Available online at www.sciencedirect.com
ScienceDirect
Journal of Mathematical Economics 44 (2008) 157–166
JOURNAL OF Mathematical ECONOMICS
www.elsevier.com/locate/jmateco

# Trading constraints penalizing default: A recursive approach

Luis H.B. Braido *

*Graduate School of Economics, Getulio Vargas Foundation, Praia de Botafogo 190, s.1100, Rio de Janeiro, RJ 22253-900, Brazil*

Received 6 July 2006; received in revised form 20 October 2006; accepted 23 October 2006
Available online 18 December 2006

**Abstract**

This paper proves existence of an ergodic Markov equilibrium for a class of general equilibrium economies with infinite horizon, incomplete markets, and default. Agents may choose to deny their liabilities and face trading constraints that depend on the adjusted amount of past default on each asset. These constraints replace the usual utility penalties and explore intertemporal tie-ins that appear in dynamic economies. The equilibrium prices and solvency rates present stationary properties that are usually required in econometric models of credit risk.


*JEL Classification:* D52; D80; D90; G10

*Keywords:* Trading constraints; General equilibrium; Incomplete markets; Risk; Default; Markov; Stationary; Time-homogeneous; Ergodic

## 1. Introduction

General equilibrium is a world representation intended to explain how markets coordinate egoistic individuals acting independently. This school of economic thought was initiated by Leon Walras in the 19th century, and its modern synthesis developed by Kenneth J. Arrow, Gerard Debreu, and Lionel W. Mackenzie during the 1950s. This methodology implicitly assumes the presence of a strong enforcement power that makes individuals always honor their contracts—that is, each agent faces a budget constraint in which all liabilities must be paid. However, the widespread occurrence of default makes that assumption rather counterfactual, and some recent models have attempted to incorporate this into the economic general equilibrium analysis.

The economic literature has used two distinct approaches to model default. In a first perspective, Kehoe and Levine (1993) study infinite-horizon economies with a complete set of Arrow-Debreu claims that are transacted under imperfect commitment. Agents need not honor their financial promises in which case they are excluded forever from the financial markets. Trades are not anonymous and rational traders anticipate potential defaulters. In equilibrium, agents are able to sell claims up to the point at which denying debts becomes preferable to smoothing consumption through financial trades. This limits the risk-sharing possibilities in the economy. In this line, Zhang (1997) and Alvarez and Jermann (2000) explore the exclusion penalty to endogenize borrowing constraints through a system of non-default equations; while Dutta and Kapur (2002) study an economy with the exclusion penalty in which lenders are able to capture a fraction of the borrower's investments in case of default.

* Tel.: +55 21 2559 5868.
*E-mail address:* lbraido@fgv.br.

0304-4068/$ – see front matter 
doi:10.1016/j.jmateco.2006.10.003

In a second perspective, Dubey et al. (1988, 2005) and Zame (1993) present default as an institution that improves risk sharing in economies with incomplete markets. In these models, agents are allowed not to pay their debts and face a utility penalty that is linearly proportional to the amount of default. Araujo et al. (1996) extend this class of models to an infinite-horizon environment. An appealing result obtained by these papers is that default effectively happens in equilibrium.

This paper connects those two perspectives in the literature by introducing trading constraints as an intertemporal structure to penalize default. The model consists of a general equilibrium economy with incomplete financial markets and infinite periods. Agents can deny their liabilities and face constraints on financial transactions that depend on the adjusted amount of their past default on each asset. The trading constraints balance the economy's risk-sharing possibilities and incentives to induce financial repayment. They replace the utility penalties used in Dubey et al. (2005) – which punished default in one period and canceled the debt thereafter – by a mechanism that explores dynamic tie-ins. Moreover, these constraints are smoother than the exclusion penalty explored by Kehoe and Levine (1993), since they use the amount of default occurred in the past to gradually restrict individual participation in the financial markets. Consequently, default is not necessarily ruled out in equilibrium, which may be welfare increasing in economies with incomplete markets, as shown by Dubey et al. (2005).

Given the intertemporal nature of the trading penalties, one is led to explore the recursive properties of the model. An ergodic Markov equilibrium, in the spirit of Duffie et al. (1994), is proven to exist. This result can be viewed as a theoretical foundation for econometric models of credit risk, which typically require stationary conditions on asset prices and default rates.

There are two other contemporaneous working papers that use credit constraints to penalize default. Sabarwal (2000) uses this idea in a finite-horizon economy with a continuum of agents, and Torres-Martínez (2002) models long-lived securities in infinite-horizon economies. Our works are independent, and there are many interesting differences among them. This paper, in particular, focuses on infinite-horizon economies with finitely many agents to study recursive features generated by the trading constraints. It explores an ergodic Markov equilibrium concept, which implies fundamental differences in the equilibrium properties and proof strategy.

The remainder of the paper is organized as follows. Section 2 describes the model, and Section 3 proves the existence of an ergodic Markov equilibrium. A metric for selecting optimal trading constraints is suggested in Section 4. Individual optimality conditions and the equilibrium proof are presented in the appendixes.

## 2. An economy with default

Consider a pure exchange economy with uncertainty, infinite many periods $t \in \mathbb{T} = \{0, 1, \ldots, \infty\}$, multiple consumption goods $l = 1, \ldots, L$, and heterogeneous agents $i \in \mathbb{I} = \{1, \ldots, I\}$. In each period $t$, an exogenous shock $s_t \in \mathbb{S} = \{1, \ldots, S\}$ determines the endowment vector of each agent, namely, $w_i : \mathbb{S} \to \mathbb{R}^L_{++}$. The exogenous state $\{s_t\}$ follows a time-homogeneous Markov process with transition $P$ such that $P(s_{t+1}|s_t) > 0$, $\forall s_{t+1}, s_t$.

Agent $i$'s preference is numerically represented by a time-separable expected utility function $V_i$. For any $\mathbb{R}^L_+$-valued random variable $\mathbf{x}_i = \{x_{i,t}\}_{t \in \mathbb{T}}$ (defined on some arbitrary probability space), one has:

$$V_i(\mathbf{x}_i) = E\left(\sum_{t=0}^{\infty} \beta_i^t u_i(x_{i,t})\right), \tag{1}$$

where $\beta_i \in (0, 1)$, $u_i : \mathbb{R}^L_+ \to \mathbb{R} \cup \{-\infty\}$ is continuous, bounded above, concave on $\mathbb{R}^L_+$, strictly increasing on $\mathbb{R}^L_{++}$, and $\lim_{x_{i,l,t} \to 0} u_i(x_{i,t}) = -\infty$, $\forall l = 1, \ldots, L$.

Spot and financial markets open every period to trade the $L$ consumption goods and $J \geq 0$ short-lived *numeraire* assets—that is, one-period securities in zero-net supply that pay in units of good 1 according to $a : \mathbb{S} \to [0, 1]^J$. In each decision node, agents choose their personal consumption bundle and portfolio, namely, $x_i \in \mathbb{R}^L_+$ and $(\theta_i, \varphi_i) \in \mathbb{R}^{2J}_+$, where $\theta_{i,j}$ and $\varphi_{i,j}$ represent (respectively) the number of shares of asset $j$ that has been bought and sold by agent $i$. The prices for consumption goods and assets are given by $p \in \mathbb{R}^L_+$ and $q \in \mathbb{R}^J_+$. (The decision node index is omitted for notational convenience.)

The financial markets credibly enforce an infinitesimal fraction $\varepsilon > 0$ of each financial promise made. (Since assets pay in units of good 1, this is equivalent to assume existence of state-contingent physical guarantees.) Agents can deny the fraction $(1-\varepsilon)$ of their liabilities. This choice is represented by a vector $\eta_i \in \mathbb{R}^J_+$ describing the amount of default (measured in units of good 1) on each of the $J$ existing assets. Trades are anonymous, and the buyers of each asset receive according to the market solvency rate $\gamma_j \in [\varepsilon, 1]$. The $\varepsilon$-enforceability assumption guarantees that the solvency rates are never 0, which rules out the possibility that some assets were not traded due to a pessimistic belief that they would not be repaid if traded.

Defaulters are punished by facing narrow trading possibilities in the following periods. Financial markets access information about the agents' adjusted amount of past default on each asset, represented by $h_{i,-1} \in \mathbb{R}^J_+$, $i \in \mathbb{I}$. (For each decision node, $-1$ is used to represent the immediately preceding node, while $+1$ indicates the successor nodes in the next period.) This variable evolves over time according to the following law of motion:

$$h_i = \eta_i + \delta h_{i,-1}, \tag{2}$$

where $\delta \in [0, 1)$ is the rate at which past default is forgiven by the financial markets. Economies in which the parameter $\delta$ equals 0 are those where default is penalized for one single period. On the other hand, default will be registered forever in economies with $\delta > 0$. This recursive formulation resembles some risk-rating methodologies used in credit markets.

Agents face trading constraints represented by functions mapping the current realization of $s \in \mathbb{S}$ and the individual past default $(h_{i,-1})$ into limits for long and short transactions on each of the $J$ existing assets. Formally, define $\bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$ and let the limits for long and short trades be given by $\lambda_{\text{buy},j} : \mathbb{S} \times \mathbb{R}^J_+ \to \bar{\mathbb{R}}_+$ and $\lambda_{\text{sell},j} : \mathbb{S} \times \mathbb{R}^J_+ \to \mathbb{R}_+$, $\forall j$. By making these functions contingent on the current shock, one allows the financial markets to alleviate the penalties in periods with bad aggregate shocks. Trading constraints depending on $h_{i,-1}$ works as an incentive mechanism to induce repayment.

It is possible to have $\lambda_{\text{buy},j}(\cdot) = +\infty$. However, $\lambda_{\text{sell},j}$ is assumed to be bounded above by $\bar{\lambda} \geq 0$, $\forall j$. This rules out Ponzi schemes and is used to guarantee existence of a recursive equilibrium.[1] Moreover, in order to avoid discontinuities and nonconvexities in the budget set, the functions $\lambda_{\text{buy},j}$ and $\lambda_{\text{sell},j}$ are assumed to be continuous, nonincreasing, and concave on the subset of $\mathbb{R}^J_+$ where the default histories will live—as defined later in Assumption 5.

Agents maximize their expected utility taking as given their previous-period portfolio and default history $(\theta_{i,-1}, \varphi_{i,-1}, h_{i,-1}) \in \mathbb{R}^{3J}_+$, the current prices and market solvency rates $(p, q, \gamma) \in \mathbb{R}^{L+J}_+ \times [\varepsilon, 1]^J$, and the stochastic process describing future prices and solvency rates. In each decision node, agent $i$'s choices $(x_i, \theta_i, \varphi_i, \eta_i, h_i) \in \mathbb{R}^{L+4J}_+$ must satisfy condition (2) and the following budget constraints:

$$p \cdot (x_i - w_{i,s}) + q \cdot (\theta_i - \varphi_i) \leq p_1 \sum_{j=1}^{J} (\gamma_j a_{j,s} \theta_{i,j,-1} - a_{j,s} \varphi_{i,j,-1} + \eta_{i,j}); \tag{3}$$

$$\theta_{i,j} \leq \lambda_{\text{buy},j}(s, h_{i,-1}), \quad \forall j; \tag{4}$$

$$\varphi_{i,j} \leq \lambda_{\text{sell},j}(s, h_{i,-1}), \quad \forall j; \tag{5}$$

$$\eta_{i,j} \leq (1-\varepsilon) a_{j,s} \varphi_{i,j,-1}, \quad \forall j. \tag{6}$$

Eq. (3) states that agents can use their endowments and financial transfers to purchase consumption goods and a new portfolio. Inequalities (4) and (5) are the trading constraints, and (6) imposes that the amount of default on each asset cannot be larger than the non-enforceable part of the liability.

From conditions (5) and (6), one has $\eta_{i,j} \leq \bar{\lambda}$. Therefore, the adjusted amount of past default on each asset is restricted to lie in $\mathbb{H} = \{h_i \in \mathbb{R}^J_+ : \max(h_i) \leq 1 + \bar{\lambda}/(1-\delta)\}$.[2] Moreover, each agent $i$ will be forced to transfer $\varepsilon \sum_{j=1}^{J} a_{j,s} \varphi_{i,j,-1}$ units of good 1 in each decision node. In order to guarantee that net endowments are strictly positive in each node, the term $\varepsilon$ must be such that $\varepsilon \sum_{j=1}^{J} a_{j,s} \varphi_{i,j,-1} < w_{i,1,s}$, $\forall i, s$. Therefore, it is assumed that $0 < \varepsilon \leq \min(1, \bar{\varepsilon})$, where $0 < \bar{\varepsilon} < \min_{i,s}(w_{i,1,s}) / \max(1, J\bar{\lambda})$. Naturally, the economy's parameters could be chosen in such a way that $\varepsilon = 1$. In this case, one would have the standard general equilibrium model without default.

The key assumptions of the model can be now summarized as follows.

[1] This assumption would also allow us to extend the existence result for economies with real securities, $\hat{a} : \mathbb{S} \to [0, 1]^{JL}$.

[2] Note that $h_{i,-1} \in \mathbb{H} \Rightarrow \max(h_i) < 1 + \frac{\bar{\lambda}}{1-\delta}$.

**Assumption 1.** There are infinite periods, $L > 0$ consumption goods, $I > 0$ agents, and $S > 0$ possible realizations for the exogenous shock in each period ($s_t$). The endowment vectors in each decision node are strictly positive and depend on the current realization of $s_t$, that is, $w_{i,s_t} \in \mathbb{R}^L_{++}$, $\forall i$. The shock $s_t$ follows a time-homogeneous Markov process with transition $P(s_{t+1}|s_t) > 0$, $\forall s_{t+1}, s_t$.

**Assumption 2.** Preferences are represented by $V_i(\mathbf{x}_i) = E(\sum_{t=0}^{\infty}\beta_i^t u_i(x_{i,t}))$, where $\beta_i \in (0, 1)$, $u_i : \mathbb{R}^L_+ \to \mathbb{R} \cup \{-\infty\}$ is continuous, bounded above, concave on $\mathbb{R}^L_+$, strictly increasing on $\mathbb{R}^L_{++}$, and $\lim_{x_{i,l,t}\to 0} u_i(x_{i,t}) = -\infty$, $\forall l = 1, \ldots, L$.

**Assumption 3.** In each decision node, there are $J \geq 0$ short-lived *numeraire* securities in zero-net supply.

**Assumption 4.** The information about each agent's history of default is represented by the vector $h_{i,-1} \in \mathbb{R}^J_+$, where $h_i = \eta_i + \delta h_{i,-1}$, for a given $\delta \in [0, 1)$.

**Assumption 5.** Agents face trading constraints $\lambda_{\text{buy}} : \mathbb{S} \times \mathbb{R}^J_+ \to \bar{\mathbb{R}}^J_+$ and $\lambda_{\text{sell}} : \mathbb{S} \times \mathbb{R}^J_+ \to \mathbb{R}^J_+$. For each $j$, $\lambda_{\text{sell},j}$ is bounded above by $\bar{\lambda} \geq 0$, and both $\lambda_{\text{buy},j}$ and $\lambda_{\text{sell},j}$ are continuous, nonincreasing, and concave on $\mathbb{H} = \{h_i \in \mathbb{R}^J_+ : \max(h_i) \leq 1 + \bar{\lambda}/(1-\delta)\}$.

**Assumption 6.** A fraction $\varepsilon$ of each promise is credibly enforceable, where $0 < \varepsilon \leq \min(1, \bar{\varepsilon})$ and $0 < \bar{\varepsilon} < \min_{i,s}(w_{i,1,s})/\max(1, J\bar{\lambda})$.

This economy is characterized by $\xi = \{\mathbb{I}, \mathbb{T}, \mathbb{S}, P, (V_i, w_i)_{i\in\mathbb{I}}, a, \varepsilon, \delta, \lambda_{\text{buy}}, \lambda_{\text{sell}}\}$. In any competitive equilibrium for $\xi$, the markets for consumption goods and financial assets must clear and the solvency rates must be consistent with the effective payments. Therefore, in each decision node, one must have:

$$\sum_{i\in\mathbb{I}}(x_i - w_{i,s}, \theta_i - \varphi_i) = \mathbf{0}; \tag{7}$$

and $\gamma_j \in [\varepsilon, 1]$ such that:

$$\gamma_j \sum_{i\in\mathbb{I}} a_j\varphi_{i,j,-1} = \sum_{i\in\mathbb{I}}(a_j\varphi_{i,j,-1} - \eta_{i,j}), \quad \forall j. \tag{8}$$

Equality (8) states that lenders have the correct expectation about the fraction of promises that are actually delivered. Since $\sum_{i\in\mathbb{I}}\varphi_{i,j,-1} = \sum_{i\in\mathbb{I}}\theta_{i,j,-1}$ in equilibrium, this condition also implies that the amount of good 1 transferred to buyers equals the amount of resources transferred from sellers.

## 3. Ergodic Markov equilibrium

The equilibrium analysis follows the structure formulated by Duffie et al. (1994), where the main elements are the state space and the expectations correspondence.

### 3.1. State space

The first element of an ergodic equilibrium is a nonempty Borel space $\mathbb{Z}$ in which the equilibrium process is valued. A state $z$ is a variable that completely describes the current endogenous and exogenous variables and is a sufficient statistic for the future evolution of the model. The exogenous state is $s \in \mathbb{S}$. The endogenous state is composed of the previous-period portfolios and default histories $(\theta_{-1}, \varphi_{-1}, h_{-1}) \in \mathbb{Y}_1 = \mathbb{R}^{2JI}_+ \times \mathbb{H}^I$, and the current individual choices, prices, and solvency rates $(x, \theta, \varphi, \eta, h, p, q, \gamma) \in \mathbb{Y}_2 = \mathbb{R}^{(L+3J)I}_+ \times \mathbb{H}^I \times \mathbb{R}^{L+J}_+ \times [\varepsilon, 1]^J$. The state variable is then given by:

$$z = (s, \theta_{-1}, \varphi_{-1}, h_{-1}, x, \theta, \varphi, \eta, h, p, q, \gamma) \in \mathbb{S} \times \mathbb{Y}_1 \times \mathbb{Y}_2. \tag{9}$$

The feasibility conditions are imposed on the definition of the state space:

$$\mathbb{Z} = \{z \in \mathbb{S} \times \mathbb{Y}_1 \times \mathbb{Y}_2 : (7)\text{–}(8)\}. \tag{10}$$

### 3.2. Expectations Correspondence

The second element of the equilibrium analysis is the expectations correspondence $G(z)$ that defines a set of probability measures for the state variable in the following period conditional on the current state $z \in \mathbb{Z}$. This correspondence embeds the key features of a competitive equilibrium. Formally, let $\mathcal{B}_{\mathbb{Z}}$ be the Borel sigma-algebra over $\mathbb{Z}$ and $\mathcal{P}_{\mathbb{Z}}$ be the set of probability measures on $(\mathbb{Z}, \mathcal{B}_{\mathbb{Z}})$. The expectations correspondence $G : \mathbb{Z} \to \mathcal{P}_{\mathbb{Z}}$ is then constructed as follows.

First, let $w_{\inf} = \min_{i,s}(w_{i,1,s} - \varepsilon \max(1, J\bar{\lambda}), w_{i,2,s}, \ldots, w_{i,L,s})$ be a lower bound for the amount of endowment received by any agent after denying all previous liabilities. Note that $w_{\inf} \in \mathbb{R}^L_{++}$, thanks to Assumptions 1 and 6. Thus, given Assumption 2, there exists $x_{\inf} \in (0, \min(w_{\inf}))$ such that $\min(x_i) \leq x_{\inf}$ implies $u_i(x_i) + E(\sum_{t=1}^{\infty} \beta_i^t u_i(\cdot)) < u_i(w_{\inf}) + E(\sum_{t=1}^{\infty} \beta_i^t u_i(w_{i,t}))$, $\forall i \in \mathbb{I}$. Any optimal plan $\mathbf{x}_i = \{x_{i,t}\}_{t \in \mathbb{T}}$ must present $\min(x_{i,t}) > x_{\inf}$ a.e., $\forall t \in \mathbb{T}$. (Otherwise, $\mathbf{x}_i$ will be dominated by the process where the original consumption vector is replaced by $w_{\inf}$ in the nodes where $\min(x_i) \leq x_{\inf}$ and then by $w_i$ in all subsequent nodes.) Therefore, without loss of generality, one can restrict $x_i$ to lie in $\mathbb{X} = \{x_i \in \mathbb{R}^L_+ : \min(x_i) \geq x_{\inf}\}$.

Next, define the set-valued function $g : \mathbb{Z} \to \mathcal{P}_{\mathbb{Z}}$ to be such that, for each $z = (s, \theta_{-1}, \varphi_{-1}, h_{-1}, x, \theta, \varphi, \eta, h, p, q, \gamma) \in \mathbb{Z}$, the measure $\pi \in g(z)$ if and only if:

(a) the support of $\pi$ is the graph of some function mapping $\mathbb{S}$ into $\mathbb{Y}_1 \times \mathbb{Y}_2$;
(b) the marginal of $\pi$ on $\mathbb{S}$ is $P(s_{+1}|s)$, and the marginal of $\pi$ on $\mathbb{Y}_1$ is degenerated into $(\theta, \varphi, h)$ almost surely;
(c) $x_i \in \mathbb{X}$, $\forall i$;
(d) the endogenous choices in $z$ satisfy the budget-constraint inequalities (2)–(6) and the optimality conditions (B.1)–(B.13) derived in Appendix B.

Condition (a) implies that the all measures in $g(z)$ have finite support with $S$ mass points. Those measures are spotless in the sense that all uncertainty is related to fundamentals of the economy (i.e., $s \in \mathbb{S}$). Conditions (b)–(d) imply that the measures in $g$ are consistent with the exogenous-shock probability and the agents' optimal decisions. The set $g(z)$ may be empty for some values of $z \in \mathbb{Z}$.

Finally, following (Duffie et al., 1994, p. 767), let the expectations correspondence $G(z)$ be the closure of the convex hull of $g(z)$, $\forall z \in \mathbb{Z}$. The expectations in $G(z)$ are conditionally spotless in the sense that sunspots are used only to randomize over spotless transitions.

### 3.3. Equilibrium

The economy $\xi$ is represented by the state space $\mathbb{Z}$, the expectations correspondence $G$, and some arbitrary initial condition $(\hat{s}, \hat{\theta}_{-1}, \hat{\varphi}_{-1}, \hat{h}_{-1}) \in \mathbb{S} \times \mathbb{Y}_1$. The following definitions describe the equilibrium concept used here.

**Definition 1.** A stationary Markov equilibrium for economy $\xi$ is a pair $(Z, \Pi)$, where $Z$ is a measurable subset of $\mathbb{Z}$ and $\Pi : Z \to \mathcal{P}_Z$ is a transitional probability such that $\Pi_z \in G(z)$, $\forall z \in Z$.

**Definition 2.** An invariant ergodic measure for a transition $(Z, \Pi)$ is a measure $\mu \in \mathcal{P}_Z$ such that: (i) $\mu(\mathcal{A}) = \int_Z \Pi_z(\mathcal{A})\, \mathrm{d}\mu(z)$, for any measurable set $\mathcal{A} \subset Z$; and (ii) either $\mu(\mathcal{A}) = 1$ or $\mu(\mathcal{A}) = 0$, for any measurable set $\mathcal{A} \subset Z$ such that $\Pi_z \in \mathcal{P}_{\mathcal{A}}$ for $\mu$-a.e. $z \in \mathcal{A}$.

**Definition 3.** An ergodic Markov equilibrium for $\xi$ is a stationary Markov equilibrium $(Z, \Pi)$ with an invariant ergodic measure $\mu \in \mathcal{P}_Z$.

A stationary Markov equilibrium (Definition 1) is composed of a set of states and a law of motion such that the current realization of $z$ determines the future stochastic equilibrium path. An ergodic Markov equilibrium (Definition 3) adds to the previous concept the condition that, if the initial state is drawn with distribution $\mu$, then the distribution of all future realization of the system is also $\mu$. This is the analogue of the deterministic notion of a steady state.

The Markov equilibrium concepts encompass the Walrasian notion of equilibrium. A few definitions are necessary in order to explain this. A $\mathbb{Z}$-valued stochastic process $\{z_t\}_{t \in \mathbb{T}}$ is said to be consistent for $\xi$ if the conditional probability of $s_{t+1}$ given $(z_0, \ldots, z_t)$ is $P_{s_t}$ almost surely, $\forall t \in \mathbb{T}$. A policy $\{x_{i,t}, \theta_{i,t}, \varphi_{i,t}, \eta_{i,t}, h_{i,t}\}_{t \in \mathbb{T}}$ is feasible if it is

$(z_0, \ldots, z_t)$-measurable and satisfies (2)–(6) a.e., $\forall t \in \mathbb{T}$. It is optimal for agent $i$ if there is no other feasible policy $\{\tilde{x}_{i,t}, \tilde{\theta}_{i,t}, \tilde{\varphi}_{i,t}, \tilde{\eta}_{i,t}, \tilde{h}_{i,t}\}_{t \in \mathbb{T}}$ such that $V_i(\tilde{\mathbf{x}_i}) > V_i(\mathbf{x}_i)$. Hence, a Walrasian equilibrium is a consistent state process with the property that the policy $\{x_{i,t}, \theta_{i,t}, \varphi_{i,t}, \eta_{i,t}, h_{i,t}\}_{t \in \mathbb{T}}$ is individually optimal, $\forall i \in \mathbb{I}$. It follows from condition (b) that any stationary Markov $Z$-valued process $\{z_t\}_{t \in \mathbb{T}}$ with transition $\Pi$ is consistent. Moreover, from the Bellman's principle of optimality, condition (d) implies that the agents' plans are individually optimal.

**Proposition 1.** *There exists an ergodic Markov equilibrium $(Z, \Pi, \mu)$ for $\xi$ with the property that, for any initial condition $(\hat{s}, \hat{\theta}_{-1}, \hat{\varphi}_{-1}, \hat{h}_{-1}) \in \mathbb{S} \times \mathbb{Y}_1$, there is a vector $(x, \theta, \varphi, \eta, h, p, q, \gamma) \in \mathbb{Y}_2$ such that $(\hat{s}, \hat{\theta}_{-1}, \hat{\varphi}_{-1}, \hat{h}_{-1}, x, \theta, \varphi, \eta, h, p, q, \gamma) \in Z$.*

**Proof.** See Appendix C. □

## 4. Selecting the trading-constraint structure

The assumption that agents face exogenous transaction constraints has been extensively used in finance, and the motivation for that is frequently associated to the possibility of default. This paper introduces trading constraints as an intertemporal penalty for default. Under proper assumptions, an equilibrium exists for a large class of trading-constraint functions. However, the equilibrium risk-sharing properties will depend on the design of these func- tions.

A metric for endogenously selecting the economy's penalty structure is suggested here. The procedure follows the idea behind the classical Ramsey-taxation problem. There exists a ergodic Markov equilibrium associated with each fixed trading-constraint structure $\sigma = (\delta, \lambda_{\text{buy}}, \lambda_{\text{sell}})$. Anticipating this, a market designer chooses a structure that attains some optimal welfare criterion.

Formally, let $\sigma = (\delta, \lambda_{\text{buy}}, \lambda_{\text{sell}})$ lie in $\Omega = [0, 1) \times \mathcal{F}_{\text{buy}} \times \mathcal{F}_{\text{sell}}$, where $\mathcal{F}_{\text{buy}}$ (resp. $\mathcal{F}_{\text{sell}}$) represents the space of functions (resp. bounded functions) mapping $\mathbb{S} \times \mathbb{R}^J_+$ into $\bar{\mathbb{R}}^J_+$ (resp. $\mathbb{R}^J_+$) that are continuous, nonincreasing, and concave on $\mathbb{H} \subset \mathbb{R}^J_+$. Fix an arbitrary initial condition $(\hat{s}, \hat{\theta}_{-1}, \hat{\varphi}_{-1}, \hat{h}_{-1}) \in \mathbb{S} \times \mathbb{Y}_1$. For each $\sigma \in \Omega$, let $\mathbb{M}(\sigma)$ be the set of stochastic consumption plans $\mathbf{x}_\sigma = (\mathbf{x}_{\sigma,i})_{i \in \mathbb{I}}$ derived from some ergodic Markov equilibrium $(Z, \Pi, \mu)$ associated with $\sigma$. Next, fix a vector of Pareto weights $\alpha \in \mathbb{R}^I_{++}$, and let $\mathbb{W}(\sigma)$ be the convex hull of the set $\{W \in \mathbb{R} : \exists \mathbf{x}_\sigma \in \mathbb{M}(\sigma);\ W = \sum_{i \in \mathbb{I}} \alpha_i V_i(\mathbf{x}_{\sigma,i})\}$, $\forall \sigma \in \Omega$.[3] Then, select a welfare function $W : \Omega \to \mathbb{R}$ from $\mathbb{W} : \Omega \to \mathbb{R}$, and let the quasi-optimal trading-constraint structure be:

$$\sigma^* \in arg \sup_{\sigma \in \Omega} W(\sigma). \tag{11}$$

The selection $W$ and the quasi-optimal $\sigma^* = (\delta^*, \lambda^*_{\text{buy}}, \lambda^*_{\text{sell}})$ always exist, but one cannot guarantee that the maximum will be achieved. The quasi-optimal trading constraints must balance risk-sharing possibilities and incentives to prevent default. The history of default will affect the liquidity faced by agents, but defaulters will not be abruptly excluded from the financial markets.

## Acknowledgements

I am thankful for comments from Aloisio Araujo, Jose Fajardo, Cristian Huse, Andrew McLennan, Mario R. Pascoa, Abdelkrim Seghir, an anonymous referee, and seminar participants at the University of Chicago (2000), 24th SBE Meeting (2002), PUC-Rio (2004), and first General Equilibrium Workshop in Rio (2006). I am especially indebted to Juan P. Torres-Martínez for important remarks. Financial support from CNPq is gratefully acknowledged.

## Appendix A. Finite-horizon optimal conditions

A $T$-horizon economy $\xi_T = \{\mathbb{I}, \mathbb{T}_T, \mathbb{S}, P, (V_{i,T}, w_i)_{i \in \mathbb{I}}, a, \varepsilon, \delta, \lambda_{\text{buy}}, \lambda_{\text{sell}}\}$ is defined as follows. Fix an initial condition $(\hat{s}, \hat{\theta}_{-1}, \hat{\varphi}_{-1}, \hat{h}_{-1}) \in \mathbb{S} \times \mathbb{Y}_1$, and let $t \in \mathbb{T}_T = \{0, 1, 2, \ldots, T\}$. There are $S^* = (S^{T+1} - 1)/(S - 1)$ decision nodes

[3] This convex hull defines the set of all expected weighted-utilities that can be obtained from beliefs on the set of ergodic Markov equilibria allocations associated with each $\sigma$.

indexed by $s^t = (\hat{s}, s_1, \ldots, s_t)$. (For sake of clarity, all variables will be indexed here by their respective nodes.) Preferences are represented by $V_{i,T}(\mathbf{x}_i) = E\left(\sum_{t=0}^{T} \beta_i^t u_i(x_{i,s^t})\right)$, $\forall \mathbf{x}_i \in \mathbb{R}_+^{LS^*}$. The following terminal condition is assumed to hold: $(\theta_{i,s^T}, \varphi_{i,s^T}, \eta_{i,s^{T-1}}, \eta_{i,s^T}) = \mathbf{0}$, $\forall s^{T-1}, s^T$.

Note that $V_{i,T}(\mathbf{x}_i)$ is continuous and concave, and the budget set $\mathrm{BC}_i(\mathbf{p}, \mathbf{q}, \boldsymbol{\gamma}) = \{(\mathbf{x}_i, \boldsymbol{\theta}_i, \boldsymbol{\varphi}_i, \boldsymbol{\eta}_i, \mathbf{h}_i) \in \mathbb{R}_+^{(L+3J)S^*} \times \mathbb{H}^{S^*} : (2)\text{–}(6), \forall s^t \in \mathbb{S}^t, t \in \mathbb{T}_T; \text{ and } (\theta_{i,s^T}, \varphi_{i,s^T}, \eta_{i,s^{T-1}}, \eta_{i,s^T}) = \mathbf{0}, \forall s^{T-1}, s^T\}$ is convex. Thus, agent $i$'s optimal decisions must satisfy the Kuhn–Tucker conditions for nonsmooth concave functions (see Balder, 2001).

Formally, define the following supergradient sets:

(a) for each $x_i \in \mathbb{R}_+^L$:

$$\partial u_i(x_i) = \{d \in \mathbb{R}_{++}^L : u_i(\tilde{x}_i) \le u_i(x_i) + d \cdot (\tilde{x}_i - x_i), \quad \forall \tilde{x}_i \in \mathbb{R}_+^L\}; \tag{A.1}$$

(b) for each $j \in \{1, \ldots, J\}$, $s \in \mathbb{S}$, and $h_i \in \mathbb{H}$:

$$\partial\lambda_{\text{buy},j}(s, h_i) = \{d \in \mathbb{R}_-^J : \lambda_{\text{buy},j}(s, \tilde{h}_i) \le \lambda_{\text{buy},j}(s, h_i) + d \cdot (\tilde{h}_i - h_i), \quad \forall \tilde{h}_i \in \mathbb{R}_+^J\}; \tag{A.2}$$

$$\partial\lambda_{\text{sell},j}(s, h_i) = \{d \in \mathbb{R}_-^J : \lambda_{\text{buy},j}(s, \tilde{h}_i) \le \lambda_{\text{buy},j}(s, h_i) + d \cdot (\tilde{h}_i - h_i), \quad \forall \tilde{h}_i \in \mathbb{R}_+^J\}. \tag{A.3}$$

Then, a $T$-horizon plan $(\mathbf{x}_i, \boldsymbol{\theta}_i, \boldsymbol{\varphi}_i, \boldsymbol{\eta}_i, \mathbf{h}_i)$ is optimal for agent $i$ if and only if it belongs to $\mathrm{BC}_i(\mathbf{p}, \mathbf{q}, \boldsymbol{\gamma})$ and there exist supergradients and multipliers:

- $d_{u_i,s^t} \in \partial u_i(x_{i,s^t})$ and $A_{i,s^t} \in \mathbb{R}_+$, $\forall s^t, t \le T$;
- $d_{i,\lambda_{\text{buy},j},s^{t+1}} \in \partial\lambda_{\text{buy},j}(s_{t+1}, h_{i,s^t})$ and $(B_{i,s^t}, \hat{B}_{i,s^t}) \in \mathbb{R}_+^{2J}$, $\forall j, s^t, t < T$;
- $d_{i,\lambda_{\text{sell},j},s^{t+1}} \in \partial\lambda_{\text{sell},j}(s_{t+1}, h_{i,s^t})$ and $(C_{i,s^t}, \hat{C}_{i,s^t}) \in \mathbb{R}_+^{2J}$, $\forall j, s^t, t < T$;
- $(D_{i,s^t}, \hat{D}_{i,s^t}) \in \mathbb{R}_+^{2J}$, $\forall s^t, t < T - 1$;

such that conditions (A.4)–(A.11) hold.[4]

The notation is as follows: $\bar{P}(s^t|\hat{s})$ defines the probability of obtaining the history $s^t$ conditional on the initial state $\hat{s}$;[5] $\mathbb{S}^{t+1}|s^t = \{\tilde{s}^{t+1} \in \mathbb{S}^{t+1} : \tilde{s}^t = s^t\}$, $\forall t < T$; and $d_{i,\lambda_{\text{buy},n},j,s^t}$ (resp. $d_{i,\lambda_{\text{sell},n},j,s^t}$) is the $j$-th coordinate of the vector $d_{i,\lambda_{\text{buy},n},s^t}$ (resp. $d_{i,\lambda_{\text{sell},n},s^t}$). In Eq. (A.6), (A.7), due to the terminal conditions imposed on the default decision, define $D_{i,j,s^t} = \hat{D}_{i,j,s^t} = 0$ for $t \ge T - 1$.

*Normal Lagrange inclusion:*

$$\beta_i^t \bar{P}(s^t|\hat{s}) d_{u_i,s^t} = A_{i,s^t} p_{s^t}, \ \forall s^t, t \le T; \tag{A.4}$$

$$A_{i,s^t} q_{j,s^t} = \sum_{\mathbb{S}^{t+1}|s^t} [A_{i,s^{t+1}} p_{1,s^{t+1}} \gamma_{j,s^{t+1}} a_{j,s_{t+1}}] - B_{i,j,s^t} + \hat{B}_{i,j,s^t}, \quad \forall j, s^t, t < T; \tag{A.5}$$

$$A_{i,s^t} q_{j,s^t} = \sum_{\mathbb{S}^{t+1}|s^t} [A_{i,s^{t+1}} p_{1,s^{t+1}} a_{j,s_{t+1}} - (1-\varepsilon) D_{i,j,s^{t+1}} a_{j,s_{t+1}}] + C_{i,j,s^t} - \hat{C}_{i,j,s^t}, \quad \forall j, s^t, t < T; \tag{A.6}$$

$$\begin{aligned} A_{i,s^t} p_{1,s^t} = \sum_{\mathbb{S}^{t+1}|s^t} \Bigg\{ & A_{i,s^{t+1}} p_{1,s^{t+1}} \delta - \sum_{n=1}^{J} [B_{i,n,s^{t+1}} d_{i,\lambda_{\text{buy},n},j,s^{t+1}} + C_{i,n,s^{t+1}} d_{i,\lambda_{\text{sell},n},j,s^{t+1}}] \\ & - \delta D_{i,j,s^{t+1}} + \delta \hat{D}_{i,j,s^{t+1}} \Bigg\} + D_{i,j,s^t} - \hat{D}_{i,j,s^t}, \quad \forall j, s^t, t < T - 1; \end{aligned} \tag{A.7}$$

[4] Note that the Slater's condition hold if $\lambda_{\text{buy}}(\cdot) \gg \mathbf{0}$, $\lambda_{\text{sell}}(\cdot) \gg \mathbf{0}$, and $p \neq \mathbf{0}$. Moreover, the Kuhn–Tucker conditions are still necessary and sufficient for optimality when $p \gg \mathbf{0}$ but some or all portfolio choices are irrelevant (in the sense of being restricted to be null by the trading constraints). Finally, the optimization problem has no solution and no plan will satisfy condition (A.4) when $p_{l,s^t} = 0$ for some $l, s^t$. (Therefore, one can still say that the Kuhn–Tucker conditions are necessary and sufficient for optimality in this case.)

[5] Namely, $\bar{P}(\hat{s}|\hat{s}) = 1$, for $t = 0$; and $\bar{P}(s^t|\hat{s}) = P(s_t|s_{t-1}) \ldots P(s_1|\hat{s})$, for $t > 0$.

*Complementary slackness:*

$$A_{i,s^t}\left[p_{s^t}\cdot(x_{i,s^t}-w_{i,s_t})+q_{s^t}\cdot(\theta_{i,s^t}-\varphi_{i,s^t})-p_{1,s^t}\sum_{j=1}^{J}(\gamma_{j,s^t}a_{j,s_t}\theta_{i,j,s^{t-1}}-a_{j,s_t}\varphi_{i,j,s^{t-1}}+\eta_{i,j,s^t})\right]=0,$$
$$\forall s^t, t\le T; \tag{A.8}$$

$$B_{i,j,s^t}[\theta_{i,j,s^t}-\lambda_{\text{buy},j}(s_t,h_{i,s^{t-1}})]=0 \text{ and } \hat{B}_{i,j,s^t}\theta_{i,j,s^t}=0,\quad \forall j, s^t, t<T; \tag{A.9}$$

$$C_{i,j,s^t}[\varphi_{i,j,s^t}-\lambda_{\text{sell},j}(s_t,h_{i,s^{t-1}})]=0 \text{ and } \hat{C}_{i,j,s^t}\varphi_{i,j,s^t}=0,\quad \forall j, s^t, t<T; \tag{A.10}$$

$$D_{i,j,s^t}[\eta_{i,j,s^t}-(1-\varepsilon)a_{j,s_t}\varphi_{i,j,s^{t-1}}]=0 \text{ and } \hat{D}_{i,j,s^t}\eta_{i,j,s^t}=0,\quad \forall j, s^t, t<T-1. \tag{A.11}$$

## Appendix B. Stochastic Euler equations

Similar to the $T$-horizon Kuhn–Tucker conditions, the necessary and sufficient conditions for recursive optimality can be stated as follows. For any given state $z\in\mathbb{Z}$ and transition $\pi$ satisfying conditions (a)–(c) in the definition of $g$ (see Section 3), the policy $(x_i,\theta_i,\varphi_i,\eta_i,h_i)\in\mathbb{R}_+^{L+3J}\times\mathbb{H}$ is optimal for agent $i$ if and only if it satisfies conditions (2)–(6) and there exist:

- $d_{u_i}\in\partial u_i(x_i)$ and $A'_i\in\mathbb{R}_+$;
- $(B'_i,\hat{B}'_i,C'_i,\hat{C}'_i,D'_i,\hat{D}'_i)\in\mathbb{R}_+^{6J}$;
- $d_{u_i,s_{+1}}\in\partial u_i(x_{i,s_{+1}})$ and $A'_{i,s_{+1}}\in\mathbb{R}_+$, $\forall s_{+1}$;
- $d_{i,\lambda_{\text{buy},j},s_{+1}}\in\partial\lambda_{\text{buy},j}(s_{+1},h_i)$ and $(B'_{i,s_{+1}},\hat{B}'_{i,s_{+1}})\in\mathbb{R}_+^{2J}$, $\forall j, s_{+1}$;
- $d_{i,\lambda_{\text{sell},j},s_{+1}}\in\partial\lambda_{\text{sell},j}(s_{+1},h_i)$ and $(C'_{i,s_{+1}},\hat{C}'_{i,s_{+1}})\in\mathbb{R}_+^{2J}$, $\forall j, s_{+1}$;
- $(D'_{i,s_{+1}},\hat{D}'_{i,s_{+1}})\in\mathbb{R}_+^{2J}$, $\forall s_{+1}$;

such that conditions (B.1)–(B.13) hold.[6] (Since short sales are bounded by $\bar{\lambda}$, the consumption plans generated by the stochastic Euler equations will be uniformly bounded. Therefore, the transversality condition will hold automatically.)

*Normal Lagrange inclusion:*

$$d_{u_i}=A'_i p; \tag{B.1}$$

$$\beta_i P(s_{+1}|s)d_{u_i,s_{+1}}=A'_{i,s_{+1}}p_{s_{+1}}; \tag{B.2}$$

$$A'_i q_j=\beta_i E_\pi(d_{u_i,1,s_{+1}}\gamma_{j,s_{+1}}a_{j,s_{+1}})-B'_{i,j}+\hat{B}'_{i,j},\quad \forall j; \tag{B.3}$$

$$A'_i q_j=\beta_i E_\pi(d_{u_i,1,s_{+1}}a_{j,s_{+1}})-(1-\varepsilon)\sum_{s_{+1}=1}^{S}D'_{i,j,s_{+1}}a_{j,s_{+1}}+C'_{i,j}-\hat{C}'_{i,j},\quad \forall j; \tag{B.4}$$

$$d_{u_i,1}=\delta\beta_i E_\pi(d_{u_i,1,s_{+1}})-\sum_{s_{+1}=1}^{S}\left\{\sum_{n=1}^{J}\left[B'_{i,n,s_{+1}}d_{i,\lambda_{\text{buy},n},j,s_{+1}}+C'_{i,n,s_{+1}}d_{i,\lambda_{\text{sell},n},j,s_{+1}}\right]\right.$$
$$\left.+\delta D'_{i,j,s_{+1}}-\delta\hat{D}'_{i,j,s_{+1}}\right\}+D'_{i,j}-\hat{D}'_{i,j},\quad \forall j. \tag{B.5}$$

[6] Condition (a) in the definition of $g$ imposes that there is only one endogenous state associated with each exogenous state $s_{+1}$. Therefore, any $z_{+1}$ in the support of $\pi$ can be indexed by $s_{+1}$, i.e., $z_{+1}=(s_{+1},x_{s_{+1}},\theta_{s_{+1}},\varphi_{s_{+1}},\eta_{s_{+1}},h_{s_{+1}},p_{s_{+1}},q_{s_{+1}},\gamma_{s_{+1}})$.

*Complementary slackness:*

$$A_i'[p \cdot (x_i - w_{i,s}) + q \cdot (\theta_i - \varphi_i) - p_1 \sum_{j=1}^{J} (\gamma_j a_{j,s} \theta_{i,j,-1} - a_{j,s} \varphi_{i,j,-1} + \eta_{i,j})] = 0; \tag{B.6}$$

$$B_{i,j}'[\theta_{i,j} - \lambda_{\text{buy},j}(s, h_{i,-1})] = 0 \text{ and } \hat{B}_{i,j}' \theta_{i,j} = 0, \quad \forall j; \tag{B.7}$$

$$C_{i,j}'[\varphi_{i,j} - \lambda_{\text{sell},j}(s, h_{i,-1})] = 0 \text{ and } \hat{C}_{i,j}' \varphi_{i,j} = 0, \quad \forall j; \tag{B.8}$$

$$D_{i,j}'[\eta_{i,j} - (1-\varepsilon) a_{j,s} \varphi_{i,j,-1}] = 0 \text{ and } \hat{D}_{i,j}' \eta_{i,j} = 0, \quad \forall j. \tag{B.9}$$

For each $s_{+1} = 1, \ldots, S$:

$$A_i'[p_{s_{+1}} \cdot (x_{i,s_{+1}} - w_{i,s_{+1}}) + q \cdot (\theta_{i,s_{+1}} - \varphi_{i,s_{+1}}) \quad - \quad p_{1,s_{+1}} \sum_{j=1}^{J} (\gamma_{j,s_{+1}} a_{j,s_{+1}} \theta_{i,j} - a_{j,s_{+1}} \varphi_{i,j} + \eta_{i,j,s_{+1}})] = 0; \tag{B.10}$$

$$B_{i,j,s_{+1}}'[\theta_{i,j,s_{+1}} - \lambda_{\text{buy},j}(s_{+1}, h_i)] = 0 \text{ and } \hat{B}_{i,j,s_{+1}}' \theta_{i,j,s_{+1}} = 0, \quad \forall j; \tag{B.11}$$

$$C_{i,j,s_{+1}}'[\varphi_{i,j,s_{+1}} - \lambda_{\text{sell},j}(s_{+1}, h_i)] = 0 \text{ and } \hat{C}_{i,j,s_{+1}}' \varphi_{i,j,s_{+1}} = 0, \quad \forall j; \tag{B.12}$$

$$D_{i,j,s_{+1}}'[\eta_{i,j,s_{+1}} - (1-\varepsilon) a_{j,s_{+1}} \varphi_{i,j}] = 0 \text{ and } \hat{D}_{i,j,s_{+1}}' \eta_{i,j,s_{+1}} = 0, \quad \forall j. \tag{B.13}$$

## Appendix C. Proof of Proposition 10

A $\mathbb{Z}$-valued stochastic process $\{z_0, \ldots, z_T\}$ is said to be a finite horizon equilibrium for $G : \mathbb{Z} \to \mathcal{P}_{\mathbb{Z}}$ if the conditional distribution of $z_{t+1}$ given $(z_0, \ldots, z_t)$ is in $G(z_t)$ almost surely, $\forall t < T - 1$. Propositions 1.2 and 1.3 in (Duffie et al., 1994, pp. 752 and 757) assure that in order to prove Proposition 1 in this paper one needs to show that: (i) there exists a $T$-horizon equilibrium for $G$, $\forall T > 0$; and (ii) the equilibrium variables always lie in a compact subset of $\mathbb{Z}$.

**Lemma 1.** *There exists a T-horizon Walrasian equilibrium for $\xi_T$, for any initial condition $(\hat{s}, \hat{\theta}_{-1}, \hat{\varphi}_{-1}, \hat{h}_{-1}) \in \mathbb{S} \times \mathbb{Y}_1$ and $T > 0$.*

**Proof.** Consider the $T$-horizon economy described in Appendix A and define $K_i = \{(\mathbf{x}_i, \boldsymbol{\theta}_i, \boldsymbol{\varphi}_i, \boldsymbol{\eta}_i, \mathbf{h}_i) \in \mathbb{X}^{S^*} \times \mathbb{R}_+^{3JS^*} \times \mathbb{H}^{S^*} : \max(\mathbf{x}_i) \leq \bar{x},\ \max(\boldsymbol{\theta}_i) \leq I(1 + \bar{\lambda}), \max(\boldsymbol{\varphi}_i, \boldsymbol{\eta}_i) \leq 1 + \bar{\lambda}$, and $\eta_{i,j,s^t} \leq (1-\varepsilon) a_{j,s_t} \varphi_{i,j,s^{t-1}}, \forall j, s^t, t\}$, where $\bar{x} = 2\sum_{i \in \mathbb{I}} \max(\mathbf{w}_i)$ and $\mathbf{w}_i \in \mathbb{R}_{++}^{LS^*}$. Then, let agent $i$'s truncated budget set be $\mathrm{BC}_{K_i}(\mathbf{p}, \mathbf{q}, \boldsymbol{\gamma}) = \{(\mathbf{x}_i, \boldsymbol{\theta}_i, \boldsymbol{\varphi}_i, \boldsymbol{\eta}_i, \mathbf{h}_i) \in K_i : (2)\text{–}(6), \forall s^t, t; \text{and } (\theta_{i,s^T}, \varphi_{i,s^T}, \eta_{i,s^{T-1}}, \eta_{i,s^T}) = \mathbf{0},\ \forall s^{T-1}, s^T\}$.

Now, assume that the commodity prices lie in the simplex. From the $T$-horizon optimal conditions derived in Appendix A, there exists $\bar{q} > 0$ such that $q_{j,s^t} \geq \bar{q}$ implies $\theta_{i,j,s^t} = 0$ and $\varphi_{i,j,s^t} = \lambda_{\text{sell},j}(s, \cdot)$, $\forall i, j, s^t, t < T$.[7] Hence, one can assume that prices and solvency rates belong to $\Delta = \{(\mathbf{p}, \mathbf{q}, \boldsymbol{\gamma}) \in \mathbb{R}_+^{(L+J)S^*} \times [\varepsilon, 1]^{JS^*} : \max(\mathbf{q}) \leq \bar{q}$ and $\sum_{l=1}^{L} p_{s^t,l} = 1,\ \forall s^t, t\}$.

Next, set $K = K_1 \times \cdots \times K_I$ and define $\phi : \Delta \times K \to \Delta \times K$ to be such that $(\mathbf{p}', \mathbf{q}', \boldsymbol{\gamma}', \mathbf{x}', \boldsymbol{\theta}', \boldsymbol{\varphi}', \boldsymbol{\eta}', \mathbf{h}') \in \phi(\mathbf{p}, \mathbf{q}, \boldsymbol{\gamma}, \mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\varphi}, \boldsymbol{\eta}, \mathbf{h})$ if and only if:

---

[7] From Eqs. (A.4)–(A.6) in Appendix A, one must have: $q_{j,s^t} = \frac{\sum_{\mathbb{S}^{t+1}|s^t} \beta_i P(s_{t+1}|s_t) d_{u_i,1,s^{t+1}} \gamma_{j,s^{t+1}} a_{j,s_{t+1}}}{\sum_{l=1}^{L} d_{u_i,l,s^t}} + \frac{\hat{B}_{i,j,s^t} - B_{i,j,s^t}}{\beta_i^t \bar{P}(s^t|\hat{s}) \sum_{l=1}^{L} d_{u_i,l,s^t}}$ and $q_{j,s^t} = \frac{\sum_{\mathbb{S}^{t+1}|s^t} \beta_i P(s_{t+1}|s_t) d_{u_i,1,s^{t+1}} a_{j,s_{t+1}}}{\sum_{l=1}^{L} d_{u_i,l,s^t}} + \frac{C_{i,j,s^t} - \hat{C}_{i,j,s^t} - \sum_{\mathbb{S}^{t+1}|s^t} (1-\varepsilon) D_{i,j,s^{t+1}} a_{j,s_{t+1}}}{\beta_i^t \bar{P}(s^t|\hat{s}) \sum_{l=1}^{L} d_{u_i,l,s^t}}$, $\forall i, j, s^t, t < T$. Define $d_{u_i}^{\sup} = \sup\{\max(d_{u_i,s^t}) : d_{u_i,s^t} \in \partial u_i(x_{i,s^t}),\ x_{i,s^t} \in \mathbb{X},\ \max(x_{i,s^t}) \leq \bar{x}\}$ and $d_{u_i}^{\inf} = \inf\{\min(d_{u_i,s^t}) : d_{u_i,s^t} \in \partial u_i(x_{i,s^t}),\ x_{i,s^t} \in \mathbb{X},\ \max(x_{i,s^t}) \leq \bar{x}\}$. Since $u_i$ is strictly increasing and concave, $d_{u_i}^{\sup} < \infty$ and $d_{u_i}^{\inf} > 0$. One can then set $\bar{q} = 2S\max_{i \in \mathbb{I}} \left(d_{u_i}^{\sup}/d_{u_i}^{\inf}\right)$. (Note that $\bar{q}$ does not depend on $T$.)

(a) $(p'_{s^t}, q'_{s^t}) \in \operatorname{argmax}_{\tilde{p}_{s^t}, \tilde{q}_{s^t}} [\tilde{p}_{s^t} \cdot \sum_{i \in \mathbb{I}} (x_{i,s^t} - w_{i,s_t}) + \tilde{q}_{s^t} \cdot \sum_{i \in \mathbb{I}} (\theta_{i,s^t} - \varphi_{i,s^t})]$, $\forall s^t, t$;
(b) $\gamma'_{j,s^t} \sum_{i \in \mathbb{I}} a_{j,s_t} \varphi_{i,j,s^{t-1}} = \sum_{i \in \mathbb{I}} (a_{j,s_t} \varphi_{i,j,s^{t-1}} - \eta_{i,j,s^t})$, $\forall j, s^t, t$;
(c) $(\mathbf{x}'_i, \boldsymbol{\theta}'_i, \boldsymbol{\varphi}'_i, \boldsymbol{\eta}'_i, \mathbf{h}'_i)$ maximizes $V_{i,T}$ in $\mathrm{BC}_{K_i}(\mathbf{p}, \mathbf{q}, \boldsymbol{\gamma})$, $\forall i$.

For each given $(\mathbf{p}, \mathbf{q}, \boldsymbol{\gamma}, \mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\varphi}, \boldsymbol{\eta}, \mathbf{h}) \in \Delta \times K$, condition (a) defines the set of solutions of a linear optimization problem in which the objective function is continuous with compact domain; while (b) defines the set of solutions of a linear equation—note that $0 \le \eta_{i,j,s^t} \le (1-\varepsilon) a_{j,s_t} \varphi_{i,j,s^{t-1}}$ for any element of $K_i$. Moreover, the objective function in (c) is continuous and concave, and the correspondence $\mathrm{BC}_{K_i} : \Delta \to K_i$ is nonempty, compact, and convex valued, and continuous.[8] Therefore, it is simple to check that $\phi$ is a nonempty, convex valued, and upper hemicontinuous correspondence that maps a compact into itself. It follows from Kakutani's fixed-point theorem that there exists $(\mathbf{p}^*, \mathbf{q}^*, \boldsymbol{\gamma}^*, \mathbf{x}^*, \boldsymbol{\theta}^*, \boldsymbol{\varphi}^*, \boldsymbol{\eta}^*, \mathbf{h}^*) \in \phi(\mathbf{p}^*, \mathbf{q}^*, \boldsymbol{\gamma}^*, \mathbf{x}^*, \boldsymbol{\theta}^*, \boldsymbol{\varphi}^*, \boldsymbol{\eta}^*, \mathbf{h}^*)$.

Note then that (b) in the definition of $\phi$ implies that the solvency rates satisfy the equilibrium condition (8). Given this, (a) in the definition of $\phi$ implies: $\sum_{i \in \mathbb{I}} (\theta^*_{i,j,s^t} - \varphi^*_{i,j,s^t}) = \sum_{i \in \mathbb{I}} (x^*_{i,l,s^t} - w_{i,l,s_t}) = 0$, $\forall j, l, s^t, t$.[9] Finally, continuity of $V_{i,T}$ and $\mathrm{BC}_{K_i}$ imply that $(\mathbf{x}^*_i, \boldsymbol{\theta}^*_i, \boldsymbol{\varphi}^*_i, \boldsymbol{\eta}^*_i, \mathbf{h}^*_i)$ is optimal in $\mathrm{BC}_{K_i}(\mathbf{p}^*, \mathbf{q}^*, \boldsymbol{\gamma}^*)$. Therefore, it is also optimal in the untruncated budget set, $\mathrm{BC}_i(\mathbf{p}^*, \mathbf{q}^*, \boldsymbol{\gamma}^*)$, since $V_{i,T}$ is continuous and concave and the boundary of $K_i$ is not reached—that is, $\min(\mathbf{x}^*_i) > x_{\inf}$, $\max(\mathbf{x}^*_i) < \bar{x}$, $\max(\boldsymbol{\theta}^*_i) < I(1+\bar{\lambda})$, $\max(\boldsymbol{\varphi}^*_i, \boldsymbol{\eta}^*_i) < 1 + \bar{\lambda}$, and $\max(\mathbf{h}^*_i) < 1 + \frac{\bar{\lambda}}{1-\delta}$. □

Note from the proof of Lemma C.1 that, for any $T$-horizon equilibrium, the state variable must belong to the following compact set: $\{z \in \mathbb{Z} : \min(x) \ge x_{\inf}, \max(x) \le \bar{x}, \max(\theta_{-1}, \theta) \le I(1+\bar{\lambda}), \max(\varphi_{-1}, \varphi, \eta) \le 1 + \bar{\lambda}, \sum_{l=1}^{L} p_l = 1, \text{ and } \max(q) \le \bar{q}\}$. This concludes the proof of Proposition 1.

## References


Alvarez, F., Jermann, U.J., 2000. Efficiency, equilibrium and asset pricing with risk of default. Econometrica 68, 775–798.
Araujo, A.P., Monteiro, P.K., Pascoa, M.R., 1996. Infinite horizon incomplete markets with a continuum of states. Mathematical Finance 6, 119–132.
Balder, E.J., 2001. On subdifferential calculus. Lecture Notes, Universiteit Utrecht.
Dubey, P., Geanakoplos, J.D., Shubik, M., 1988. Default and efficiency in a general equilibrium model with incomplete markets. Cowles Foundation Discussion Paper 879R.
Dubey, P., Geanakoplos, J.D., Shubik, M., 2005. Default and punishment in general equilibrium. Econometrica 73, 1–37.
Duffie, D., Geanakoplos, J.D., Mas-Colell, A., McLennan, A.M., 1994. Stationary Markov equilibria. Econometrica 62, 745–781.
Dutta, J., Kapur, S., 2002. Default and efficient debt markets. Journal of Mathematical Economics 38, 249–270.
Kehoe, T.J., Levine, D.K., 1993. Debt-constrained asset markets. Review of Economic Studies 60, 865–888.
Sabarwal, T., 2000. Bankruptcy in general equilibrium. Mimeo, Federal Reserve Board.
Torres-Martínez, J.P., 2002. Endogenous debt constraints and default. Pontifical Catholic University of Rio de Janeiro, Mimeo.
Zame, W.R., 1993. Efficiency and the role of default when security markets are incomplete. American Economic Review 83, 1142–1164.
Zhang, H.H., 1997. Endogenous borrowing constraints with incomplete markets. Journal of Finance 52, 2187–2209.


[8] Note from Section 3 that $\min(w_{i,s,1} - \varepsilon \max(1, J\bar{\lambda}), w_{i,s,2}, \ldots, w_{i,s,L}) > x_{\inf}$, $\forall i, s$.

[9] First, note that $\sum_{i \in \mathbb{I}} (\theta^*_{i,j,s^t} - \varphi^*_{i,j,s^t}) = 0 \, \forall j, s^t, t$. This holds by definition at the terminal nodes $s^T \in \mathbb{S}^T$. For any $t < T$, the following contradictions can be derived. If $\sum_{i \in \mathbb{I}} (\theta^*_{i,j,s^t} - \varphi^*_{i,j,s^t}) > 0$, then condition (a) in the definition of $\phi$ would imply $q^*_{j,s^t} = \bar{q}$. Then, the Kuhn–Tucker conditions in Appendix A would imply $\sum_{i \in \mathbb{I}} (\theta^*_{i,j,s^t} - \varphi^*_{i,j,s^t}) = -\sum_{i \in \mathbb{I}} \lambda_{\text{sell},i,j}(s_t, h^*_{i,s^t}) \le 0$ (contradiction). Similarly, if $\sum_{i \in \mathbb{I}} (\theta^*_{i,j,s^t} - \varphi^*_{i,j,s^t}) < 0$, one would have $q^*_{j,s^t} = 0$ and then $\sum_{i \in \mathbb{I}} (\theta^*_{i,j,s^t} - \varphi^*_{i,j,s^t}) = \sum_{i \in \mathbb{I}} \lambda_{\text{buy},i,j}(s_t, h^*_{i,s^t}) \ge 0$ (contradiction). Next, by adding (3) over $i$, one obtains: $p^*_{s^t} \cdot \sum_{i \in \mathbb{I}} (x^*_{i,s^t} - w_{i,s_t}) \le 0$. Joint with condition (a) in the definition of $\phi$, this implies $\sum_{i \in \mathbb{I}} (x^*_{i,l,s^t} - w_{i,l,s_t}) \le 0$, $\forall l$. Hence, $\max(\mathbf{x}^*_i) < \bar{x}$. Thus, strict monotonicity of $V^i$ implies $p^*_{s^t} \cdot \sum_{i \in \mathbb{I}} (x^*_{i,s^t} - w_{i,s_t}) = 0$ and $p^*_{s^t} \gg \mathbf{0}$. Therefore, $\sum_{i \in \mathbb{I}} (x^*_{i,l,s^t} - w_{i,l,s_t}) = 0$, $\forall l, s^t, t$.