Long Range Dependence in Copula Models

Beatriz Vaz de Melo Mendes[1]
Instituto de Matemática/COPPEAD - UFRJ, Brazil
Silvia Regina Costa Lopes
Instituto de Matemática - UFRGS, Brazil

Abstract

Modeling short and long time dependence in univariate time series may be successfully accomplished through existing time series processes. In the multivariate setting just a few complex models exist to take care of the different marginal dynamics as well as of the dynamic covariance matrix. The copula approach factors the joint distribution into the marginals and a dependence function, its copula. This allows for tailored marginal dynamic modeling of each margin, considering all characteristics of each marginal distribution, including skewness, kurtosis and any type of short and long memory serial dependence, plus a search for the best fit for the dependence structure which is entirely determined by the copula. Assuming a conditional copula model, depending on past observations, enhances the fit. Attempts to model serial correlation in copula environment are only a few, moreover just short memory is considered. However, the dependence structure linking the series may also possess long memory. For example, the dependence structure associated with the standardized residuals from a FIGARCH fit on log-returns may still present long memory. Moreover, long memory could be present only in the dependence structure and not in the margins. In this paper we propose a proxy for detecting long memory in the copula function. We show how to simulate from a copula possessing long memory, we discuss inference methods, and provide an application using real data.

JEL subject classifications: $C22$ $C51$ $G10$

Keywords: Long Memory, Conditional Copulas, Time Series, Financial Applications.

---

[1] Corresponding author. Coppead/UFRJ, PO Box 68514, Rio de Janeiro, RJ 21941-972, Brazil. `beatriz@im.ufrj.br`

1 Introduction

Modeling short and long time dependence found in a univariate time series may be successfully accomplished through many popular models including ARIMA (Box e Jenkins, 1976), ARFIMA (Granger and Joyeux (1980), Hosking (1981)), GARCH (Engle, 1982), FIGARCH (Baillie, Bollerslev and Mikkelsen (1996), Bollerslev and Mikkelsen (1996)), and FISV (Breidt et al., 1998) models. The statistical properties of most of these processes are relatively well established, and computer packages are avaliable for practitioners. In the multivariate setting just a few complex conditional models exist to take care of the dynamics in the marginal distributions as well as in the covariance matrix. For example, multivariate GARCH models combined with a conditional specification for the mean. The copula approach factors the joint distribution into the marginals and a dependence function, its copula. This allows for tailored marginal dynamic modeling of each margin, considering all characteristics of each marginal distribution, including the mean, standard deviation, skewness, kurtosis and any type of short and long memory serial dependence, plus a search for the best fit for the dependence structure which is entirely determined by the copula. A large number of copula families may be considered. Assuming a conditional copula model, depending on past observations, enhances the fit. Attempts to model serial correlation in copula environment are only a few, moreover just short memory is considered (Patton (2006), Rockinger and Jondeau (2005), Van Den Goorbergh, Genest, and Werkerc (2005), among others). However, the dependence structure linking the series may also possess long memory. For example, the dependence structure associated with the standardized residuals from a FIGARCH fit on log-returns may still present long memory. Moreover, long memory could be present only in the dependence structure and not in the margins. To the best of our knowledge, no model has been proposed so far to address long memory in copula models. Accordingly, in this paper we propose a proxy for detecting long memory in the copula function, we show how to simulate from a copula possessing long memory, we discuss inference methods, and provide an application using real data.

There are many ways to define the copula function, see Nelsen (2006). Simply stating, a copula is function that links univariate marginals resulting in a proper multivariate probability density function. From the probability theory we know that a multivariate density contains all the information about the random vector, in particular all the information about the dependence structure linking the components. Therefore the copula contains all the information about the dependence structure of the random vector. Additionally, copulas can be employed in probability theory to characterize dependence concepts, see Joe (1997). A copula based measure of dependence is invariant to any increasing transformation of individual series.

In the last decade copulas have gained popularity in the areas of finance and insurance because of the flexibility they offer when dealing with multivariate problems. Curiously, the most important theorem in copula theory dates back to the fifties (Sklar, 1959). It states that any multivariate distribution can be expressed by its copula function evaluated at its marginal distribution functions. Conversely, a copula function allows for constructing flexible multivariate distributions with different margins. An important example is the family of the *meta*-elliptical distributions (Fang, Fang and Kotz, 2002, 2005) which, unlike the family of elliptical distributions, do not impose any constraints on their margins. Taking care of dependencies becomes important in order to extend standard models towards more efficient ones.

Empirical studies have revealed that financial returns often show two stylized facts: non-normality and presence of non-linear forms of dependence, in particular tail dependence. Measuring dependence only through the linear correlation coefficient is adequate in the Gaussian world or when assessing linear dependence. Neglecting other forms of association may result in underestimation of risks. All this have motivated applications of copulas in finance: They include asset allocation, credit scoring, default risk modeling, derivative pricing, and risk management, see Bouyè, Durrleman, Bikeghbali, Riboulet, and Roncalli (2000), Li (2000), Costinot, et al. (2000), McNeil, Frey e Embrechts (2005), Cherubini and Luciano (2002), Hu (2002), Embrechts, Lindskog, and McNeil (2003), Cherubini, Luciano, and Vecchiato (2004).

Most of the above mentioned applications deal with *unconditional* copulas. In practice, conditional distributions with respect to past observations are more powerful when describing the underlying model. This is also true for copulas, which may efficiently fit static and dynamic forms of dependence. Patton in 2001 extend Sklar's theorem and introduced the conditional copula in the bivariate case. Modeling exchange rates, he assumed a bivariate Gaussian conditional copula with the correlation coefficient following an ARMA model based on a logistic transformation. This paper was later circulated as Patton (2003) and finally published as Patton (2006).

Rockinger and Jondeau (2001) assumed a parametric copula conditional to the position of past joint observations in the unit square, combined with previous marginal estimation of GARCH-type models with time varying skewness and kurtosis. Fermanian and Wegkamp (2004) introduced the concept of *pseudo-copulas* and unified previous attempts in the direction of modeling time varying dependence structures using copulas. Cherubini, Luciano, and Vecchiato (2004) fitted a GARCH(1,1) model to the margins and modeled the dependence structure with a Gaussian copula with a time varying correlation coefficient based on a modified logistic transformation. Dias and Embrechts (2004) applied univariate GARCH models and the t-copula model with time varying correlation to high frequency data. Mendes (2005) extended Rockinger and Jondeau (2001) model and assumed a parametric pseudo-copula conditional to the position of lag 1 past joint and lag 2 past joint observations in the unit square, combined with marginal FIGARCH estimation. Van Den Goorbergh, Genest and Werker (2005) studied the behavior of bivariate option prices when the dependence structure of the underlying financial assets follows a dynamic copula model. Mendes and Melo (2008) proposed to assess the full dynamic dependence structure existing among assets applying local maximum likelihood estimation to copula parameters.

Note that *all* above cited papers just considered *short memory* in the copula model.

Models for *long memory in mean* were first introduced by Granger and Joyeux (1980) and Hosking (1981), following the seminal work of Hurst (1951). The important characteristic of an Autoregressive Fractionally Integrated Moving Average (ARFIMA) process is its autocorrelation function decay rate. In an ARFIMA process, the autocorrelation function exhibits a hyperbolic decay rate, differently from an ARMA model which presents a geometric rate. Long memory in mean has been observed in data from areas such as meteorology, astronomy, hydrology, and economics, as reported in Beran (1994).

The ARFIMA framework was naturally extended towards volatility models. The Fractionally Integrated Generalized Autoregressive Conditionally Heteroskedastic (FIGARCH) models were introduced by Baillie, Bollerslev and Mikkelsen (1996) and Bollerslev and Mikkelsen (1996), motivated by the fact that autocorrelation function of the squared, log-squared, or the absolute value series of an asset return decays slowly, even when the return series has no serial correlation. Also aiming to model long memory in the second moment, Breidt et al. (1998) introduced the Fractionally Integrated Stochastic Volatility (FISV) model.

Models for heteroskedastic time series with long memory are of great interest in econometrics and finance, where empirical facts about asset returns have motivated the several extensions of GARCH type models (FIGARCH, FIEGARCH, TGARCH, SW-ARCH, LM-ARCH, among many others). Many empirical papers have detected the presence of long memory in the mean and in the volatility of risky assets, market indexes and exchange rates (for example, Crato (1994), Saqdique and Silvapulle (2001), Lobato and Savin (1998)).

However, the existing long memory models are restricted to the univariate ones.

In this paper we define long memory bivariate processes. We empirically show that long memory may exist in the univariate processes and/or in the dependence structure. We show how to simulate from long memory copula models and, given real bivariate data, we show how to identify the presence of long memory in the copula and how to estimate it. The remaining of the paper is as follows: In Section 2 we review definitions of dynamic copulas, of long memory processes, and study how to detect and estimate long memory in copula models. In Section 3 we provide an illustration using real data. In Section 4 we discuss extentions of this work, its strengths and weaknesses.

2 Long Memory Bivariate Processes

Consider a stationary continuous process $(X_{1,t}, X_{2,t})_{t\in\mathcal{Z}}$. Let F_t, $F_{1,t}$, and $F_{2,t}$ represent their joint and marginal cdf's at time t. In the case the joint law of $(X_{1,t}, X_{2,t})$ is independent of t, the dependence structure of (X_1, X_2) is given by its constant copula C. In the time series context, the Sklar's theorem (Sklar, 1959) may be extended:

$$F_t(x_{1,t}, x_{2,t} \mid \mathcal{A}_t) = C_t(F_{1,t}(x_{1,t} \mid \mathcal{A}_t), F_{2,t}(x_{2,t} \mid \mathcal{A}_t) \mid \mathcal{A}_t), \tag{1}$$

where C_t is a copula at times t, and

$$\mathcal{A}_t = \sigma\{x_{1,t-1}, x_{2,t-1}, x_{1,t-2}, x_{2,t-2}, \cdots\} \quad t = 1, \cdots, T,$$

represents the σ-algebra generated by all past joint information up to time t provided by the sample $(x_{1,1}, x_{2,1}), \cdots, (x_{1,T}, x_{2,T})$. The same conditioning set for each marginal and for the conditional copula guarantees that each transformed variable is independent of the information in the conditioning set of its marginal distribution, see theoretical details in Fermanian and Wegkamp (2004). As specified, the common σ-algebra may take into account several lagged values.

Let $(U_1, V_1), \cdots, (U_T, V_T) = (F_{1,1}(X_{1,1} \mid \mathcal{A}_t), F_{2,1}(X_{2,1} \mid \mathcal{A}_t)), \cdots, (F_{1,T}(X_{1,T} \mid \mathcal{A}_t), F_{2,T}(X_{2,T} \mid \mathcal{A}_t))$ be the probability integral transformed random variables. Now suppose that C_t is a member of the large family of elliptical copulas (Fang, Kotz, and Ng (1990), Frahm, Junker, and Sziymayer (2003)), being unknown just the correlation coefficient ρ. To model the short range dynamics in the parameter ρ one could assume an autoregressive model combined with an appropriate transformation to keep the parameter within its boundaries. Patton (2006), Cherubini et al. (2004) and also Fantazzini (2007) used this approach and considered the modified logistic transformation $\Lambda(x) = \frac{1-e^{-x}}{1+e^{-x}}$. More generally, one can assume $\{\rho\}_{t=1}^{T}$ is an ARMA(p, q) process. We now extend this idea to incorporate long memory in the copula parameter process, and assume $\{\rho\}_{t=1}^{T}$ follows an ARFIMA(p, d, q) process[2].

---

[2] Granger and Joyeux (1980) proposed the fractionally integrated autoregressive and moving average model, denoted by ARFIMA (p, d, q), defined by

$$\phi(B)(1-B)^d \rho_t = \theta(B)\epsilon_t$$

where $\{\rho_t\}$, $t = 1, 2, ..., T$, is the time series; B is the backshift operator, that is, $(x_t) = x_{t-1}$, $\phi(B) = 1 - \phi_1 B - \cdots \phi_p B^p$ and $\theta(B) = 1\theta_1 B - \cdots - \theta_q B^q$ represent the ordinary autoregressive and moving average components; ϵ_t is a white noise process with zero mean and unit variance. When $-0.5 < d < 0.5$, the ARFIMA(p, d, q) process is stationary, and if $0 < d < 0.5$ the process presents long-memory behavior. In a stationary long memory process the autorrelation function at lag k, as $k \to \infty$, is equal to $[\Gamma(1-d)/\Gamma(d)]k^{2d-1}$ for $\mid d \mid \leq 0.5$.

2.1 *Identification and simulation of copula models possessing long memory*

We claim that $U_1 * V_1, \cdots, U_T * V_T$ may be used as a proxy for detecting long memory in the copula function. This means that whenever ρ varies according to some long memory process, the sample autocorrelation function (acf) of the proxy would show the hyperbolic decay rate (Granger and Joyeux (1980), Hosking (1981)).

To check this assumption we simulated from an ARFIMA$(0, d, 0)$ process and applied the logistic transformation to the simulated series, obtaining a series of values for the correlation coefficient. More specifically, we simulated $Y_1, Y_2, \cdots, Y_T$, from an ARFIMA$(0, d, 0)$ model with $d = 0.35$, obtained $\rho_1, \rho_2, \cdots, \rho_T = \Lambda(Y_1), \Lambda(Y_2), \cdots, \Lambda(Y_T)$, to finally simulate from the Gaussian copula where, at each time t, $t = 1, 2, \cdots, T$, the copula parameter was given by ρ_t in the simulated path. The following simulated series were considered for inspection: $\{Y\}_{t=1}^T$, $\{\Lambda(Y)\}_{t=1}^T$, the proxy $\{U * V\}_{t=1}^T$, a tentative proxy $\{U + V\}_{t=1}^T$, and the marginal series $\{U\}_{t=1}^T$ and $\{V\}_{t=1}^T$. The Lo's modified rescaled adjusted range test, R/S test (Lo, 1991) for long range dependence was applied to the simulated series and an estimate for d was computed according to the periodogram based Whittle's method (Taqqu et al., 1995)[3].

A concern about this simulation experiment is that for each ρ_t, just one pair of values is generated to represent the copula at time t, and this is of course a source of large variability. There is no way to check if data generated reflects the *true* ρ_t. To overcome this problem we set T large, $T = 10000$ (the large sample also helps increasing the power of the R/S test).

Long memory in the copula parameter was detected by the proxy $U * V$ in 65.80% of the 500 repetitions of the experiment, providing an average d estimate of 0.123. The tentative proxy $U + V$ was not able to indicate presence of long memory, being successful in only in 4% of the cases. As expected, both marginal series did not rejected the null of *no* long range dependence in 94.80% and 94.52% of the repetitions. So, the long memory process is in the copula and it is not carried on to the marginal processes.

For example, for a particular simulated series, the R/S test for long range dependence applied to the proxy $U * V$ provided test statistic 3.6853, significant at the 1% level, and $d = 0.1253$. The R/S test applied to $U + V$ resulted significant at 5% (2.1417). Based on of this simulated data set, Figure 1 shows the acfs' computed for the ARFIMA series, for the correlation coefficient path, for the copula proxies, and for the marginal processes.

Therefore, it is possible for long memory to exist only in the dependence structure,

---

[3] We applied the Lo's modified rescaled adjusted range test (R/S test) (Lo, 1991), and computed an approximate maximum likelihood estimator, the Whittle's estimator, implemented in S-Plus based on the algorithm of Haslett and Raftery (1989). All the existing tests and estimators for long range dependence are very tricky to use, heavily dependent on arguments. To get around this problem in the simulation experiments we rejected the null only when it was rejected by the R/S test and the Whittle estimator was statistically significant at the 5% level.

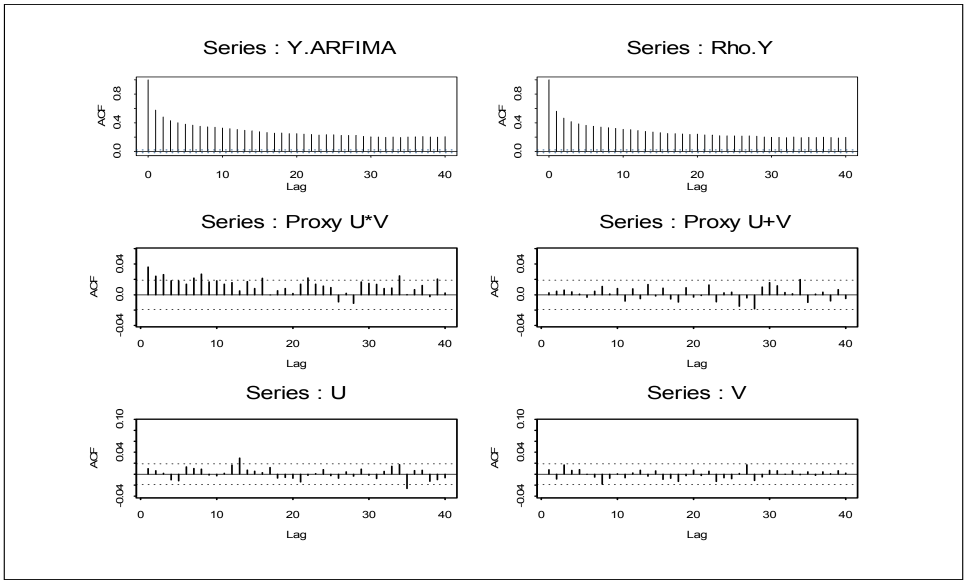

Figure 1: *The acfs' of the simulated data: the ARFIMA series, the correlation coefficient path, the proxies and the marginal processes.*

possessing the marginal processes just short memory or no temporal structure at all. In the case $\{X_{1,t}, X_{2,t}\}$ do have short and/or long memory, they would be implied by their marginal processes, and not by the copula. In summary, we have now a tool for detecting long memory in copula parameters for real data. Simulations from different long memory processes, and different copula families indexed by real valued parameters, may be carried on to assess the performance of the exploratory tools.

2.2 *Estimation of long memory in copula parameters*

In this subsection we treat the case of elliptical copulas assuming that the linear correlation coefficient ρ_t, $t = 1, \cdots, T$, varies on time according to a long range process. We recall the great flexibility of elliptical copulas which allow for tail dependence and extends to the meta-elliptical family of distributions (see Fang, Kotz, and Ng (1990)). In addition, they can be easily implemented. We note that for elliptical copulas ρ is the canonical measure of dependence, being sufficient to describe the dependence structure. As usual, the final objective is to obtain forecasts for the linear correlation coefficient and to estimate joint events of interest, for example the Value-at-Risk one step ahead.

Let θ_t represent the vector of all parameters indexing a bivariate distribution F_t at time t. This means $\theta_t = (\delta_t, \theta_{1,t}, \theta_{2,t})$, where δ_t is the copula vector parameter, and $\theta_{i,t}$, $i = 1, 2$, represent the marginal distributions parameters. The conditional joint density may be written as

$$f(x_{1,t}, x_{2,t}; \theta_t \mid \mathcal{A}_t) = c_t(u_t, v_t; \delta_t \mid \mathcal{A}_t) \quad f_{1,t}(x_{1,t}; \theta_{1,t} \mid \mathcal{A}_t) \quad f_{2,t}(x_{2,t}; \theta_{2,t} \mid \mathcal{A}_t) \tag{2}$$

where c_t is the copula density function, $u_t = F_{1,t}(x_{1,t}; \theta_{1,t} \mid \mathcal{A}_t)$ and $v_t = F_{2,t}(x_{2,t}; \theta_{2,t} \mid \mathcal{A}_t)$, and $f_{i,t}(x_{i,t}; \theta_{i,t} \mid \mathcal{A}_t)$ is the conditional density of each marginal distribution, $i = 1, 2$. For the expressions of the densities and details on the two most important members of the elliptical family, namely the Gaussian and t-copula, see Demarta and McNeill (2004) or Fantazzini (2007).

Parametric estimation of bivariate data under the copula approach may be accomplished in two steps, the so called IFM method, see Joe (1997). In the first step univariate marginal models are fitted and the estimated cdf's $\widehat{F}_{i,t}$, $i = 1, 2$, are used to compute the pseudo uniform$(0, 1)$ data. In the second step the copula parameters are estimated.

From now on in this section we assume that the marginal distributions are known or have been estimate, so that we have observations from two sequences of standard uniform random variables $\{U_t\}_{t=1}^T$, and $\{V_t\}_{t=1}^T$. We propose to estimate the time varying correlation coefficient using a maximum likelihood based estimate of the proxy.

We fit by maximum likelihood an ARFIMA(p, d, q) model (note we are allowing also for short memory) to the proxy $\{U_t * V_t\}_{t=1}^T$, obtaining the fitted values, denoted by $\{\widehat{U_t * V_t}\}_{t=1}^T$. Since the linear correlation coefficient between U_t and V_t at time t is given by

$$\rho_t = 12E[U_t * V_t] - 3,$$

we propose to estimate the path of correlation coefficients using

$$\{\widehat{\rho}\}_{t=1}^T = 12\{\widehat{U_t * V_t}\}_{t=1}^T - 3. \tag{3}$$

The method also yields an estimate for d and are also able to predict k-steps ahead using the predictions provided the the ARFIMA methodology.

Alternatively, we could fix the values $\widehat{U_t * V_t}$, $t = 1, 2, \cdots, T$, assume $\rho_t = \rho_0 + \widehat{U_t * V_t}$, and maximize the copula log-density "globally". The fluctuations of the correlation coefficient would be given by the estimated values $\widehat{U_t * V_t}$ which possess long memory. For example, in the case of the t-copula, $\delta_t = (\nu_t, \rho_t)$ (degrees of freedom and correlation coefficient), and the optimization would be with respect to ν_t and ρ_0. The degrees of freedom may be considered constant on time and, in that case one has just two parameters to estimate, ν_0 and ρ_0. Again, we obtain the path of the estimated correlation coefficients and are able to predict.

Note this method may be applied to estimate any copula parameter δ_t as long as it is possible to write the copula parameter δ_t as a function of ρ_t, $\delta_t = \delta_t(\rho_t)$. Even when there is no closed form solution for this equation, iterative methods may be used.

3 Real Data

In this section we provide an application of the new modeling strategy. We consider a 5-years sample composed by 1827 pairs of daily log-returns on the SP500 and Nasdaq, from June 1, 2000 to June 1, 2007, obtained from Datastream. As usual, exploratory analysis of the bivariate data starts with the inspection of their scatter plot (in the original scale), along with their univariate plots along time. Figure 2 shows the evolution through time of the log-returns from SP500 and Nasdaq, from now on respectively denoted by $X_{1,t}$ and $X_{2,t}$, $i = 1, \cdots, T$, $T = 1827$.

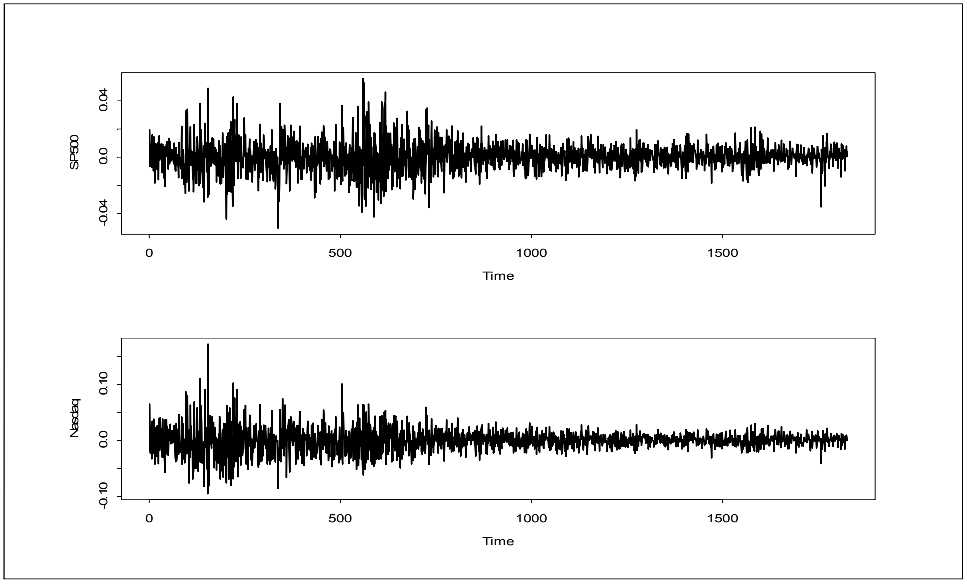

Figure 2: *Time series plot of daily log-returns on the SP500 (top) and Nasdaq (bottom).*

We apply the two-steps estimation methodology described in the previous section and start by fitting models to the univariate margins. Both return series are stationary. Typically, financial returns show linear and non-linear forms of dependence, which evolve through time according to some ARFIMA-FIEGARCH type model (Granger and Joyeux (1980), Hosking (1981), Engle (1982), Bollerslev (1986), Baillie, Bollerslev and Mikkelsen (1996), and Bollerslev and Mikkelsen (1996)). Usually, one finds significant autocorrelations in $\{X_{i,t}^2\}$ and, for few lags, in $\{X_{i,t}\}$, and also long memory in the mean and in the volatility. The acf is the most widely used tool to identify the orders of the ARFIMA-FIEGARCH model. Figure 3 shows the acf of the SP500 and Nasdaq log-returns in the upper panel, and the acf of their squares on the lower panel. The plots suggest the presence of short and long memory in the returns series and in their squares.

For the SP500, the best fit was an ARFIMA(0, 1)-FIEGARCH$(2, d, 1)$ plus leverage term, with $d = 0.355$ and all estimates highly statistically significant. The p-values of the

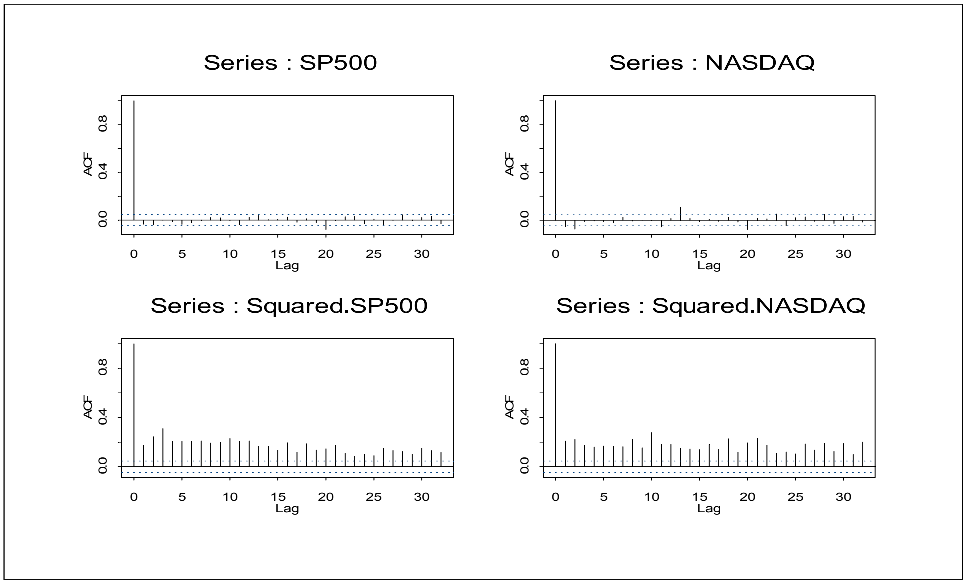

Figure 3: *The acf of SP500 and Nasdaq log-returns (top), and the acf of their squares (bottom).*

Ljung-Box test applied to the standardized residuals and their squares were respectively 0.599 and 0.458, indicating that the residuals are free of volatility clusters and temporal dependences. For the Nasdaq the best model found was ARFIMA$(0, 1)$-FIEGARCH$(2, d, 1)$ plus leverage term, with $d = 0.718$ and all estimates highly statistically significant. Again, the Ljung-Box test applied to the standardized and squared standardized residuals accepted the null hypotheses of no autocorrelation. In addition, the R/S test for long range dependence applied to each series of standardized residuals *accepted* the null hypothesis of no long memory. The test statistics were, respectively, 1.685 and 1.597.

Then we obtain the series of transformed data $\{U_t\}_{t=1}^T$, and $\{V_t\}_{t=1}^T$ by ranking the standardized residuals (empirical distribution)[4]. The R/S test applied to the copula proxies $\{U_t * V_t\}_{t=1}^T$ and $\{U_t + V_t\}_{t=1}^T$, *rejected* the null hypothesis of *no* long memory. The test statistics were respectively equal to 1.908 ($d = 0.0859$) and 2.051 ($d = 0.0988$), both significant at the 5% level. Figure 4 shows the acf of the proxies $U * V$ and $U + V$.

Having found long range dependence in the proxy we fitted an ARFIMA$(0, d, 0)$ to the $U * V$ proxy and applied the estimation methods suggested in the previous section. The simple estimates defined in (3) are shown in the first row of Figure 5.

To obtain the estimates based on a copula model we must first select a copula family. We examine the plot of the support set for the empirical copula, and note positive association as well as tail dependence. To account for tail dependence we assume a t-copula,

---

[4] Another possibility for computing the transformed data is to assume a mixture model for the univariate distributions, where the Generalized Pareto distribution would be used for the extreme lower and upper tails and the empirical cdf for the rest of the data.

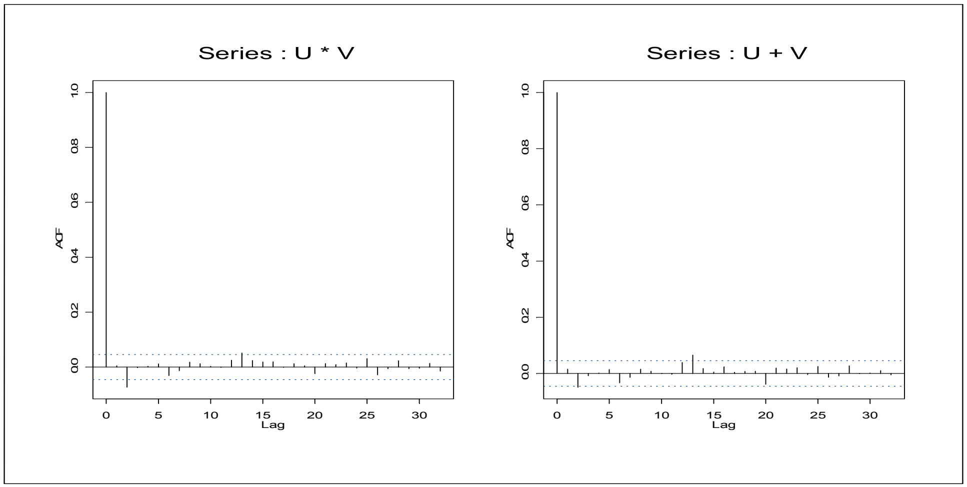

Figure 4: *The autocorrelation function of the proxies $U * V$ and $U + V$ for SP500 & Nasdaq data.*

but for the sake of comparisons we also fit a Gaussian copula. The log-likelihood value for the time varying t-copula model was 1316.55 ($\widehat{\rho}_0 = 0.5533, \widehat{\nu} = 11$), and for the Gaussian copula it was 1304.05 ($\widehat{\rho}_0 = 0.5514$). The t-copula based correlation coefficients estimated path is shown in second row of Figure 5. Of course the pattern is the same, since the ρ_t values under the two estimation methods are linear combinations of each other. Assuming $\nu = 11$ and the path of simple estimates, we computed the t-copula based log-likelihood and found 1270.03. Therefore, for this data set our best estimates will be given by the t-copula based ones (second row of Figure 5). The third row of of this figure shows the acf applied to these estimates[5].

Usually, the estimation of the time varying dependence and marginal parameters is not the final objective in a time series application. Each problem will require the forecast of a different functional of the joint conditional distribution. For example, estimation of the one-step ahead conditional Value-at-Risk (VaR) of a portfolio of assets requires the computation of the one-step ahead prediction of a quantile of some linear combination of the component variables. This can be easily accomplished using our suggested modeling strategy and estimation methods. Moreover, bootstrap confidence intervals may be constructed based on simulations from the one-step ahead predicted copula and marginal distributions.

---

[5] For the sake of completeness we also fit a static (constant ρ) t-copula to the data, obtaining the global constant maximum likelihood estimates of ($\widehat{\rho} = 0.8744, \widehat{\nu} = 10$), being the log-likelihood equal to 1315.99. It would be interesting to compare the constant ρ with the mean of the estimates paths of the correlation coefficients: 0.8631 for the simple estimates, and 0.8752 for the t-copula based estimates.

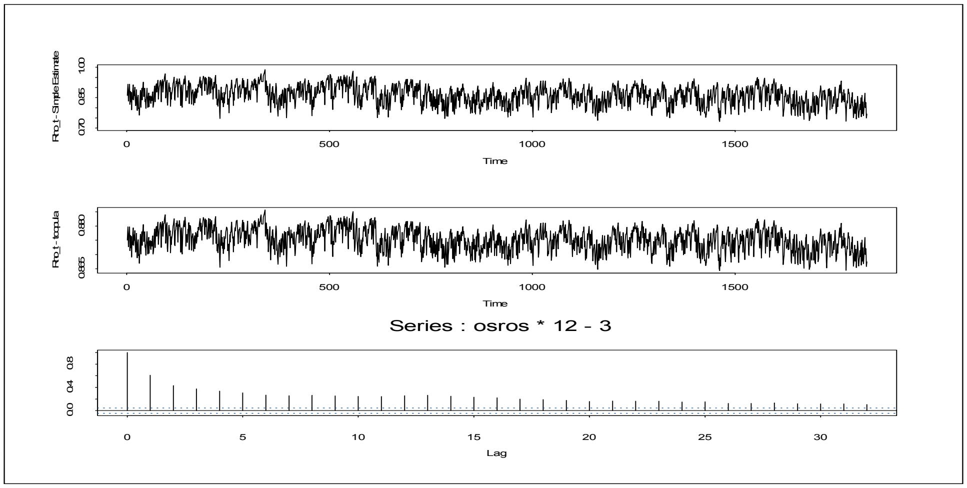

Figure 5: *The estimated paths of the correlation coefficients and the acf applied to the second row path.*

4 Discussions

In this paper we studied long range dependence in copula models. We proposed a proxy for detecting long memory in the copula function, showed how to simulate from a copula possessing long memory, discussed inference methods, and provided an application using real data. We proposed two estimation methods based on the maximum likelihood procedure. Simulations may be carried on in a further work to assess the best estimates. Uncertainty may be easily assessed by parametric bootstrap methods, and boostrap confidence intervals may be constructed for the functionals of interest.

We mainly treated the case of elliptical copulas, for which the correlation coefficient is the canonical measure of dependence. In a further work we may carry on simulations from different copula families indexed by real valued parameters, to assess the performance of the new methods proposed here. For other copula families we may consider using the Kendall correlation coefficient, which may be expressed as a function of copula parameters and is scale invariant. For dependence structures allowing for types of non-linear dependence, the long memory may manifest itself in some other measure of association, for example in the tail dependence coefficient. Since this measure may be expressed as a function of the copula parameters, we could then model long memory through a appropriate transformation of the tail dependence coefficient.

We showed that it is possible for long memory to exist only in the dependence structure, possessing the marginal processes just short memory or no temporal structure at all. Another possible extention of this work is the investigation of presence of long memory at different lags of the marginal variables.

We were initially motivated by problems from the area of finance, but the methodology may be applied to data from any other environment, and we encourage applications in other fields to exploit the potentialities of the new model.

References

Baillie, R.T., Bollerslev, T. and Mikkelsen, H.O. (1996). "Fractionally Integrated Generalized Autoregressive Conditional Heteroskedasticity". *Journal of Econometrics*, Vol. **74**, 3-30.

Bollerslev, T. and Mikkelsen, H.O. (1996). "Modeling and pricing long memory in stock market volatility". *Journal of Econometrics*, Vol. **73**, 151-184.

Bouyé, E., Durrleman, V. Nikeghbali, A., Riboulet, G., Roncalli, T. (2000). "Copulas for finance: a reading guide and some applications". Working paper. Available at http://gro.creditlyonnais.fr/content/rd/home_copulas.htm.

Box, G.E.P., and Jenkins, G.M. *Yime Series Analysis: Forecasting and Control*, Revised Edition, San Francisco: Holden Day, 1976.

Breidt, F.J., Crato, N. and de Lima, P. (1998). "The detection and estimation of long memory in stochastic volatility". *Journal of Econometrics*, Vol. **83**, 325-348.

Cherubini, U., Luciano, E. (2002). "Multivariate Option Pricing with Copulas". ICER Working Papers - Applied Mathematics Series 05-2002, ICER - International Centre for Economic Research.

Cherubini, U., Luciano, E., and Vecchiato, W. (2004). *Copula Methods in Finance*, John Wiley.

Costinot, A., Roncalli, T., & Teiletche, J. (2000). Revisiting the dependence between financial markets with copulas. Working paper.

Crato, N. (1994). Some international evidence regarding the stochastic behavior of stock returns. Applied Financial Economics, 4, 33-39.

Demarta, S. and McNeill, A.J. (2004). The t-copula and related copulas. ETH. Zurich.

Dias, A., Embrechts, P. (2004). Dynamic copula models for multivariate high-frequency data in finance. Discussion paper.

Engle, R.F. (1982). "Autoregressive conditional heteroskedasticity with estimates of the variance of U.K. inflation". *Econometrica*, Vol. **50**, 987-1008.

Fang, H-B., Fang, K-T., and Kotz, S. (2002). "The meta-elliptical distributions with given marginals". *J. Multivar. Analysis*, 82, 1-16.

Fang, H., Fang, K-T, Kotz, S. (2005). "Corrigendum to The Meta-Elliptical distribution with given marginal, [J. Multiv. Anal., 82, 2002, 1-16]. *Journal of Multivariate Analysis*, 94, 222-223.

Fantazzini, D. (2007). "Dynamic Copula Modelling for Value at Risk". *Frontiers in Finance and Economics*, forthcoming. Available at SSRN: http://ssrn.com.

Fermanian, J-D., Wegkamp, M. (2004). "Time dependent copulas. Working paper. Available at http://www.crest.fr/pageperso/lfa/fermanian/fermanian.htm.

Frahm, G., Junker, M., and Sziymayer, A. (2003). Elliptical copulas: applicability and limitations *Statistics & Probability Letters*, V. 63, 3, 275-286.

Fermanian, J-D., Wegkamp, M. (2004). "Time dependent copulas. Working paper. Available at http://www.crest.fr/pageperso/lfa/fermanian/fermanian.htm.

McNeil, A. J., Frey, R. and Embrechts, P. (2005). *Quantitative Risk Management: Concepts, Techniques and Tools*. Princeton University Press.

Granger, C.W.J. and Joyeux, R. (1980) "An Introduction to Long Memory Time Series Models and Fractional Differencing". *Journal of Time Series Analysis*, Vol. **1**, 15-29.

Hosking, J. (1981). “Fractional Differencing”. *Biometrika*, Vol. **68**, 165-167.

Joe, H. (1997). *Multivariate Models and Dependence Concepts.* Chapman & Hall: London.

Li, D. (2000). On default correlation: A copula function approach. *Journal of Fixed Income* **9**, 43-54.

Lo, A. W. (1991). Long-term memory in stock market prices. Econometrica, 59:1279:1313.

Lobato, I., & Savin, N. E. (1998). ”Real and spurious long-memory properties of stock market data”. *Journal of Business and Economic Statistics*, 16, 261-268.

Mendes, B.V.M. (2005). “Computing Conditional VaR Using Time-varying Copulas”, *Brazilian Finance Review* Vol. 3, No. 2, pp. 249263.

Mendes, B.V.M. and Melo, E.F.L. (2008). ”Local Estimation of Dynamic Copula Models”. In press.

Nelsen, R. B. (2006). *An introduction to copulas.* New York: Springer.

Patton, A. (2006). ”Modelling Asymmetric Exchange Rate Dependence”, *International Economic Review*, 47(2), 527-556. This paper was previously circulated as Patton, A. (2001) Modelling Time-Varying Exchange Rate Dependence Using the Conditional Copula , University of California, San Diego, Discussion Paper 01-09.

Rockinger, M. and Jondeau, E. (2001). Conditional dependency of financial series: An application of copulas. HEC Paris DP 723.

Saqdique, S., & Silvapulle, P. (2001). Long-term memory in stock market returns: International evidence. *International Journal of Finance and Economics*, 6, 59-67.

Sklar, A. (1959). “Fonctions de repartition an dimensions et leurs marges”. *Publications de l’Institut de Statistique de l’Universit de Paris*, 8, 229-231.

Taqqu, M. S., Teverovsky, V., Willinger, W. (1995). Estimators for long range dependence: an empirical study. Fractals, 3(4):785-798.

Van Den Goorbergh, R. W. J., Genest, C., Werkerc, J. (2005). Bivariate option pricing using dynamic copula models. *Insurance: Mathematics and Economics*, 37 p.101-114.