, \quad p_n \in C, \\
\mathbf{q} &= ( q_t )_{t=2}^\infty, \quad q_t : Z^t \to R^+, \\
\mathbf{\omega} &= ( ( \omega_j^t )_{j=1}^J )_{t=2}^\infty, \quad \omega_j^t : Z^{t-1} \to R .
\end{align*}
\]

The first component \( c \) is the allocation of consumption across agents. The second component \( a \) describes the allocation of long-lived assets across agents. The third component \( b \) is the allocation of state-contingent claims to consumption. The fourth component \( p \) is the price of the long-lived assets, and the fifth component \( q \) is the price of the one-period ahead state-contingent claims to consumption. The final component \( \omega \) describes the agents’ endogenously determined solvency constraints in every date and state.

A given \((c^*, a^*, b^*, p^*, q^*, \omega^*)\) is an A–J equilibrium if it satisfies the following three conditions. The first is individual optimality for each \( j \), so that \((c_j^*, a_j^*, b_j^*, \omega_j^*)\) is a solution to the maximization problem:

\[
\begin{align*}
\max_{(c^*, a^*, b^*)} & \quad \sum_{t=1}^\infty \sum_{z^t} \beta^{t-1} \text{Pr}(z^t) u^j (c_j^t (z^t)) \\
\text{s.t.} & \quad c_j^t (z^t) + \sum_{n=1}^N a_{n,t+1}^j (z^t) p_{nt}^* (z^t) + \sum_{z' \in Z} b_{t+1}^j (z^t, z') q_{t+1}^* (z^t, z') \\
& \quad \leq y_j^t (z^t) + \sum_{n=1}^N a_{nt}^j (z^{t-1}) \{ p_{nt}^* (z^t) + d_{nt} (z^t) \} + b_j^t (z^t) \quad \text{for all } t, z^t \\
& \quad \sum_{m=1}^M \sum_{z_s \in Z} a_{nt}^j (z^{t-1}) \{ p_{ns}^* (z^t) + d_{ns} (z^t) \} + b_j^t (z^t) \geq \omega_j^t (z^t) \quad \text{for all } z^t, t > 1 \\
& \quad c_j^t (z^t) \geq 0, \quad a_{n1}^j (z^*) = \overline{a}_{n1}^j, \quad b_j^t (z^*) = 0.
\end{align*}
\]

The second condition is that the solvency limits are determined endogenously so that for any \((t, z^t)\), agent \( j \) is indifferent between staying in the economy or not when he has wealth \( \omega_j^t (z^t) \). Alvarez and Jermann [1] refer to this condition as requiring solvency constraints to be not too tight. Formally, for all \( t, z^t \):

\[
\begin{align*}
\overline{U}_t^j (z^t) = \max_{(c^*, a^*, b^*)} & \quad \sum_{s=t}^\infty \sum_{z^s \geq z^t} (\beta) \text{Pr}(z^s | z^t) u^j (c_j^s (z^s)) \\
\text{s.t.} & \quad c_j^s (z^s) + \sum_{n=1}^N a_{n,s+1}^j (z^s) p_{ns}^* (z^s) + \sum_{z' \in Z} b_{s+1}^j (z^s, z') q_{s+1}^* (z^s, z') \\
& \quad \leq y_j^s (z^s) + \sum_{n=1}^N a_{ns}^j (z^{s-1}) \{ p_{ns}^* (z^s) + d_{ns} (z^s) \} + b_j^s (z^s) \quad \forall s \geq (t + 1), \quad \forall z^s \geq z^t, \\
& \quad c_j^t (z^t) + \sum_{n=1}^N a_{n,t+1}^j (z^t) p_{nt}^* (z^t) + \sum_{z' \in Z} b_{t+1}^j (z^t, z') q_{t+1}^* (z^t, z') \\
& \quad \leq y_j^t (z^t) + \omega_j^t (z^t),
\end{align*}
\]
\[ \sum_{n=1}^{N} a_{n,t}^j(z^{t-1})\{p_{n,s}^*(z^s) + d_{n,s}(z^s)\} + b_{s}^j(z^s) \geq \omega_{s}^j(z^s) \]

\[ \forall s > t, \forall z^s \geq z^t, \]

\[ c_{t+s}^j(z^{t+s}) \geq 0 \quad \text{for all } s \geq 0 \text{ and all } z^{t+s} \geq z^t. \]

This condition says that in history \( z^t \), agent \( j \) is indifferent between having financial wealth \( \omega_{j}^* (z^t) \) and leaving the economy. It captures the notion that no lender will be willing to offer agent \( j \) a loan that reduces his financial wealth below this level, because the loan will not be re-paid. Note that the level of \( \omega_{j}^* (z^t) \) that satisfies this restriction depends on the process generating the future solvency constraints \( \omega_{j}^* (z^t) \).

The final condition is market-clearing for consumption, long-lived assets, and state-contingent claims:

\[ \sum_{j=1}^{J} c_{t}^j(z^t) = \sum_{j=1}^{J} y_{t}^j(z^t) + \sum_{n=1}^{N} d_{nt}(z^t) \quad \text{for all } t, z^t, \]

\[ \sum_{j=1}^{J} a_{nt}^j(z^{t-1}) = 1 \quad \text{for all } n, t, z^{t-1}, \]

\[ \sum_{j=1}^{J} b_{t}^j(z^t) = 0 \quad \text{for all } t, z^t. \]

The following theorem shows that bubbles are an intrinsic feature of A–J equilibria.

**Theorem 5.** Let \( (c^*, a^*, b^*, p^*, q^*, \omega^*) \) be an A–J equilibrium. Let \( \varepsilon = ((\varepsilon_{nt})_{t=1}^{\infty})_{n=1}^{N} \) be any element of \( C^N \) such that

\[ \varepsilon_{n,t}(z^t) = \sum_{z'_{t+1}} q_{t+1}^*(z^t, z') \varepsilon_{n,t+1}(z', z') \]

for all \( n, t, z^t \). Then, \( (c^*, a^*, b^*, p^*, q^*, \omega) \) is an A–J equilibrium, where

\[ \hat{p}_{nt}(z^t) = p_{nt}^*(z^t) + \varepsilon_{nt}(z^t), \]

\[ \hat{b}_{t}^j(z^t) = b_{t}^j(z^t) + \sum_{n=1}^{N} \varepsilon_{nt}(z^t) (a_{nt}^j(z^{t-1}) - a_{nt}^j(z^t)), \]

\[ \hat{\omega}_{t}^j(z^t) = \omega_{t}^j(z^t) + \sum_{n=1}^{N} \varepsilon_{nt}(z^t) a_{nt}^j. \]

**Proof.** It is straightforward to show that \( (c^*, a^*, \hat{b}) \) satisfies market-clearing because

\[ \sum_{j=1}^{J} \hat{b}_{t}^j(z^t) = 0 - \sum_{n=1}^{N} \varepsilon_{nt}(z^t) + \sum_{n=1}^{N} \varepsilon_{nt}(z^t) = 0. \]

I can show that \( (c_{t}^j, a_{t}^j, \hat{b}_{t}^j) \) is individually optimal using the same argument as in the proof of Theorem 4. The key part of the proof is to demonstrate that the solvency constraints \( \hat{\omega} \) are not too
tight, given \((\hat{p}, \hat{q}^*)\). In the original equilibrium, we know that for all \(t, z')

\[
U^j_t(z') = \max_{(c^j, a^j, b^j)} \sum_{s=0}^{\infty} \sum_{z^{t+s} \geq z'} \beta^s \Pr(z^{t+s} | z') u^j(c^j_{t+s}(z^{t+s}))
\]

s.t.

\[
c^j_{t+s}(z^{t+s}) + \sum_{n=1}^{N} a^j_{n,t+s+1}(z^{t+s}) p^*_n,t+s(z^{t+s})
\]

\[
+ \sum_{k \in Z} b^j_{t+s+1}(z^{t+s}, z') q^*_t+s+1(z^{t+s}, z')
\]

\[
\leq y^j_t(z') + \sum_{n=1}^{N} a^j_{n,t+s}(z^{t+s-1})(p^*_n,t+s(z^{t+s}) + d_{n,t+s}(z^{t+s}))
\]

\[
+ b^j_{t+s}(z^{t+s}) \quad \forall z^{t+s} \geq z', \quad s \geq 1,
\]

\[
c^j_t(z') + \sum_{n=1}^{N} a^j_{n,t+1}(z') p^*_n,t(z')
\]

\[
\times q^*_t+1(z', z') \leq y^j_t(z') + \omega^j_t(z'),
\]

\[
+ \sum_{n=1}^{N} a^j_{n,t+s-1}(z^{t-1})(p^*_n,t+s(z') + d_{n,t+s}(z'))
\]

\[
+ b^j_{t+s}(z^{t+s}) \geq \omega^j_{t+s}(z^{t+s}) \quad \forall z^{t+s} \geq z', \quad s \geq 1,
\]

\[
c^j_{t+s}(z^{t+s}) \geq 0 \quad \text{for all } s \geq 0 \text{ and } z^{t+s} \geq z'.
\]

Call the constraint set to this maximization problem \(P_t\), and label the corresponding constraint set in the hatted equilibrium \(\hat{P}_t\). If \((c^j, a^j, b^j)\) lies in \(P_t\), then \((c^j, a^j, b^j')\) lies in \(\hat{P}_t\), where

\[
b^j_{t+s}(z^{t+s}) = b^j_{t+s}(z^{t+s}) - \sum_{n=1}^{N} a^j_{n,t+s}(z^{t+s}) e_{n,t+s}(z^{t+s}) + \sum_{n=1}^{N} e_{n,t+s}(z^{t+s}) \hat{a}^j_{n1}.
\]

As well, if \((c^j, b^j, a^j)\) lies in \(\hat{P}_t\), then \((c^j, b'^j, a^j)\) lies in \(P_t\), where

\[
b^j'_{t+s}(z^{t+s}) = b^j_{t+s}(z^{t+s}) + \sum_{n=1}^{N} a^j_{n,t+s}(z^{t+s}) e_{n,t+s}(z^{t+s}) - \sum_{n=1}^{N} e_{n,t+s}(z^{t+s}) \hat{a}^j_{n1}.
\]

Hence, \(P_t\) and \(\hat{P}_t\) are payoff-equivalent, and the solvency constraints are not too tight in the hatted economy. □

Because solvency constraints are endogenous, bubbles are an intrinsic feature of A–J equilibria. In an A–J equilibrium, agents can borrow until they are indifferent between defaulting and not defaulting. But this condition depends on what agents anticipate their solvency constraints to be. In equilibrium, assets can have bubbles because solvency constraints have bubbles, and vice versa.\(^1\)

\(^1\) Kiyotaki and Moore [4] endogenize solvency constraints in a different way. They assume that agents can use a subset of assets as collateral, and that the value of loans cannot exceed the value of collateral. Theorem 5 does not apply to this way of endogenizing solvency constraints.
It is important to note that the solvency constraints need not bind for the above theorem to apply. As long as agents have some outside option available in every date and state, regardless of how unpleasant that option, bubbles exist. One way to understand this point is to re-consider the representative agent example. Define

$$U_j^j(z_t) = u(y(1 - \lambda))/(1 - \beta)$$

so that an agent can exit the economy with a fraction \((1 - \lambda)\) of his income. Then, for any process \(\varepsilon\) such that

$$\hat{p}_t(z_t) = d/(1 - \beta) + \varepsilon_t,$$
$$\hat{o}_t(z_t) = -\lambda y/(1 - \beta) + \varepsilon_t$$

is an A–J equilibrium in which the infinitely lived asset has a bubble \(\varepsilon\). In all of these A–J equilibria, the solvency constraints do not bind.

The theorem also makes clear that bubbles and efficiency are not inconsistent. In this environment with limited enforcement, Kehoe–Levine [3] define a notion of Arrow–Debreu equilibrium in which the agents’ ex-post participation constraints are embedded into their budget sets. Any constrained Pareto optimal allocation (given the constraint that agents can leave the society at any date and state) is a Kehoe–Levine [3] equilibrium, and any Kehoe–Levine equilibrium is an A–J equilibrium. Hence, Theorem 5 shows that given any constrained Pareto optimal allocation, it is part of an equilibrium in which long-lived assets having bubbles in their prices.

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