Welfare costs of inflation when interest-bearing deposits are disregarded: A calculation of the bias*

Rubens Penha Cysne, †David Turchick‡

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Abstract

Most estimates of the welfare costs of inflation are devised considering only noninterest-bearing assets, ignoring that since the 80’s technological innovations and new regulations have increased the liquidity of interest-bearing deposits. We investigate the resulting bias. Sufficient and necessary conditions on its sign are presented, along with closed-form expressions for its magnitude. Two examples dealing with bidimensional bilogarithmic money demands show that disregarding interest-bearing monies may lead to a non-negligible overestimation of the welfare costs of inflation. An intuitive explanation is that such assets may partially make up for the decreased demand of noninterest-bearing assets due to higher inflation.

1 Introduction

Feldstein (1997), Lucas (2000), Mulligan and Sala-i-Martin (2000) and Attanasio et al. (2002) are examples of papers providing theoretical foundations and/or estimates of welfare

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†Corresponding author. Rubens Penha Cysne is a Professor at the Graduate School of Economics of the Getulio Vargas Foundation (EPGE/FGV). Address: Praia de Botafogo 190, 11º andar – Rio de Janeiro, RJ – 22250-900 – Brazil. Telephone number: +55 (21) 3799-5871. Fax: +55 (21) 2553-8821. E-mail address: rubens.cysne@fgv.br.
‡David Turchick is a researcher at the Getulio Vargas Foundation (FGV).
costs of inflation (for unconstrained consumers) based on Bailey’s (1956) unidimensional area-under-the-inverse-money-demand-function formula.\(^1\) Attanasio et al.’s welfare calculations, following the work of Mulligan and Sala-i-Martin, also consider adoption decisions by households concerning the different financial technologies available in the economy. But the underlying welfare formula is still Bailey’s.

To simplify the notation throughout this paper, let \(M_1^*\) denote the noninterest-bearing component of \(M_1\), and \(M_2 \setminus M_1^*\) a monetary aggregate composed of all interest-bearing deposits belonging to \(M_2\) (according to some definition), but not to \(M_1^*\). Both Bailey’s (1956) and Lucas’s (2000, sections 3 and 5) welfare measures have been devised for monetary aggregates which do not pay interest. That is, irrespective of considering only the monetary base or \(M_1^*\) as the relevant monetary aggregate, they ignore a fact that has been thoroughly documented since the 80’s: the presence of several interest-bearing deposits \((M_2 \setminus M_1^*)\) providing monetary services.

As Teles and Zhou (2005) put it, "technological innovation and changes in regulatory practices in the past two decades have made other monetary aggregates as liquid as \(M_1\)." Attanasio et al. (2002, p. 341), for instance, report that 58.7 percent of the households in their sample (originated from the Italian economy) hold, besides the two monetary assets currency and interest-bearing bank deposits, at least one other interest-bearing non-monetary asset (e.g., bonds). Regarding the United States, Mulligan and Sala-i-Martin (2000, p. 962) report, following the 1989 Survey of Consumer Finances, that 35 percent of all households hold bank deposits and at least one additional interest-bearing asset.

In such settings, Bailey’s (1956) or Lucas’s (2000) unidimensional formulas may be misleading because they disregard, in the calculation of the welfare costs of inflation, the existence of a possible trade-off between more-liquid noninterest-bearing and less liquid

\(^1\)The dimension here corresponds to the number of monetary assets considered in the economy. See, e.g., Cysne (2003).
interest-bearing monies. This paper investigates the bias in the welfare calculations arising from this fact. Put in another way, we investigate the error arising from using unidimensional, rather than bidimensional (or multidimensional, since the extension to \( n \) different interest-bearing monies is straightforward) measures of the welfare costs of inflation.

Monetary aggregates are here classified and aggregated solely according to their user costs. Let \( m \) and \( x \) stand for the real quantities of \( M_1^* \) and \( M_2 \setminus M_1^* \), respectively. The aggregation of all interest-bearing deposits in one single asset \( x \), as is done here, implicitly assumes that they all have approximately the same user cost (defined as their opportunity cost relative to holding bonds). Otherwise, additional dimensions may be used, an extension that requires no significant cost. The present analysis concerns using only \( m \) in the calculations of the welfare costs of inflation, vis-à-vis using an aggregation of \( m \) and \( x \). We argue that the latter should become the standard procedure.

We assume that all monies are costlessly issued by the government. When it comes to the closed-form solutions for the bias arising from disregarding interest-bearing deposits, our calculations assume, as Jones et al. (2004) do in their theoretical model and Attanasio et al. (2002) verify in their empirical assessment with Italian data, that the user cost of the interest-bearing money is constant. The underlying approach to the problem, though, could be extended to the more general case in which this fact is not taken for granted, using for example the multidimensional formulas in Cysne and Turchick (2007).

To make our point clear right from the outset, let \( R \) stand for the interest rate on bank deposits, and \( R_B \) for the interest rate on a non-monetary financial asset. Since bank deposits provide monetary services, we must have \( R_B > R \geq 0 \). Bailey’s unidimensional

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\(^2\)Lucas (2000) acknowledges this point in the concluding section of his work.

\(^3\)Cysne and Turchick (2007) provide an upper bound to another type of error: the one arising from using one of the different types of multidimensional measures available in the literature, vis-à-vis the others.

\(^4\)Output is taken to be equal to 1, so that \( m \) and \( x \) may also represent fractions of GDP.
welfare formula $B_U$, which reads\(^{5}\)

$$B_U(R_B) = - \int_0^{R_B} \theta m'(\theta) d\theta, \quad (1)$$

has been widely employed in the literature in the past fifty years, a recent contribution being that of Ireland (2009).

We shall see that even when the opportunity cost of holding bank deposits ($R_B - R$) is constant, welfare formulas for households holding both bank deposits and bonds (in addition to currency) should also take into account the effect of variations of the interest rate paid by bonds on the demand for bank deposits ($x'(R_B)$). The use of equation (1) should therefore lead to a bias, the sign of which depends on the sign of $x'$, i.e., whether currency and deposits are substitutes or complements.

In the remainder of this text we will proceed as follows. Section 2 shortly presents multidimensional measures of the welfare costs of inflation based, respectively, on the McCallum-Goodfriend (1987) shopping-time dynamic framework and the Sidrauski (1967) money-in-the-utility-function (MUF) framework. They are developed for households holding currency, bank deposits and bonds. The main purpose here is to show why a second integral (besides the one in Bailey’s unidimensional formula) has to be considered in the calculations of the welfare costs of inflation.

Section 3 concentrates on the sign of the bias arising from the use of unidimensional measures instead of formulas which also take into consideration the existence of interest-bearing deposits. The comparison could be based on the multidimensional formulas arising from the use of the McCallum-Goodfriend or of the Sidrauski model. To simplify, we define the bias and proceed with the calculations using an approximation to both of these formulas. For analogy reasons, we shall call this approximation formula "Bailey’s multidimensional formula for the welfare costs of inflation", $B_M$. Next, we

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\(^{5}\)The subscript "U" used here stands for unidimensional. The subscript "B" in $R_B$ may stand for bonds.
determine when Bailey’s unidimensional measure $B_U$ should be expected to overestimate or underestimate $B_M$.

Section 4 introduces particular functional forms of the utility and monetary-aggregator function in order to offer closed-form expressions of the bias. Two examples of a particular case leading to bilogarithmic money-demand functions are developed in order to illustrate the fact that the size of the relative bias can be far from negligible. Section 5 concludes.

2 Multidimensional approaches to the welfare costs of inflation

The two next subsections reproduce (with minor adjustments) results obtained in Simonsen and Cysne (2001), Cysne (2003) and Cysne and Turchick (2007). They are included here for two reasons: first, to introduce the notation and assumptions necessary for the derivations of our main results in the forecoming section; and second, for convenience of the reader.

2.1 The shopping-time approach

In this section we shall not consider growth (it makes no difference concerning our final results). Let $y$ stand for real output and normalize it to 1. The consumer has a (fixed) time endowment equal to unity and gains utility from real consumption ($c = C/P$, $P$ standing for the price index) of a single non-storable consumption good, with preferences determined by

$$
\int_0^\infty e^{-gt}U(c)dt, \quad (2)
$$

where $U$ is a strictly increasing and concave function and $g > 0$.

Consumers can accumulate three assets: currency ($M$), interest-bearing deposits ($X$) and bonds ($B$). The interest rates paid by each one of these assets are, respectively, zero, $R$
and $R_B$. Let $b = B/P$, $m = M/P$, $x = X/P$, $h = H/P$ ($H$ indicates the (exogenous) flow of money transferred by the government to such consumers). Let $s$ stand for shopping time and $\pi = \dot{P}/P$ for the inflation rate (the dot over the variable represents its time derivative). Then, the budget constraint reads:

$$\dot{m} + \dot{x} + \dot{b} = 1 - c - s + h + (R_B - \pi) b + (R - \pi) x - \pi m. \tag{3}$$

The transacting technology is given by

$$c = F(m, x, s) = G(m, x)\phi(s), \tag{4}$$

with $F_m > 0$, $F_x > 0$ and $F_s > 0$. The twice-differentiable monetary-aggregator function $G$ is assumed to be first-degree homogeneous, concave and such that $G_{mm} < 0$, $G_{xx} < 0$. The microfoundations for a transacting technology of type (4) are based on the inventory technology found in the works of Baumol (1952), Tobin (1956) and Miller and Orr (1966). Lucas (2000, p. 265 – see, in particular, ft. 13) discusses this point.

Households maximize (2) subject to the budget constraint (3) and to the time-transacting technology (4). In the steady state, assuming interior solutions, Euler equations lead to the necessary conditions

$$\begin{cases} R_B = \pi + g \\ R_B = \frac{F_m}{F_s} = \frac{G_m(m, x)}{\phi'(s)} \\ R_B - R = \frac{F_x}{F_s} = \frac{G_x(m, x)}{\phi'(s)} \end{cases}.$$  

The equilibrium in the goods market is described by

$$1 - s = F(m, x, s).$$

When the functions $G$ and $\phi$ are known, the three last equations can be used to determine
As shown in Cysne (2003), in this model the welfare costs of inflation are given by the differential form of a Divisia index:

\[
\frac{ds}{1 - s + R_B m + (R_B - R)x},
\]

with \( s(0,0) = 0 \). In this 2-dimensional context, a reasonable approximation to (5) for low values of \( R_B \) can be given by the Bailey-like multidimensional differential formula

\[
\frac{dB_M}{1 - s + R_B m + (R_B - R)x},
\]

with \( B_M(0,0) = 0 \). Expressions (6) and (5) extend the 1-dimensional ones offered by Bailey (1956) and by Lucas (2000, section 5), respectively. Note that as \( R_B \to 0 \), provided \( R_B m \) and \( (R_B - R)x \) also go to 0, we get \( ds \approx dB_M \) and, upon integration and usage of the initial conditions, \( s(R_B, R_B - R) \approx B_M(R_B, R_B - R) \). As an example, if currency demand takes the conventional log-log form with respect to \( R_B \), the assumption \( \lim_{R_B \to 0} R_B m = 0 \) amounts to the condition that this demand be inelastic with respect to its opportunity cost (and similar reasoning applies to \( \lim_{R_B - R \to 0} (R_B - R)x = 0 \)).

### 2.2 The MUF approach

The purpose of this subsection is to establish the robustness of the results derived in the previous one concerning the general form of the expressions which should be used for

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6 As a counterpart to the subscript "U" used in (1), the subscript "M" used here stands for "multidimensional".

7 Here "\( f \approx g \)" means asymptotic equivalence, that is, \( f/g \to 1 \) in the limiting process considered. Lucas (2000) shows in the unidimensional case that, in practice, the difference between \( B_U \) and \( s \) is negligible.
welfare calculations in economies with more than one type of monetary asset. We show that the use of an alternative setting (Sidrauski’s money-in-the-utility-function setting, rather than the shopping-time one) leads to welfare expressions similar to (5) and (6), but now extending the unidimensional measure given in Lucas (2000, section 3).

We incorporate growth in the model by normalizing $y_0$ to 1 and making (as in Lucas 2000)

$$y_t = e^{\gamma t},$$

with $\gamma > 0$.

Also as in Lucas (2000), the utility function (here written in terms of the output shares $c, m$ and $x$) is given by

$$U(c, m, x) = \left( \frac{c \varphi \left( \frac{G(m, x)}{e} \right)}{1 - \sigma} \right)^{1-\sigma},$$

where $\sigma > 0$, the twice-differentiable function $\varphi : [0, +\infty] \to [0, +\infty]$ is such that $\varphi' > 0$ and $\varphi'' < 0$ over $(0, \bar{M})$ (where $\bar{M} \in (0, +\infty]$) and constant thereafter, and $G : [0, +\infty]^2 \to [0, +\infty]$ is as in the previous subsection. Given (7), consumers maximize

$$\int_0^{+\infty} e^{-\left(g-(1-\sigma)\gamma\right)t} U(c, m, x)dt$$

subject to

$$\dot{m} + \dot{x} + \dot{b} = 1 - c + h + (R_B - (\pi + \gamma)) b + (R - (\pi + \gamma)) x - (\pi + \gamma)m. \quad (9)$$

Euler equations for this problem give

$$\begin{cases}
R_B = \frac{U_m}{U_c} \\
R_B - R = \frac{U_x}{U_c}.
\end{cases} \quad (10)$$
Define the domains

\[ D_\leq := \{(m, x) \in [0, +\infty]^2 : G(m, x) \leq \bar{M}\}, \]

\[ D_\geq := \{(m, x) \in [0, +\infty]^2 : G(m, x) \geq \bar{M}\} \]

and

\[ D := D_\leq \cap D_\geq. \]

The points \( \mathbf{d} \in D_\geq \) are those corresponding to the maximum available value for \( \varphi \). Call it \( \varphi^* := \sup_{m, x} \varphi(G(m, x)) = \varphi(\bar{M}) \). Note that \( \varphi^* \), as \( \bar{M} \), is allowed to equal infinity.

Repeating the procedure used in Cysne (2003), we shall henceforth work with the welfare-cost function \( \overline{w} \) defined as a function of the vector of monetary aggregates, rather than as a function of the interest rates. The initial condition reads \( \overline{w}(\mathbf{d}) = 0 \) for any \( \mathbf{d} \in D_\geq \). We shall be specifically interested in paths \( \chi : t \mapsto (m, x) \) such that \( \chi(0) = \mathbf{d} \in D \) (note that each coordinate \( d_i \) of \( \mathbf{d} \) can be equal to infinity) and \( \dot{\chi} \ll 0 \). The reason for taking \( \chi(0) \in D \) is that this is how one gets the lowest possible values for the opportunity costs, making this the benchmark used for measuring the welfare cost of inflation.

Making \( c = 1 \), (10) becomes

\[
\begin{cases}
R_B = \frac{\varphi'(G(m,x))}{\varphi(G(m,x)) - \varphi'(G(m,x))} G(m, x) =: \psi^m(m, x) \\
R_B - R = \frac{\varphi'(G(m,x))}{\varphi(G(m,x)) - \varphi'(G(m,x))} G_x(m, x) =: \psi^x(m, x)
\end{cases}
\]  

(11)

Note that (11) gives the opportunity cost of holding each asset as a function of \( m \) and \( x \). Call this function \( \psi : D_\leq \to [0, +\infty]^2 \), with \( \psi := (\psi^m(m, x), \psi^x(m, x)) \). It is a differentiable function, given the twice-differentiability of \( \varphi \).

Write down the equations defining Lucas’s measure of the welfare cost of inflation using the definition of \( \varphi^* \):

\[ U(1 + \overline{w}(m, x), m, x) = U(1, \mathbf{d}) \]
for some $d \in D$, or
\[
(1 + \overline{w})\varphi \left( \frac{1}{1 + \overline{w}} G(m, x) \right) = \varphi (\overline{M}) = \varphi^*.
\]

Totally differentiating this expression gives
\[
\left[ \varphi \left( \frac{1}{1 + \overline{w}} G(m, x) \right) - \frac{1}{1 + \overline{w}} G(m, x) \varphi' \left( \frac{1}{1 + \overline{w}} G(m, x) \right) \right] d\overline{w} + \\
\varphi' \left( \frac{1}{1 + \overline{w}} G(m, x) \right) (G_m (m, x) dm + G_x (m, x) dx) = 0,
\]
or, using (11) together with $G$'s homogeneity,
\[
d\overline{w} = - \left[ \psi^m \left( \frac{1}{1 + \overline{w}(m, x)}(m, x) \right) dm + \psi^x \left( \frac{1}{1 + \overline{w}(m, x)}(m, x) \right) dx \right]. \tag{12}
\]

(12) is the differential form of the Divisia index which gives the welfare costs of inflation for this economy. Obviously, as $(m, x)$ approaches $D$, $\overline{w}(m, x) \to 0$, and
\[
\psi^m \left( (1 + \overline{w}(m, x))^{-1} (m, x) \right) \approx \psi^m (m, x) \text{ and } \psi^x \left( (1 + \overline{w}(m, x))^{-1} (m, x) \right) \approx \psi^x (m, x).
\]
Therefore, from (11) and (6), $d\overline{w} \approx dB_M$ and $\overline{w} \approx B_M$, as shown for $s$ in the previous subsection. In words (and again, as shown concerning the unidimensional case in Lucas 2000), as $\overline{w}$ gets closer and closer to zero, the welfare measure which emerges from the multidimensional money-in-the-utility-function approach leads to (6), the generalization of Bailey’s formula to a multidimensional setting.

The main conclusion to be drawn from this section is that when either the shopping-time or the Sidrauski setting is considered, (5) or (12) are the correct welfare measures to be used for households holding currency, bank deposits and bonds, rather than (1). And that, when reasonable interest rates are considered, (6) can be used as a good approximation to both.\(^8\)

\(^8\)It can even be shown that, irrespective of interest rates, one always gets $s < B < \overline{w}$ (see Cysne and Turchick, 2007).
3 The sign of the bias

From the previous section we conclude that, in economies with currency, interest-bearing deposits and bonds, at least three alternative formulas can be used to calculate the welfare costs of inflation: (5), (6) or (12). In such economies, the use of Bailey’s (1956) or Lucas’s (2000) unidimensional formulas may be misleading. Even when $R_B - R$ is independent of inflation, as considered to be the case in Attanasio et al. (2002), the use of unidimensional measures overlooks the fact that the effect of changes in the nominal interest rate $R_B$ on the demand for bank deposits also has to be taken into consideration in the welfare calculations (since such deposits also provide monetary services).

Indeed, by assuming $R_B - R = \bar{K}$ constant in the previous multidimensional formulas, and taking into consideration that $R_B \geq \bar{K}$ (since $R \geq 0$), one obtains the following particular cases:

\begin{align}
  s(R_B) := s(R_B, \bar{K}) &= - \int_{\bar{K}}^{R_B} \frac{(1 - s(\theta)) \left[ \theta m'(\theta) + \bar{K} x'(\theta) \right]}{1 - s(\theta) + \theta m(\theta, \bar{K}) + \bar{K} x(\theta, \bar{K})} d\theta \quad (13)
\end{align}

when the shopping-time model is taken as reference;

\begin{align}
  B_M(R_B) := B_M(R_B, \bar{K}) &= - \int_{\bar{K}}^{R_B} \left[ \theta m'(\theta) + \bar{K} x'(\theta) \right] d\theta \quad (14)
\end{align}

when a multidimensional Bailey-like approximation is taken as reference; and

\begin{align}
  \bar{w}(R_B) := \bar{w}(m(R_B, \bar{K}), x(R_B, \bar{K})) &= - \int_{\bar{K}}^{R_B} \left[ \psi^m \left( \frac{1}{1 + \bar{w}^m(\theta)} m(\theta, \bar{K}), x(\theta, \bar{K}) \right) m'(\theta) + \psi^x \left( \frac{1}{1 + \bar{w}^x(\theta)} m(\theta, \bar{K}), x(\theta, \bar{K}) \right) x'(\theta) \right] d\theta \quad (15)
\end{align}

when a money-in-the-utility-function approach is taken as reference.

\(^9\)For notational ease only, we write $m'(\theta)$ and $x'(\theta)$ for $(\partial m/\partial R_B)(\theta, \bar{K})$ and $(\partial x/\partial R_B)(\theta, \bar{K})$, respectively.
There are three conceptual differences between (1) and any one of formulas (13), (14) and (15).

First, since \( R \geq 0 \), the dummy variable \( \theta \) in the formulas above should run from \( \bar{K} \) to \( R_B \), rather than from 0 to \( R_B \). Note, though, that \( \bar{K} \) can be equal to zero.

Second, one has to take into consideration the effects of changes in \( R_B \) on the demand for bank deposits \( (x) \).

Third, and in general as well, all functions should be written as depending both on \( R_B \) and \( R_B - R \), rather than on one variable only.

Note that (13), (14) and (15) can be split into two Riemann integrals, the second one based on \( x'(\theta) \) (evaluated for a fixed value of \( R_B - R \) equal to \( \bar{K} \)), whereas (1) cannot.

Since (13), (14) and (15) are asymptotically equivalent (for small interest rates or large amounts of monies demanded; and under the assumptions at the end of Section 2.1), a comparison between (1) and any one of these three measures also conveys relevant information about the difference between (1) and the other two. For simplicity, we shall in the remainder of this section compare (1) with its multidimensional extension (14) (accordingly, the bias will be defined in the next section as the absolute value of the relative difference between these two formulas).

(14) will lead to welfare figures less than or greater than (1), depending if \( x'(\theta) \), the cross derivative of the demand for interest-bearing deposits with respect to the opportunity cost of holding currency, is, respectively, positive or negative. In the former case, currency and interest-bearing deposits are said to be substitutes; in the latter case, to be complementary to each other. It is therefore important to study the sign of \( x'(\theta) \) which may emerge from the shopping-time or the MUF model.

From this point on, we work this out in the context of the MUF model. This choice bears on generality reasons, since it can be shown that money demand specifications derived from the shopping-time model can also be obtained through the Sidrauski model, or an adequate extension including interest-bearing deposits (Cysne and Turchick 2009).
Totally differentiating the equilibrium equations (11) yields:

\[
\begin{align*}
\frac{dR_B}{dt} &= \delta_{mn} dm + \delta_{mx} dx, \\
\frac{d (R_B - R)}{dt} &= \delta_{mx} dm + \delta_{xx} dx,
\end{align*}
\]

where

\[\delta_{ij} = \frac{1}{\varphi - G\varphi'} \left( \varphi' G_{ij} + \frac{\varphi''}{\varphi - G\varphi'} G_i G_j \right)\]

and the functions \( G, \varphi, \varphi', \varphi'' \) are calculated at the equilibrium.

Inversion of this system gives:

\[
\begin{align*}
\frac{dm}{dR_B} &= \frac{1}{\delta_{mm} \delta_{xx} - \delta_{mx}^2} \left[ \delta_{xx} dR_B - \delta_{mx} d(R_B - R) \right], \\
\frac{dx}{dR_B} &= \frac{1}{\delta_{mm} \delta_{xx} - \delta_{mx}^2} \left[ -\delta_{mx} dR_B + \delta_{mm} d(R_B - R) \right],
\end{align*}
\]

implying the equilibrium derivatives

\[
\begin{align*}
\frac{\partial m}{\partial R_B} &= \frac{\delta_{xx}}{\delta_{mm} \delta_{xx} - \delta_{mx}^2}, \\
\frac{\partial x}{\partial R_B} &= -\frac{\delta_{mx}}{\delta_{mm} \delta_{xx} - \delta_{mx}^2}.
\end{align*}
\]

This can be rewritten as (see Appendix A for details)

\[
m' (R_B) = -\frac{m x (\varphi - G\varphi')}{G^2 G_{mx} \varphi' \varphi''} \left( G_x \varphi'' + G_{xx} \varphi' (\varphi - G\varphi') \right) < 0
\]

and

\[
x' (R_B) = \frac{m x (\varphi - G\varphi')}{G^2 G_{mx} \varphi' \varphi''} \left( G_m G_x \varphi'' + G_{mx} \varphi' (\varphi - G\varphi') \right).
\]

From the subgradient inequality applied to the concave function \( \varphi \), we know that \( \varphi (G) - G\varphi' (G) > 0 \) if \( G > 0 \). Since \( G_m \) is 0-degree homogeneous, Euler’s formula gives, for any \((m, x) \in \mathbb{R}^2_{++}, 0 = G_{mm} (m, x) m + G_{mx} (m, x) x\), so that \( G_{mx} > 0 \). This explains the negativity of \( m' \) on the one hand, but the sign-indeterminacy of \( x' \) on the other.
3.1 Results under general $G$ and $\varphi$ functions

We now look for a sufficient condition regarding functions $G$ and $\varphi$ so that $B_U > B_M$, irrespective of interest rates. By comparing (1) and (14), one easily concludes that a sufficient condition for $B_U > B_M$ is that currency and interest-bearing deposits are substitutes to each other ($x' > 0$).\textsuperscript{10} Indeed, for any $R_B$, we have

$$B_U (R_B) - B_M (R_B) = K \int_K^{R_B} x' (\theta) d\theta. \quad (19)$$

Therefore,

$$B_U > B_M \text{ if, and only if, } x (R_B) > x (\bar{K}), \forall R_B > \bar{K}.$$

This leads to Proposition 1.

**Proposition 1** Consider an economy with currency, interest-bearing bank deposits and bonds, where the spread on bank deposits can be considered constant. Then a sufficient condition for Bailey's unidimensional measure of the welfare costs of inflation to overestimate (underestimate) the true welfare costs of inflation is that the following expression is negative (positive):

$$G_m G_x \varphi'' + G_{mx} \varphi' (\varphi - G \varphi'), \quad (20)$$

where $G$, $G_m$, $G_x$ and $G_{mx}$ are evaluated at each pair $(m, x) \in [0, +\infty]^2$, and $\varphi$, $\varphi'$ and $\varphi''$ are evaluated at $G (m, x)$.

**Proof.** This comes directly from (18) and (19). $\blacksquare$

The condition given in the above proposition, which makes $x'(R_B) > 0$, can also be written in terms of elasticities in the following way. Let $\varepsilon_{R_B}^m$ and $\varepsilon_{R_B}^x$ be the elasticities of,$^\text{10}$Following the explanation above, in this economy an adaptation in Bailey’s formula (1) must take place: the lower limit of integration is now $\bar{K}$. 14
respectively, cash and bank-deposits demands, with respect to the nominal interest rate paid by government bonds. Let $\xi$ represent the elasticity of substitution between cash and bank deposits, all three elasticities being evaluated at the same point. That is,

$$
\varepsilon_{RB}^m = \frac{d \log m}{d \log R_B},
$$

$$
\varepsilon_{RB}^x = \frac{d \log x}{d \log R_B},
$$

and

$$
\xi = \frac{d \log (x/m)}{d \log (R_B/(R_B - R))} = \frac{d \log (x/m)}{d \log (G_m/G_x)}.
$$

Since $R_B - R$ is constant, we have

$$
\xi = \frac{d \log (x/m)}{d (\log R_B - \log K)} = \frac{d (\log x - \log m)}{d \log R_B} = \varepsilon_{RB}^x - \varepsilon_{RB}^m. \tag{21}
$$

Therefore, $\varepsilon_{RB}^x > 0$ if, and only if, $\xi + \varepsilon_{RB}^m > 0$, that is, $|\varepsilon_{RB}^m| < \xi$ (since $\varepsilon_{RB}^m < 0$, by (17)). This leads to Proposition 2.

**Proposition 2** Consider an economy with currency, interest-bearing bank deposits and bonds, where the spread on bank deposits can be considered constant. Then Bailey’s unidimensional measure of the welfare costs of inflation overestimates the true welfare costs when the elasticity of demand for currency is (in absolute value) always lower than the elasticity of substitution between currency and bank deposits:

$$
|\varepsilon_{RB}^m| < \xi. \tag{22}
$$

Conversely, an underestimation occurs when the elasticity of demand for currency is (in absolute value) always greater than that elasticity of substitution:

$$
|\varepsilon_{RB}^m| > \xi. \tag{23}
$$
Proof. Done.

Remark 1 (The Cobb-Douglas case, $\xi = 1$.) Note that $|\varepsilon_{R_B}^m| \geq 1$ is unusual in theoretical monetary models. In models with constant real interest rates, like the two presented in Section 2, it would make the inflation tax negatively correlated with inflation, leading to potential instability. Lucas (2000), for instance, presents welfare formulas which are well defined only when $|\varepsilon_{R_B}^m| < 1$. This implies that in the Cobb-Douglas case one should expect, from Proposition 2, an overestimation of the welfare costs of inflation when Lucas’s or Bailey’s unidimensional formulas are used.

3.2 Results under a CES $G$ function and a general $\varphi$ function

Note that condition (20) establishes a property which depends on both functions $G$ and $\varphi$. In order to get further results and insights, this subsection still assumes a general $\varphi$, but particularizes to the case in which the aggregator function $G$ has a constant elasticity of substitution $\xi$:

Assumption $G$. $G(m, x) = \begin{cases} \left( \beta m^{\frac{\xi-1}{\xi}} + (1 - \beta) x^{\frac{\xi-1}{\xi}} \right)^{\frac{\xi}{\xi-1}}, & \text{if } \xi > 0 \text{ and } \xi \neq 1 \\ m^{\beta} x^{1-\beta}, & \text{if } \xi = 1 \end{cases}$, where $\beta \in (0, 1)$.

Proposition 3 gives a sufficient condition for overestimation related to $\varphi$ only, provided the elasticity of substitution between $m$ and $x$ is high enough.

Proposition 3 Consider an economy with currency, interest-bearing bank deposits and bonds, where the spread on bank deposits can be considered constant, and Assumption $G$ is valid. Then, a sufficient condition for Bailey’s unidimensional measure of the welfare costs of inflation to overestimate (underestimate) the true welfare costs is that, $\forall G \in (0, \bar{M})$, the

\[11\] For example, the second expression from the bottom on page 251.
following expression is positive (negative):
\[ \xi + \frac{\varphi' (\varphi - G \varphi')}{G \varphi''}, \quad (24) \]
where \( \varphi, \varphi' \) and \( \varphi'' \) are evaluated at \( G \).

**Proof.** Using the expressions obtained in Appendix B for the partial derivatives of \( G \), one has

\[
\begin{align*}
\text{sgn} \left( G_mG_x \varphi'' + G_{mx} \varphi' (\varphi - G \varphi') \right) &= \text{sgn} \left( \frac{\beta}{m} \left( \frac{G'}{x} \right)^{\frac{1}{2}} (1 - \beta) \left( \frac{G'}{x} \right)^{\frac{1}{2}} \varphi'' + \frac{\beta (1 - \beta) G^{-1 + \frac{2}{x}}}{\xi m^{\frac{1}{2}} x^{\frac{1}{2}}} \varphi' (\varphi - G \varphi') \right) \\
&= \text{sgn} \left( \varphi'' + \frac{\varphi' (\varphi - G \varphi')}{\xi G} \right) = -\text{sgn} \left( \xi + \frac{\varphi' (\varphi - G \varphi')}{G \varphi''} \right).
\end{align*}
\]

Proposition 1 finishes the job. ■

**Remark 2** The sign of the expression in (24) coincides with that of
\((1 - \xi) G \varphi''/\varphi - (G \varphi'/\varphi)'\). Thus, in the \( \xi = 1 \) case (that is, if \( G \) is simply a
Cobb-Douglas-type aggregator function), \( B_U \geq B_M \) if \((G \varphi'/\varphi)' \leq 0\).

At this point it is natural to ask what condition would be necessary so that \( B_U = B_M \).

**Proposition 4** Consider an economy with currency, interest-bearing bank deposits and
bonds, where the spread on bank deposits can be considered constant, and Assumption \( G \) is
valid. Then \( B_U = B_M \) if, and only if, the expression in (24) equals zero. Moreover, in this
case, if \( \xi = 1 \) (the Cobb-Douglas \( G \) case), the demand for \( m \) and for \( x \) depend only on their
respective user costs, as in\(^{12}\)
\[
\begin{align*}
m &= \frac{\delta}{1 - \delta} \frac{\beta}{R_B} \\
x &= \frac{\delta}{1 - \delta} \frac{1 - \beta}{R_B - R}
\end{align*}
\]
\(^{12}\)This functional form is the one used by Bali (2000, equations 9a and 9b).
Proof. From (19), $B_U$ and $B_M$ are identical if and only if $x'$ is identically null, which, from (18), happens if and only if the expression in (20) is identically null. Or still, from the math done in the proof of Proposition 3, $\xi + \varphi' (\varphi - G') / (G'\varphi'') = 0$.

In particular, if $\xi = 1$, Remark 2 gives $(G'\varphi')' = 0$, whence $\varphi$ is such that $\varphi (G) = \eta G^\delta$, for some $\eta > 0$ and $\delta \in (0, 1)$. Given (11), this leads to

$$\begin{cases} R_B = \frac{\delta \beta}{1-\delta \alpha}, \\ R_B - R = \frac{\delta}{1-\delta} \frac{1-\beta}{x}. \end{cases}$$

3.3 Results under a CES $G$ function and a CES $\varphi$ function

For definiteness, we impose, besides Assumption $G$, a functional form on $\varphi$ implying the general CES-type expression for the utility function:

$$U (c, m, x) = \left( \left( \rho c^{\alpha-1} + (1-\rho) \eta \frac{\alpha-1}{\alpha} G (m, x) \frac{\alpha-1}{\alpha} \right) \right)^{1-\sigma}. $$

That is,

Assumption $\varphi$. $\varphi (G) = \begin{cases} \left( \rho + (1-\rho) \eta G \right)^{\frac{\alpha}{\alpha-1}}, & \text{if } \alpha > 0 \text{ and } \alpha \neq 1, \\
(\eta G)^{1-\rho}, & \text{if } \alpha = 1 \end{cases}$, where $\rho \in (0, 1)$ and $\eta > 0$.

Here $\bar{M} = +\infty$, and properties $\varphi' > 0$ and $\varphi'' < 0$ are straightforward. We then obtain the following very simple test:

Proposition 5 Consider an economy with currency, interest-bearing bank deposits and bonds, where the spread on bank deposits can be considered constant, and Assumptions $G$ and $\varphi$ are valid. Then $B_U \gtrless B_M$ if (and only if) $\xi \gtrless \alpha$. In words, $B_U$ overestimates (underestimates) the true welfare costs of inflation if the monies under consideration share
a greater (lesser) degree of substitutability than that between the consumption good and the monetary aggregate.

**Proof.** As shown in Appendix C, (24) becomes $\xi - \alpha$. Thus Propositions 3 and 4 provide the result. ■

The relation between Propositions and is established by equation (29), to be derived in the next section.

Under the additional Assumption $\varphi$, since (24) depends on parameters only, either $B_U$ always overestimates $B_M$, or always underestimates $B_M$, or always equals it.

### 4 Gauging the bias

The last section presented a simple reason why unidimensional calculations of the welfare costs of inflation may be biased. We now shift our concern from the direction of the bias to its size. That is, we are interested in assessing the measure

$$
\Omega (R_B) := \left| \frac{B_U (R_B) - B_M (R_B)}{B_M (R_B)} \right| = \left| \frac{K (x (R_B) - x (K))}{B_M (R_B)} \right| , \tag{25}
$$

the error one incurs when taking $B_U$ instead of $B_M$. The second equal sign is derived from (19). Evidently, in the extreme $K = 0$ case, in which bonds and deposits are perfect substitutes, (1) and (14) would yield $B_M (R_B) = B_U (R_B)$, so that $\Omega = 0$. From now on, $K > 0$ is assumed. Assumptions $G$ and $\varphi$ are also supposed to hold. For the reader’s sake, auxiliary expressions relative to the $\xi = 1$ case are left to the appendices.

As shown in Appendix C, given Assumptions $G$ and $\varphi$, the demand system (11) in the $\xi = 1$ case becomes, after inversion (regardless of whether $\alpha$ is equal to unity or not),

$$
\begin{align*}
\left\{ 
\begin{array}{l}
m = \left( \frac{1 - \rho}{\rho} \right)^{\alpha} \eta^{\alpha-1} (\beta R_B^{-1})^{1+\alpha(\alpha-1)} \left( (1 - \beta) (R_B - R)^{-1} \right)^{(\alpha-1)(1-\beta)} \\
x = \left( \frac{1 - \rho}{\rho} \right)^{\alpha} \eta^{\alpha-1} (\beta R_B^{-1})^{(\alpha-1)} \left( (1 - \beta) (R_B - R)^{-1} \right)^{1+(\alpha-1)(1-\beta)},
\end{array}
\right.
\end{align*}
$$

(26)
the bidimensional log-log money-demand specification.

Putting $R_B - R = \bar{K}$, we get $m$ and $x$ as functions of $R_B$ only. Formula (1) (with the lower limit of the integral adjusted to $\bar{K}$ as before) gives (see Appendix D)

$$B_U(R_B) = \begin{cases} \frac{1-\rho}{\rho} \beta \left( \bar{K} - R_B + \ln \frac{R_B}{K} \right), & \text{if } \alpha = 1 \\ \left( \frac{1-\rho}{\rho} \right)^{\alpha} (1-\alpha) \beta \left( \eta \beta \left( 1 - \beta \right)^{1-\beta} \right)^{\alpha-1} \times \\ \tilde{K}^{(1-\alpha)(1-\beta)} (R_B^{(1-\alpha)\beta} - \bar{K}^{(1-\alpha)\beta}), & \text{if } \alpha \neq 1 \end{cases}. $$

From (14), we have

$$B_M(R_B) := B_M(R_B, \bar{K}) = B_U(R_B) - \bar{K} \int_{K}^{R_B} x'(\theta) d\theta = B_U(R_B) - \bar{K} \left( x(R_B) - x(\bar{K}) \right),$$

whence

$$B_M(R_B) = \begin{cases} B_U(R_B) - \frac{1-\rho}{\rho} (1 - \beta) (1 - 1) = \frac{1-\rho}{\rho} \beta \left( \bar{K} - R_B + \ln \frac{R_B}{K} \right), & \text{if } \alpha = 1 \\ B_U(R_B) - \left( \frac{1-\rho}{\rho} \right)^{\alpha} \eta^{\alpha-1} \beta^{(\alpha-1)\beta} \left( 1 - \beta \right)^{1+(\alpha-1)(1-\beta)} \times \\ \tilde{K}^{(1-\alpha)(1-\beta)} \left( R_B^{(1-\alpha)\beta} - \bar{K}^{(1-\alpha)\beta} \right) \\ = \left( \frac{1-\rho}{\rho} \right)^{\alpha} \frac{1}{1-\alpha} \left( \eta \beta \left( 1 - \beta \right)^{1-\beta} \right)^{\alpha-1} \times \\ \tilde{K}^{(1-\alpha)(1-\beta)} (R_B^{(1-\alpha)\beta} - \bar{K}^{(1-\alpha)\beta}), & \text{if } \alpha \neq 1 \end{cases}. $$

Therefore, we get

**Proposition 6** Consider an economy with currency, interest-bearing bank deposits and bonds, where the spread on bank deposits can be considered a positive constant, and
Assumptions $G$ and $\varphi$ are valid. Then

\[
\Omega(R_B) = \begin{cases} 
0, & \text{if } \xi = 1 \\
\frac{(1-\alpha)(1-\beta)}{\alpha}, & \text{if } \xi = 1 \\
(1 - \xi)(1 - \beta)^{\xi-1} K^{1-\xi} \left( \frac{\beta \xi R_B^{1-\xi} + (1-\beta)\xi K^{1-\xi}}{\ln(\beta \xi R_B^{1-\xi} + (1-\beta)\xi K^{1-\xi}) - \ln(K^{1-\xi} (\beta \xi + (1-\beta)\xi))} \right), & \text{if } \xi = 1 \\
\frac{(1-\alpha)(1-\beta)}{\alpha} \left( \frac{\beta \xi R_B^{1-\xi} + (1-\beta)\xi K^{1-\xi}}{\ln(\beta \xi R_B^{1-\xi} + (1-\beta)\xi K^{1-\xi}) - \ln(K^{1-\xi} (\beta \xi + (1-\beta)\xi))} \right), & \text{if } \xi = 1
\end{cases}
\]

Proof. It’s just a matter of noting that the amount $\bar{K} \left( x(R_B) - x(\bar{K}) \right)$ was conveniently left in explicit form in the calculation of $B_M$ above (as well as in Appendix D), and then applying the second expression for $\Omega$ in (25).

It may be noted that, in any of the four cases presented in Proposition 6, neither $\rho$ nor $\eta$ (the parameters associated with the share of importance the individual gives to consumption of the good vis-à-vis the money aggregate) have any effect on this bias.

Obviously, the absolute-value bars may be dispensed with when $B_U > B_M$. From Proposition 2, we know this will be the case when $|\varepsilon_{R_B}^m| < \xi$. From (26) and $R_B - R = \bar{K}$,

\[
|\varepsilon_{R_B}^m| = -\varepsilon_{R_B}^m = -\frac{\partial \log m}{\partial \log R_B} = 1 + (\alpha - 1) \beta = \beta \alpha + (1 - \beta) \xi
\]

when $\xi = 1$. As calculated in Appendix C, also in the $\xi \neq 1$ case one obtains for $|\varepsilon_{R_B}^m|$ a weighted average between $\alpha$ and $\xi$:

\[
|\varepsilon_{R_B}^m| = \frac{\beta \xi R_B^{1-\xi} + (1-\beta)\xi \bar{K}^{1-\xi}}{\beta \xi R_B^{1-\xi} + (1-\beta)\xi K^{1-\xi}}.
\]  

(29)

Thus $\xi \geq \alpha$ if and only if $\xi \geq |\varepsilon_{R_B}^m|$, whence Proposition 2 provides another proof for
Proposition 5.

The two examples ahead illustrate the two main possibilities, overestimation and underestimation, of the welfare cost of inflation, when interest-bearing deposits are not taken into account\(^{13}\). For simplicity, they use the Cobb-Douglas case \((\xi = 1)\), where neither \(R_B\) nor \(\bar{K}\) affect the bias. For \(\bar{K}\), we use the value 0.0242.\(^{14}\)

**Example 1** \(\xi = 1, \beta = 0.5, \alpha = 0.2, \bar{K} = 0.0242, \text{ and any } (\rho, \eta) \in (0, 1) \times \mathbb{R}_{++}, \text{ making } B_U(0.1) = 0.016 \) (as in Lucas 2000, p. 258).\(^{15}\) Formula 26 gives, in this case, \((m, x) \approx (0.062R_B^{-0.6}, 2.557R_B^{0.4})\). Propositions 5 and 6 imply that, in this case, \(B_U\) would overestimate the welfare cost of inflation in \(\Omega = (1 - 0.2)(1 - 0.5)/0.2 = 200\%\).

That is, while for yearly interest rates of 10\%, \(B_U\) is at 1.6\% of GDP, the measure \(B_M\) leads to just 0.53\% of GDP, a threefold difference.

\(^{13}\)Although, from a theoretical point of view, the partition we propose here is between \(M_1^*\) and \(M_2 \setminus M_1^*\), from this point on, when using monetary data (all of it related to the U.S. economy), we will not distinguish between \(M\) and \(M^*\). This is to say that we will associate \(m\) with the available definition of \(M_1\) (disregarding the possible existence of interest-bearing deposits in it, such as the NOW accounts) and \(x\) with \(M_2 \setminus M_1\), both as fractions of the U.S. GDP (\(M_1\) was entirely noninterest-bearing only before 1980). We do that for practical reasons only.

\(^{14}\)Considering the 6 1/2 year-period from 01/2003 to 06/2009, the yearly average interest rate on the ten-year constant-maturity Treasury Bill was 4.23\%, whereas that on \(M_2 \setminus M_1\) was approximately 1.53\%. The relative share of \(M_2 \setminus M_1\) in \(M_2\) was around 0.81. The average spread and the unconditional standard deviation (both calculated using monthly data) were 2.42\% and 0.70\%, respectively (data available from the Federal Reserve Bank of St. Louis - FRED).

\(^{15}\)For example, \(\rho = 0.5\) and \(\eta = 4.24\) will do.
Figure 1. Overestimation of the welfare costs of inflation due to disregarding other monetary assets.

Although our purpose here is more focused at providing an illustration of the application of our theoretical results than empirical estimates of the bias, a word of comparison with previous investigations of this same issue may be valuable.

Bali (2000) concludes, akin to the numbers above, that assuming $M_1$ is noninterest bearing leads to an overestimate of the welfare cost by a factor of three. Bali’s result, though, is not comparable with ours, for he assumes a utility function separable in currency and deposits, equivalent to making $\xi \to +\infty$.

The overestimation obtained by Bali is a consequence of assuming that the user cost of interest-bearing deposits is proportional to the nominal interest rate (determined by a competitive banking system operating under constant reserve requirements). Under such an assumption it is easy to show that the welfare costs are proportional to the real value of the monetary aggregate used in the calculations.\footnote{See Proposition 4 in Cysne (2003).} Since the historical ratio of $M_1$ to the
monetary base is equal to three, an overestimation by a factor of three emerges.

We present below an example where the use of Bailey’s welfare formula leads to an underestimation, rather than an overestimation, of the true welfare costs of inflation ($B_U < B_M$). As noted in Remark 1, because it implies (when $G$ is Cobb-Douglas) an elasticity of demand for currency greater than one, this case can be considered to be less likely to hold in the real world.

**Example 2** $\xi = 1$, $\beta = 0.5$, $\alpha = 2$, $K = 0.0242$, and any $(\rho, \eta) \in (0, 1) \times \mathbb{R}_{++}$ making $B_U (0.1) = 0.016$ (as in Lucas 2000, p. 258). Formula 26 gives, in this case, $(m, x) \approx (1.633 \times 10^{-3} R_B^{-1.5}, 6.748 \times 10^{-2} R_B^{-0.5})$. Propositions 5 and 6 imply that, in this case, $B_U$ would underestimate the welfare cost of inflation in $\Omega = |(1 - 2) (1 - 0.5) / 2| = 25\%$. That is, while for yearly interest rates of 10%, $B_U$ is at 1.6% of GDP, the measure $B_M$ leads to 2.13% of GDP.

![Figure 2. Underestimation of the welfare costs of inflation due to disregarding other monetary assets.](image)

---

For example, $\rho = 0.9$ and $\eta = 0.0823$. 
As previously mentioned, setting $\xi = 1$ in Examples 1 and 2 was arbitrary, done for illustrative reasons only. Starting with Chetty (1969), empirical assessments of the U.S. elasticity of substitution between noninterest-bearing and interest-bearing monies found in the literature have varied in a very wide range. First, because in the last 30 years financial systems all over the world have gone through considerable technological innovations. Second, different definitions of $M1$ and $M2\backslash M1$ (which we are taking in this Section as proxies for $M1^*$ and $M2\backslash M1^*$) have been used in this period\textsuperscript{18}. Third, on account of different model and econometric specifications and methods.

Values reported for this parameter have been as low as 0.024 (Husted and Rush, 1984, p. 179) and as high as 30.864 (Chetty, 1969, p. 278). Edwards (1972, p. 566) reports 2.417; Boughton (1981, p. 383), 4.63; and Gauger (1992, p. 250), 0.1400. Estimates for this same elasticity using post-80’s U.S. data include 9.73 (Sims et al., 1987, p. 123); 0.0981 (Gauger, 1992, p. 251); 1.691 (Fisher, 1992, p. 150); and $1/(1 - 0.269) \approx 1.368$ (Poterba and Rotemberg, 1987, p. 229).

Figure 3 uses the same set of parameters used in Example 1 (except $\xi$) and depicts the behavior of the bias function $\Omega$ for $\xi \in \{0.02, 0.05, 0.1, 0.2, 0.5, 1, 2, 5, 10, 20, 50\}$.

\textsuperscript{18}Not to mention the fact that some of the measurements of the elasticity of substitution found in the literature refer to partitions (of the monetary aggregates) other than this one.
Figure 3. Bias function for several values of the elasticity of substitution between $m$ and $x$.

The 200% (or 2) bias corresponding to the Cobb-Douglas-G case, already calculated in Example 1, is repeated in Figure 3. As anticipated in Proposition 5, when $\xi = 0.2$ (only because we have set $\alpha = 0.2$) there is no bias at all. The figure suggests that in the low-$\xi$/left-side (high-$\xi$/right-side) case a larger interest rate tends to reduce (increase) the relative difference between these two measures.

Although it is not our purpose here to make a sensitivity analysis regarding $\Omega$, it is clear from Figure 3 that the bias is usually of a relevant magnitude. Consider, for instance, a 10% yearly interest rate. Then taking $\xi = 0.02$ leads to a $B_U$ about 28% lower than $B_M$, whereas with $\xi = 2$ $B_U$ turns out to be about 585% higher than $B_M$.

5 Conclusions

Several papers in the literature on the welfare costs of inflation use the one-dimensional formulas provided by Bailey (1956) and by Lucas (2000), in which interest-bearing monies
are not explicitly considered. This procedure (as acknowledged in Lucas 2000) is at odds with the huge process of financial innovations which has occurred in several economies since the 80’s, and which has led to the widespread use of different interest-bearing monies.

This paper has built upon the literature on the welfare costs of inflation with different monies to provide the sign and a measure of the error when interest-bearing monies are not explicitly introduced in the calculations.

Specializing to the case in which the user cost of the interest-bearing asset is constant, we have provided conditions under which the use of unidimensional formulas can lead to overestimation or underestimation of the true welfare costs. The general results assume only a homothetic utility function. The discussion was then further specialized to a CES utility, including the case in which the money demand has a bilogarithmic structure. We have used the bilogarithmic money demand to illustrate with reasonable parameter values that Bailey’s and Lucas’s measures of the welfare costs of inflation may easily overestimate the true welfare costs of inflation by a factor as high as 3.

Underestimation may also occur, but only if the interest-rate elasticity of \( M1 \) is, in absolute value, sufficiently high (as stipulated by Proposition 2). For instance, in the Cobb-Douglas \( G \) case, leading to the bilogarithmic money demand (26), it would have to be greater than one, an assumption usually not supported by empirical evaluations. The intuition for the overestimation implied by the use of unidimensional formulas is clear: such measures capture the welfare costs caused by inflation due to a decrease in the equilibrium money demand. However, they ignore the fact that the existence of interest-bearing monetary assets in the economy may partially offset the drop in the real equilibrium value of \( M1 \), by these means leading to welfare costs that may be lower than those calculated by such formulas.

Finally, we have also depicted the behavior of the relative bias between Bailey’s uni- and multidimensional welfare measures as a function of the interest rate for several values of the elasticity of substitution between monetary aggregates found in the literature. Although
highly dependent on the value of this elasticity, the bias is usually of a relevant magnitude.

Acknowledgement

We thank Robert E. Lucas for helpful discussions. The usual disclaimer applies.
Appendix A

Here we derive (17) and (18) from (16).

Firstly, note that

\[
\delta_{mm}\delta_{xx} - \delta_{mx}^2 = \left( \frac{1}{\varphi - G'\varphi'} \right)^2 \left[ \left( \varphi' G_{mm} + \frac{\varphi''}{\varphi - G'\varphi'} G_m^2 \right) \left( \varphi' G_{xx} + \frac{\varphi''}{\varphi - G'\varphi'} G_x^2 \right) \right]
\]

\[
= \left( \frac{1}{\varphi - G'\varphi'} \right)^2 \left[ \varphi'^2 \left( G_{mm} G_{xx} - G_{mx}^2 \right) + \frac{\varphi''}{\varphi - G'\varphi'} \left( G_{mm} G_x^2 + G_{xx} G_m^2 - 2G_{mx} G_m G_x \right) \right].
\]

From $G_m$’s and $G_x$’s 0-homogeneity, Euler’s relation for homogeneous functions gives

\[
\begin{cases}
0 = mG_{mm} + xG_{mx} \\
0 = mG_{mx} + xG_{xx},
\end{cases}
\]

so that $G_{mx} = -mG_{mm}/x$ and $G_{mx} = -xG_{xx}/m$, and $G_{mm} G_{xx} - G_{mx}^2 = 0$.

Also, $G_{mm} G_x - G_{mx} G_m = G_{mx} (-xG_x/m - G_m) = -(G_{mx}/m) (mG_m + xG_x) = -GG_{mx}/m$, where the last equality comes again from Euler’s relation. Similarly,

$G_{xx} G_m - G_{mx} G_x = -GG_{mx}/x$, so that $G_{mm} G_x^2 + G_{xx} G_m^2 - 2G_{mx} G_m G_x = -G_x G G_{mx}/m - G_m G G_{mx}/x = -G^2 G_{mx}/(mx)$, where once again Euler’s relation is used in the last step.

Thus

\[
\delta_{mm}\delta_{xx} - \delta_{mx}^2 = - \frac{G^2 G_{mx}}{mx} \frac{\varphi' \varphi''}{(\varphi - G'\varphi')^3}
\]

which, together with (16), gives (17) and (18).

Appendix B

We here calculate the partial derivatives of $G$ under Assumption $G$. Regardless of whether $\xi$ is 1 (Cobb-Douglas case) or not, one has:
\[ \cdot \text{Case } \xi \neq 1: G(m, x) = \left( \beta m^{\frac{\xi - 1}{\xi}} + (1 - \beta) x^{\frac{\xi - 1}{\xi}} \right)^{\frac{1}{\xi - 1}} \Rightarrow G_m = \beta m^{\frac{1}{\xi}} \times \left( \beta m^{\frac{\xi - 1}{\xi}} + (1 - \beta) x^{\frac{\xi - 1}{\xi}} \right)^{\frac{1}{\xi - 1}} = \beta (G/m)^{\frac{1}{\xi}} \text{ and, analogously, } G_x = (1 - \beta) (G/x)^{\frac{1}{\xi}}. \]

\[ \cdot \text{Case } \xi = 1: G(m, x) = m^{\beta - 1} \Rightarrow G_m = \beta m^{\beta - 1} x^{1 - \beta} = \beta G/m = \beta (G/m)^{\frac{1}{\xi}} \text{ and } G_x = (1 - \beta) (G/x)^{\frac{1}{\xi}}. \]

So one can continue with the calculations from this point on as if it were a single case:

\[ G_{mm} = \frac{\beta}{\xi} \left( \frac{G}{m} \right)^{\frac{1}{\xi} - 1} G_m m - G = \frac{-\beta}{\xi} \left( \frac{G}{m} \right)^{\frac{1}{\xi} - 1} G_x m = \frac{-\beta (1 - \beta) G^{-1 + \frac{2}{\xi}}}{m^{\frac{1}{\xi} + \frac{1}{\xi} x^{-1 + \frac{1}{\xi}}}}; \]

\[ G_{xx} = -\frac{\beta (1 - \beta) G^{-1 + \frac{2}{\xi}}}{m^{-1 + \frac{1}{\xi} x^1 + \frac{1}{\xi}}}; \]

\[ G_{mx} = \frac{\beta}{\xi} \left( \frac{G}{m} \right)^{\frac{1}{\xi} - 1} G_x m = \frac{\beta (1 - \beta) G^{-1 + \frac{2}{\xi}}}{m^1 \frac{1}{\xi} x^\xi}. \]

\[ \text{Appendix C} \]

Our objective here is to show how the money-demand specification (26) arises from (11), Assumptions $G$ and $\varphi$, as well as evaluating expression (24) under these assumptions.

From

\[ \varphi(G) = \begin{cases} 
\left( \rho + (1 - \rho) (\eta G)^{\frac{\alpha - 1}{\alpha}} \right)^{\frac{\alpha}{\alpha - 1}}, & \text{if } \alpha > 0 \text{ and } \alpha \neq 1 \\
(\eta G)^{1 - \rho}, & \text{if } \alpha = 1 
\end{cases}, \]

we get

\[ \varphi'(G) = \begin{cases} 
\left( \rho + (1 - \rho) (\eta G)^{\frac{\alpha - 1}{\alpha}} \right)^{\frac{1}{\alpha - 1}} (1 - \rho) \eta^{\frac{\alpha - 1}{\alpha}} G^{-\frac{1}{\alpha}} & \text{if } \alpha > 0 \text{ and } \alpha \neq 1 \\
(1 - \rho) \eta^{\frac{\alpha - 1}{\alpha}} G^{-\rho} & \text{if } \alpha = 1 
\end{cases}, \]

\[ = (1 - \rho) \eta^{\frac{\alpha - 1}{\alpha}} \left( \frac{\varphi(G)}{G} \right)^{\frac{1}{\alpha}} \]
and

\[
\varphi (G) - G \varphi' (G) = \begin{cases} 
\varphi (G)^{\frac{1}{\alpha}} \left( \varphi (G)^{\frac{\alpha-1}{\alpha}} - G (1 - \rho) \eta^{\frac{\alpha-1}{\alpha}} G^{-\frac{1}{\alpha}} \right) & \text{if } \alpha > 0 \text{ and } \alpha \neq 1 \\
\rho \varphi (G)^{\frac{1}{\alpha}} , & \text{} \\
\varphi (G) - G (1 - \rho) \eta^{1-\rho} G^{-\rho} = \rho \varphi (G) , & \text{if } \alpha = 1 \\
= \rho \varphi (G)^{\frac{1}{\alpha}} . & \text{}
\end{cases}
\]

Then (11) gives

\[
\begin{aligned}
R_B &= \frac{(1-\rho) \frac{\alpha-1}{\alpha} (\varphi (G) \frac{1}{G})^{\frac{1}{\alpha}}}{\rho \varphi (G)^{\frac{1}{\alpha}}} G_m = \beta^{1-\rho} \eta^{\frac{\alpha-1}{\alpha}} G^{\frac{1}{\alpha}} m^{-\frac{1}{\xi}} \\
R_B - R &= \frac{(1-\rho) \frac{\alpha-1}{\alpha} (\varphi (G) \frac{1}{G})^{\frac{1}{\alpha}}}{\rho \varphi (G)^{\frac{1}{\alpha}}} G_x = (1 - \beta) \frac{1-\rho}{\rho} \eta^{\frac{\alpha-1}{\alpha}} G^{\frac{1}{\alpha}} m^{-\frac{1}{\xi}} x^{-\frac{1}{\xi}},
\end{aligned}
\]

which can be inverted (first in the more general $\xi \neq 1$ case) by noting that

\[
\begin{aligned}
\beta^{\xi} R_B^{1-\xi} + (1 - \beta)^{\xi} (R_B - R)^{1-\xi} &= \left( \frac{1-\rho}{\rho} \frac{\alpha-1}{\alpha} \right)^{1-\xi} G^{(1-\xi)(\frac{1}{\xi} - \frac{1}{\xi})} \left( \beta m^{\frac{\xi-1}{\xi}} + (1 - \beta) x^{\frac{\xi-1}{\xi}} \right) \\
&= \left( \frac{1-\rho}{\rho} \frac{\alpha-1}{\alpha} \right)^{1-\xi} G^{\frac{\xi-1}{\alpha}},
\end{aligned}
\]

whence

\[
G = \left( \frac{1-\rho}{\rho} \right)^{\alpha} \eta^{\alpha-1} \left( \beta^{\xi} R_B^{1-\xi} + (1 - \beta)^{\xi} (R_B - R)^{1-\xi} \right)^{\frac{\alpha}{\xi-1}}.
\]

Plugging this into (C1) yields the demand system

\[
\begin{aligned}
m &= \left( \frac{1-\rho}{\rho} \right)^{\alpha} \eta^{\alpha-1} \left( \beta R_B^{1} \right)^{\xi} \left( \beta^{\xi} R_B^{1-\xi} + (1 - \beta)^{\xi} (R_B - R)^{1-\xi} \right)^{\frac{\xi-\alpha}{1-\xi}} \\
x &= \left( \frac{1-\rho}{\rho} \right)^{\alpha} \eta^{\alpha-1} \left( (1 - \beta) (R_B - R)^{-1} \right)^{\xi} \left( \beta^{\xi} R_B^{1-\xi} + (1 - \beta)^{\xi} (R_B - R)^{1-\xi} \right)^{\frac{\xi-\alpha}{1-\xi}}.
\end{aligned}
\]
For the $\xi = 1$ case, (C1) reads

\[
\begin{cases}
R_B = \beta^{1-\rho} \eta^{\alpha-1} m^\beta \eta^{\alpha-1} \cdot x^{(1-\beta)\alpha-1} \\
R_B - R = (1-\beta) \frac{1-\rho}{\rho} \eta^{\alpha-1} m^\beta \eta^{\alpha-1} \cdot x^{(1-\beta)\alpha-1},
\end{cases}
\]

readily implying (26).

We also have, from the expression above for $\varphi'(G)$,

\[
\varphi''(G) = \frac{1}{\alpha} (1-\rho) \eta^{\alpha-1} \left( \varphi(G) \right)^{\frac{1}{\alpha}-1} \frac{\varphi'(G) G - \varphi(G)}{G^2} 
= -\frac{1-\rho}{\alpha} \eta^{\alpha-1} \left( \varphi(G) \right)^{\frac{1}{\alpha}-1} \frac{\rho \varphi(G)^{\frac{1}{\alpha}}}{G^2} 
= -\frac{\rho (1-\rho)}{\alpha} \eta^{\alpha-1} \varphi(G)^{\frac{2}{\alpha}-1} \frac{G^{1+\frac{1}{\alpha}}}{G^{1+\frac{1}{\alpha}}},
\]

so that

\[
\frac{\varphi'(G) G \varphi''}{G \varphi'(G)} = \frac{(1-\rho) \eta^{\alpha-1} \left( \varphi(G) \right)^{\frac{1}{\alpha}} \rho \varphi(G)^{\frac{1}{\alpha}}}{G \varphi(G) \left( -\frac{\rho (1-\rho)}{\alpha} \eta^{\alpha-1} \varphi(G)^{\frac{2}{\alpha}-1} \frac{G^{1+\frac{1}{\alpha}}}{G^{1+\frac{1}{\alpha}}} \right)} = -\alpha,
\]

and the expression in (24) equals $\xi - \alpha$.

Finally, for $\xi \neq 1$, (C2) gives

\[
\varepsilon_{R_B} = \frac{\partial \log m}{\partial \log R_B} \left( \frac{1-\rho}{\rho} \eta^{\alpha-1} \beta^\xi \right) - \xi \log R_B + \frac{\xi-\alpha}{1-\xi} \log \left( \beta^\xi R_B^{1-\xi} + (1-\beta)^\xi K^{1-\xi} \right)
= \frac{\partial \log (\log \left( \frac{1-\rho}{\rho} \eta^{\alpha-1} \beta^\xi \right) - \xi \log R_B + \frac{\xi-\alpha}{1-\xi} \log \left( \beta^\xi R_B^{1-\xi} + (1-\beta)^\xi K^{1-\xi} \right))}{\partial \log R_B}
= -\xi + \frac{\xi-\alpha}{1-\xi} \frac{\beta^\xi (1-\xi) R_B^{-\xi}}{\beta^\xi R_B^{1-\xi} + (1-\beta)^\xi K^{1-\xi}}
= -\xi + \frac{\xi-\alpha}{1-\xi} \frac{\beta^\xi R_B^{1-\xi}}{\beta^\xi R_B^{1-\xi} + (1-\beta)^\xi K^{1-\xi}},
\]

which coincides with (29).
Appendix D

Here we calculate the welfare cost measures $B_U$ and $B_M$. We are still under Assumptions $G$ and $\varphi$.

- Case $\xi = 1$ and $\alpha = 1$:

$$B_U (R_B) = - \int_{R_B} \theta dm (\theta) = - \theta m (\theta)|_K^{R_B} + \int_{K}^{R_B} m (\theta) d\theta$$

$$= \frac{1 - \rho \beta}{\rho} \left[ - 1|_K^{R_B} + \int_{K}^{R_B} \theta^{-1} d\theta \right] = \frac{1 - \rho \beta}{\rho} \left( \tilde{K} - R_B + \ln \frac{R_B}{K} \right).$$

$B_M$ is shown in (27).

- Case $\xi = 1$ and $\alpha \neq 1$:

$$B_U (R_B) = - \int_{R_B} \theta dm (\theta) = - \theta m (\theta)|_K^{R_B} + \int_{K}^{R_B} m (\theta) d\theta$$

$$= \left( \frac{1 - \rho}{\rho} \right)^{\alpha} \eta^{\alpha - 1} \beta^{1 + (\alpha - 1) \beta} \left( (1 - \beta) \tilde{K}^{-1} \right)^{(\alpha - 1)(1 - \beta)} \times$$

$$\left( - \theta^{(1 - \alpha) \beta} \bigg|_K^{R_B} + \int_{K}^{R_B} \theta^{-1 + (1 - \alpha) \beta} d\theta \right)$$

$$= \left( \frac{1 - \rho}{\rho} \right)^{\alpha} \eta^{\alpha - 1} \beta^{1 + (\alpha - 1) \beta} \left( (1 - \beta) \tilde{K}^{-1} \right)^{(\alpha - 1)(1 - \beta)} \left[ - \theta^{(1 - \alpha) \beta} + \frac{\theta^{(1 - \alpha) \beta}}{(1 - \alpha) \beta} \bigg|_K^{R_B} \right]$$

$$= \left( \frac{1 - \rho}{\rho} \right)^{\alpha} \frac{1 - (1 - \alpha) \beta}{1 - \alpha} \left( \eta \beta \left( 1 - \beta \right)^{\alpha - 1} \tilde{K}^{(1 - \alpha)(1 - \beta)} \times \right.$$

$$\left. (R_B)^{(1 - \alpha) \beta} - \tilde{K}^{(1 - \alpha) \beta} \right).$$

Again, $B_M$ is shown in (27).
\cdot \text{Case } \xi \neq 1 \text{ and } \alpha = 1:

\begin{align*}
B_U(R_B) &= -\int_K \theta dm(\theta) = -\theta m(\theta)\bigg|_K^{R_B} + \int_K^{R_B} m(\theta) d\theta \\
&= \beta^\xi \frac{1 - \rho}{\rho} \left[ -\theta^{1 - \xi} \left( \beta^\xi \theta^{1 - \xi} + (1 - \beta)^\xi \bar{K}^{1 - \xi} \right)^{-1} \bigg|_K^{R_B} + \int_K^{R_B} \theta^{-\xi} \left( \beta^\xi \theta^{1 - \xi} + (1 - \beta)^\xi \bar{K}^{1 - \xi} \right)^{-1} d\theta \right] \\
&= \frac{1 - \rho}{\rho} \beta^\xi \left[ \frac{1}{\beta^\xi (1 - \xi)} - \frac{R_B^{1 - \xi}}{R_B^{1 - \xi} + (1 - \beta)^\xi \bar{K}^{1 - \xi}} \right] \\
&= \frac{1 - \rho}{\rho} \beta^\xi \left[ \frac{1}{\beta^\xi (1 - \xi)} - \frac{R_B^{1 - \xi}}{R_B^{1 - \xi} + (1 - \beta)^\xi \bar{K}^{1 - \xi}} \right].
\end{align*}

Using \( B_M(R_B) = B_U(R_B) - \bar{K} \left( x(R_B) - x(\bar{K}) \right) \):

\begin{align*}
B_M(R_B) &= B_U(R_B) - \frac{1 - \rho}{\rho} (1 - \beta)^\xi \bar{K}^{1 - \xi} \left[ \frac{1}{\beta^\xi R_B^{1 - \xi} + (1 - \beta)^\xi \bar{K}^{1 - \xi}} - \frac{1}{\beta^\xi + (1 - \beta)^\xi} \right] \\
&= \frac{1 - \rho}{\rho} \frac{1}{1 - \xi} \ln \frac{\beta^\xi R_B^{1 - \xi} + (1 - \beta)^\xi \bar{K}^{1 - \xi}}{\bar{K}^{1 - \xi} \left( \beta^\xi + (1 - \beta)^\xi \right)}.
\end{align*}
\* Case \( \xi \neq 1 \) and \( \alpha \neq 1 \):

\[
B_U(R_B) = - \int_K \theta dm(\theta) = - \theta m(\theta)|_{R_B} + \int_K m(\theta) d\theta
\]

\[
= \beta^\xi \left( \frac{1 - \rho}{\rho} \right)^\alpha \eta^{\alpha - 1} \left[ - \theta^{1 - \xi} \left( \beta^\xi \theta^{1 - \xi} + (1 - \beta)^\xi K^{1 - \xi} \right)^{\frac{\xi - \alpha}{1 - \xi}} |_K^{R_B} \right] + \int_K \theta^{1 - \xi} \left( \beta^\xi \theta^{1 - \xi} + (1 - \beta)^\xi K^{1 - \xi} \right)^{\frac{\xi - \alpha}{1 - \xi}} d\theta
\]

\[
= \beta^\xi \left( \frac{1 - \rho}{\rho} \right)^\alpha \eta^{\alpha - 1} \left[ - \theta^{1 - \xi} \left( \beta^\xi \theta^{1 - \xi} + (1 - \beta)^\xi K^{1 - \xi} \right)^{\frac{\xi - \alpha}{1 - \xi}} + \frac{1}{(1 - \alpha)\beta^\xi} \left( \beta^\xi \theta^{1 - \xi} + (1 - \beta)^\xi K^{1 - \xi} \right)^{\frac{\xi - \alpha}{1 - \xi}} \right] |_K^{R_B}
\]

\[
= \left( \frac{1 - \rho}{\rho} \right)^\alpha \eta^{\alpha - 1} \frac{1}{1 - \alpha} \left[ - \theta^{1 - \xi} \left( \beta^\xi \theta^{1 - \xi} + (1 - \beta)^\xi K^{1 - \xi} \right) \left( \beta^\xi \theta^{1 - \xi} + (1 - \beta)^\xi K^{1 - \xi} \right)^{\frac{\xi - \alpha}{1 - \xi}} \right] |_K^{R_B}
\]

\[
= \left( \frac{1 - \rho}{\rho} \right)^\alpha \eta^{\alpha - 1} \frac{1}{1 - \alpha} \left[ - \theta^{1 - \xi} \left( \beta^\xi R_B^{1 - \xi} + (1 - \beta)^\xi K^{1 - \xi} \right) \left( \beta^\xi R_B^{1 - \xi} + (1 - \beta)^\xi K^{1 - \xi} \right)^{\frac{\xi - \alpha}{1 - \xi}} \right] - \frac{1}{(1 - \alpha)\beta^\xi} \left( \beta^\xi R_B^{1 - \xi} + (1 - \beta)^\xi K^{1 - \xi} \right)^{\frac{\xi - \alpha}{1 - \xi}} |_K^{R_B}
\]

Again using \( B_M(R_B) = B_U(R_B) - \bar{K} (x(R_B) - x(\bar{K})) \):

\[
B_M(R_B) = B_U(R_B) - \left( \frac{1 - \rho}{\rho} \right)^\alpha \eta^{\alpha - 1} (1 - \beta)^\xi K^{1 - \xi} \left[ - \theta^{1 - \xi} \left( \beta^\xi R_B^{1 - \xi} + (1 - \beta)^\xi K^{1 - \xi} \right)^{\frac{\xi - \alpha}{1 - \xi}} \right] + \frac{1}{(1 - \alpha)\beta^\xi} \left( \beta^\xi (1 - \beta)^\xi \right)^{\frac{\xi - \alpha}{1 - \xi}} |_K^{R_B}
\]

\[
= \left( \frac{1 - \rho}{\rho} \right)^\alpha \eta^{\alpha - 1} \frac{1}{1 - \alpha} \left[ - \theta^{1 - \xi} \left( \beta^\xi R_B^{1 - \xi} + (1 - \beta)^\xi K^{1 - \xi} \right)^{\frac{\xi - \alpha}{1 - \xi}} \right] - \frac{1}{(1 - \alpha)\beta^\xi} \left( \beta^\xi (1 - \beta)^\xi \right)^{\frac{\xi - \alpha}{1 - \xi}} |_K^{R_B}
\]
References


