The \( n \)-Dimensional Bailey-Divisia Measure as a General-Equilibrium Measure of the Welfare Costs of Inflation

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Bailey’s Measure as an Approximation

- Bailey (1956), Lucas (Econometrica, 2000)
Bailey’s Measure as an Approximation

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![Graph](image_url)

**Figure 8.**—Approximate and exact welfare cost functions.
Bailey’s Measure as an Approximation

- Bailey (1956), Lucas (Econometrica, 2000)

- Simonsen and Cysne (JMCB, 2001): $w = 1 - e^{-A} < s < A < B$

Figure 8.—Approximate and exact welfare cost functions.
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\[ 1 - \sigma \]

\[ \phi(m, c) \]
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- \[ U(c, m) = g(c + \lambda(m)) \] rather than \[ U(c, m) = \frac{1}{1-\sigma} \left[ c \varphi \left( \frac{m}{c} \right) \right]^{1-\sigma} \]
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Regarding the United States, Mulligan and Sala-i-Martin (2000, p. 962) report, following the 1989 Survey of Consumer Finances, that 35% of all households hold bank deposits and at least one additional interest-bearing asset.
Using One - Rather than n - Dimensional Measures

- Suggestion by Lucas (Econometrica, 2000)
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Cysne (JMCB, 2003): A Divisia Index as a Welfare Measure
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- Bias Depends on the elasticity of substitution among the different monies
Cysne and Turchick (MD, forthcoming): \( \underline{w} = 1 - e^{-A} < s < A < B < \overline{w} \)
Abstract

This paper shows that Bailey’s multidimensional Divisia Index measure $B_D(m) = -\int_{\chi} u \cdot dm$ emerges as an exact general-equilibrium measure of the welfare costs of inflation when preferences are quasilinear.
The Model

Our model is an extension of Sidrauski’s (1967) to an economy with several monies. Let

\[ \mathbf{m} = (m_1, \ldots, m_n) \in [0, +\infty]^n \]

represent the vector of real quantities of each type of money, as a fraction of nominal GDP. Real output is supposed to be constant and equal to one. Each \( m_i \) yields a nominal interest rate of \( r_i \), and

\[ \mathbf{r} := (r_1, \ldots, r_n) \in \mathbb{R}_+^n \]

The first monetary asset \( (m_1) \) is assumed to be real currency, in which case \( r_1 = 0 \).
We shall write the vector of opportunity costs (relatively to holding bonds) as:
\[ u := (u_1, \ldots, u_n) := (r, r - r_2, \ldots, r - r_n) \in \mathbb{R}_+^n \]

Given a concave utility function \( U(c, m) \), the maximization problem of the representative consumer reads:

\[
\max_{c > 0, \ m \geq 0} \int_0^{+\infty} e^{-\rho t} U(c, m) \, dt \tag{P^n_S}
\]

subject to

\[
\dot{b} + 1 \cdot m = 1 - h - c + (r - \pi) b + (r - (\pi \cdot 1)) \cdot m \\
b_0 > 0 \text{ and } m_0 > 0 \text{ given.}
\]

where we write \( 1 \) for the vector \((1, \ldots, 1) \in \mathbb{R}^n\), and ‘.’ for the canonical inner product of \( \mathbb{R}^n \).
The Model

Given a rate of monetary expansion equal to $\theta$, in the steady state we have $\pi = \theta$ and the usual Euler equations imply:

$$r = \theta + \rho$$

$$u_i : = r - r_i = \frac{U_{mi}}{U_c}, \quad \forall i \in \{1, \ldots, n\}.$$  \hspace{1cm} (1)

In equilibrium $c = 1$ and the $n$ equations given by (2) determine the demand for the $n$ monetary assets in the economy.
This subsection is based on Cysne and Turchick (2007). Assume for a moment that our representative agent’s utility had the general form:

$$U(c, m) = \frac{1}{1 - \sigma} \left( c \varphi \left( \frac{G(m)}{c} \right) \right)^{1-\sigma}$$

(3)

where $G$ is a monetary aggregator function, $\sigma > 0$, $\sigma \neq 1$. $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a differentiable function satisfying:

**Assumption $\varphi_0$.** $(\varphi / \varphi') (0+) = 0$.

**Assumption $\varphi$.** There exists an $\bar{m} \in (0, +\infty]$ such that $\varphi \mid_{[0, \bar{m}]}$ is strictly increasing, $\varphi \mid_{[\bar{m}, +\infty)}$ is constant, $\varphi'' \mid_{(0, \bar{m})} < 0$ and $\varphi' (\bar{m}-) = 0$. 

Equations (2) now imply:

\[
u_i(m) = \frac{\phi'(G(m))}{\phi(G(m)) - G(m) \phi'(G(m))} G_{m_i}(m), \forall i \in \{1, \ldots, n\}.
\]  

Let \( C_m := \{m \in \mathbb{R}^n_{++} : G(m) = \bar{m}\} \). The consumer is satiated with monetary balances when \( G(m) = \bar{m} \). We denote by \( \bar{m} \) those \( m \) which lie in \( C_m \).

Throughout this paper we shall follow the same methodology set forth by Lucas (2000) and implicitly define the welfare costs of inflation \( w(m) \) by:

\[
U(1 + w(m), G(m)) = U(1, G(\bar{m})).
\]
The Model
A n-Dimensional (Sidrauski) General-Equilibrium Measure of the Welfare Costs of Inflation

Take the partial derivatives of (5) and divide by
\[ \varphi' \left( G \left( (1 + w(m))^{-1} m \right) \right) \] to obtain:

\[
w_i(m) \frac{\varphi \left( G \left( \frac{1}{1+w(m)} m \right) \right)}{\varphi' \left( G \left( \frac{1}{1+w(m)} m \right) \right)} + \left[ G_{m_i}(m) - w_i(m) G \left( \frac{1}{1 + w(m)} m \right) \right] = 0.
\]

Using (4):

\[
\frac{\varphi \left( G \left( \frac{1}{1+w(m)} m \right) \right)}{\varphi' \left( G \left( \frac{1}{1+w(m)} m \right) \right)} = \frac{G_{m_i}(m)}{u_i \left( \frac{1}{1+w(m)} m \right)} + G \left( \frac{1}{1 + w(m)} m \right),
\]
This leads to \( w \) being represented by the \( n \) differential equations:

\[
w_i(m) = -u_i \left( \frac{1}{1 + w(m)m} \right)\]

with initial condition \( w(m) = 0 \).

Alternatively, consider a \( C^1 \) path

\[
\chi : [0, 1] \rightarrow [0, +\infty]^n
\]

such that \( \chi(0) = m \) and \( \chi(1) = m \). Then, \( w(m) \) can be written as the line integral:

\[
w(m) = -\int_{\chi} u \left( \frac{1}{1 + w(m)m} \right) \cdot dm
\]
Note that,

\[ w(m) = - \int_\chi u \left( \frac{1}{1 + w(m)} m \right) \cdot dm \]

the measure of the welfare costs of inflation which emerges from Sidrauski’s general-equilibrium model, is not equal to the Bailey-Divisia (\( B_D \)) measure, defined by:

\[ B_D(m) = -u_i(m), \quad B_D(-m) = 0 \quad (8) \]

or, alternatively, when expressed as a line integral, and considering the path \( \chi \) defined above:

\[ B_D(m) = - \int_\chi u \cdot dm \quad (9) \]
Our main purpose here is showing that $B_D = w$ when preferences, rather than defined by the general form (3), can be expressed by the quasi-linear form:

$$U(c, m) = g(c + f(G(m)))$$  \hspace{1cm} (10)

where $c$ and $G$ are as defined before. $f : [0, +\infty] \to \mathbb{R}_+$ and $g : [0, +\infty] \to \mathbb{R}$ are twice-differentiable functions such that $f' > 0$, $f'' < 0$, $g' > 0$ and $g'' \leq 0$. Any $U$ in the class of functions represented by (10) is concave in $(c, m)$, since it is given by the composition of concave and increasing functions. This makes $e^{-gt} U$ concave with respect to $(b, \dot{b}, m, \dot{m})$, which makes the Euler equations sufficient for an optimum.
The Model
The Bailey-Divisia Measure as a General-Equilibrium Measure

In this particular case, equations (2) give, $\forall i \in \{1, \ldots, n\}$:

$$u_i = f'(G(m))G_{m_i}(m), \quad i \in \{1, \ldots, n\}$$

(11)

Here, (5) and (10) imply:

$$g(1 + w(m) + f(G(m))) = g(1 + f(G(m)))$$

(12)

Proposition

Let an economy be described as above. Then

$$B_D(m) = w(m)$$

(13)
Proof.
The demonstration follows basically the same steps as in Cysne (2009). Since $g'(.) > 0$, we can write, from (12):

$$w(m) = f(G(m)) - f(G(m))$$

Taking the derivative with respect to $m_i$:

$$w_i(m) = -f'(G(m))G_{m_i}(m), \quad \forall i \in \{1, \ldots, n\}$$

By using (11) in (15):

$$w_i(m) = -u_i(m), \quad \forall i \in \{1, \ldots, n\}$$
The Model
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Proof.

Normalize \( w(m) \) by making \( w(m) = 0 \), which is equivalent to writing \( B_D(\chi(0)) = 0 \). Now use the definition of \( B_D \) in (9) to obtain the main result:

\[
w(m) = - \int_{\chi} u \cdot dm = B_D(m) \quad (17)
\]
General-equilibrium considerations remind us that the vector $\mathbf{m}$ in (17) is a function not only of the nominal interest rate $r$, but also of all remaining spreads $u_i$, $i \in \{2, \ldots, n\}$.

**Remark**

*Note, by using the parameterized path (6), that (17) can also be written as:*

$$w(\mathbf{m}) := \int_0^1 \frac{d}{d\lambda} w(\chi(\lambda)) \, d\lambda = \int_0^1 [-u(\chi(\lambda)) \cdot \nabla \chi(\lambda)] \, d\lambda \quad (18)$$

*In (18), $\lambda$ stands for a real parameter taking values in $[0, 1]$, $\chi(\lambda)$ for the vector of monetary aggregates and $\nabla \chi(\lambda)$ for the vector of derivatives of $\chi$ with respect to $\lambda$.***
Applications

Example 1: Area Under the Inverse Demand for Monetary Base

Here we assume that the spreads $u_i, \forall i \in \{2, \ldots, n\}$ are competitively determined by a costless banking system operating under constant and non-remunerated reserve requirements. In this case $u_i := r - r_i = k_i r$, $k_i$ standing for the reserve requirements on deposit $i$. Let $z$ stand for the monetary base, $k := (k_2, \ldots, k_n)$, $m_{(-1)} := (m_2, \ldots, m_n) \in [0, +\infty]^{n-1}$. With $u_i = k_i r$ for all $i$ (17) becomes:

$$B_D(m) = w(m) = -\int_\chi r(dm_1 + k \cdot dm_{(-1)})$$

But in such an economy the monetary base (equal to currency plus non-interest-bearing reserves deposited in the Central Bank) reads $z := m_1 + k \cdot m$. Since $k$ is a vector of constants, $dz = dm_1 + k \cdot dm_{(-1)}$, in which case (19) can be written as:

$$B_D(m) = w(m) = -\int m_1 + k \cdot m_{(-1)} rdz$$
Applications

Example 1: Area Under the Inverse Demand for Monetary Base

The conclusion of this example is that under quasilinear preferences the Bailey-Divisia measure leads to an exact general-equilibrium welfare measure equal to the area under the inverse demand for monetary base (rather than the inverse demand for $M_1$, as used, for instance, by Lucas (2000)).
Let $\lambda = r$ in (18) and assume that all banking spreads $u_i$ other than that on currency ($i \in \{2, \ldots, n\}$) remain constant. This would be the case for instance if $m_2, m_3, \ldots m_n$ were issued by the government and their interest payments increased pari-passu with the interest rate on bonds, the only nonmonetary asset in the economy. In this case, using Remark 1, $B_D(m)$ can be written as a function of the nominal interest rate (which takes the role of parameter $\lambda$ by a redefinition of domain) under in the following way:

$$B_D(m) = - \int_0^r \left[ x m_1'(x) + \sum_{i=2}^n \bar{u}_i m_i'(x) \right] dx$$
\[ B_D(m) = - \int_0^r \left[ xm_1'(x) + \sum_{i=2}^n \bar{u}_i m_i'(x) \right] dx \]

Note above that having all spreads constant does not imply that Bailey’s unidimensional formula is the correct one to be used. The remaining term \( \sum_{i=2}^n \bar{u}_i m_i'(x) \) reminds us that one should also take into consideration the implications of the changes of the nominal interest rate on the demand for all other monies. Cysne and Turchick (JEDC, 2010) concentrate on calculating the bias originated by measurements which neglect this fact.
Applications

Example 3

Take, in (10), $g$ as the identity function, $f$ defined by $f(x) = \frac{x^{1-\sigma} - 1}{1-\sigma}$, where $\sigma > 0$ and $\sigma \neq 1$, $f(x) = \ln x$ when $\sigma = 1$ and $G(m_1, m_2) = m_1^\mu m_2^{1-\mu}$, with $0 < \mu < 1$. From (11) we have $u_i(m) = G(m)^{-\sigma} G_{x_i}(m)$. So, from (9),

$$B_D(m) = - \int_\chi G(m)^{-\sigma} (G_{x_1}(m) \, dm_1 + G_{x_2}(m) \, dm_2)$$

$$= - \int_{G(m)} G^{-\sigma} \, dG = \frac{1}{1-\sigma} \left( G(m)^{1-\sigma} - G(m)^{1-\sigma} \right)$$

(20)

The definition of $G$ is only used in the final step to generate:

$$B_D(m) = \frac{1}{1-\sigma} \left[ m_1^{\mu(1-\sigma)} m_2^{1-\mu(1-\sigma)} - m_1^{\mu(1-\sigma)} m_2^{1-\mu(1-\sigma)} \right]$$
When $\sigma = 1$:

$$B_D(m) = \ln \frac{G(m)}{G(\bar{m})} = \mu \ln \frac{\bar{m}_1}{m_1} + (1 - \mu) \ln \frac{\bar{m}_2}{m_2}$$

(21)

In this case the welfare costs of inflation are given by a weighted average which measures how relatively distant the representative agent is from satiation with respect to each monetary asset. Finally, note that both (20) and (21) can also be directly obtained from (14).
Conclusions

- Bailey’s and Lucas’s one-dimensional formulas do not take into consideration an important fact common to most economies nowadays: the presence of several types of money other than currency or demand deposits.

- The present work has tackled this issue and extended Cysne’s (2009) result by showing that in economies with several monies a Divisia-index version of Bailey’s original measure can also be regarded as an exact general-equilibrium measure of the welfare costs of inflation. Three applications following from this result have been presented.

- The intuition is the same as before, now applied to a higher dimension: in the absence of wealth effects, the consumers’ surplus (here, the multidimensional consumers’ surplus defined by the Bailey-Divisia measure), rather than an approximation, provides an exact measure of the deadweight loss stemming from taxation.
References


